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ANTISYMMETRY AND CONTRACTIVE REPRESENTATIONS OF FUNCTION ALGEBRAS

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ABSTRACT. In the present paper the antisymmetry of the image of a function algebra under its contractive representation is characterized. A complete solution of this problem is obtained for subnormal contractive representations. Some applications, in particular, to the von Neumann functional calculus, are given.

1. Introduction. Let A be a complex Banach algebra with unit 1; A' denotes its dual space and $K(A) = \{\psi \in A'\colon \psi(1) = \|\psi\| = 1\}$. For $a \in A$, $\mathrm{sp}_A\, a$ denotes the spectrum of a in A, $r(a)$ stands for the spectral radius of a, the set $V_A(a) = \{\psi(a)\colon \psi \in K(A)\}$ is called the *numerical range* of a and $v(a) = \sup\{|\lambda|\colon \lambda \in V_A(a)\}$ is called the *numerical radius* of a. If $a \in A$, we denote by $\hat{a}$ the functional on A' given by

$$\hat{a}(\psi) = \psi(a), \qquad \psi \in A'.$$

The set $V_A(a)$ is a convex, compact subset of the complex plane $\mathbf{C}$ and

(V1) if B is a Banach subalgebra of A such that $1, a \in B$, then

$$V_B(a) = V_A(a),$$

(V2) $r(a) \leqslant v(a)$, $e^{-1}\|a\| \leqslant v(a) \leqslant \|a\|$.

An element $a \in A$ is called *hermitian* if $V_A(a) \subset \mathbf{R}$ (the real line). Let H be a complex, Hilbert space. $L(H)$ denotes the algebra of all linear, bounder operators in H and I is the identity operator. For $T \in L(H)$ the set $W(T) = \{(Tx, x)\colon x \in H, \|x\| = 1\}$ is called the *spatial numerical range* of T. By the Hausdorff-Toeplitz theorem, [**4**, Problem 166], $W(T)$ is convex and by Theorem 8 (p 86) of [**1**],

$$\overline{W(T)} = V_{L(H)}(T). \tag{V3}$$

This implies that T is hermitian if and only if $T = T^*$ (what is also proved in terms of C^*-algebras in [**1**, Example 3, p. 47]). For the theory of numerical ranges we refer to [**1**].

For a Banach algebra A with unit 1, a *contractive representation* of A is an algebra homomorphism $\varphi\colon A \to L(H)$ such that $\varphi(1) = I$, $\|\varphi(a)\| \leqslant \|a\|$, $a \in A$. If A is a commutative Banach algebra with unit, then Spec A denotes its maximal ideal space, which is a w^* (weak*) compact subset of $K(A) \subset A'$.

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If X is a compact, Hausdorff space and $A \subset C(X)$ is a function algebra on X (see [3, p. 2]), then for $\lambda \in X$ we write τ_λ for the point-evaluation functional at λ ($\tau_\lambda(a) = a(\lambda)$, $a \in A$) and we consider X homeomorphically embedded into $\operatorname{Spec} A$ by the map $\lambda \to \tau_\lambda$. $\operatorname{Ch}(A)$ denotes the Choquet boundary of A [3, p. 87]. For a subset M of a locally convex topological vector space E we denote by $\overline{\operatorname{co}}\, M$ the closure of the convex hull of M and if M is a compact, convex subset of E, then M^e denotes the set of extreme points of M. We will often use the Krein-Milman theorem, which says that if M is a compact, convex subset of a locally convex topological vector space, then $M = \overline{\operatorname{co}}\, M^e$ [8, p. 6]. An algebra $\mathcal{Q} \subset L(H)$ containing I is called *antisymmetric* if the only selfadjoint elements of $\mathcal{Q}$ are scalar multiples of I [10].

In the present paper we prove necessary and sufficient conditions for the antisymmetry of $\varphi(A)$, where $\varphi\colon A \to L(H)$ is a contractive representation of a function algebra $A \subset C(X)$ (§3). In §1 necessary and sufficient conditions for a commutative Banach algebra A in order to $\overline{\operatorname{co}}\operatorname{Spec} A = K(A)$ are proved. §4 contains applications of the previous results to contractive subnormal representations of function algebras and to the von Neumann functional calculus for subnormal operators.

2. The convex hull of $\operatorname{Spec} A$. In this section we prove a necessary and sufficient condition for a commutative Banach algebra A with unit in order to $K(A) = \overline{\operatorname{co}}\operatorname{Spec} A$, which is needed in §4.

Let us consider two Banach algebras, not necessarily commutative, A, B with units 1 (both units are denoted by 1) and an algebra homomorphism φ: $A \to B$ such that $\|\varphi(a)\| \leqslant \|a\|$, $a \in A$, $\varphi(A)$ is dense in B, and $\varphi(1) = 1$. Let $\varphi'\colon B' \to A'$ be the map induced by φ, i.e. $\varphi'(\tau) = \tau \circ \varphi$, $\tau \in B'$. φ' is linear, one-to-one and continuous in the w^*-topologies of the duals. Put $P = \{\psi \in K(A)\colon |\psi(a)| \leqslant \|\varphi(a)\|, a \in A\}$.

PROPOSITION 1. *The restriction φ' to $K(B)$ maps homeomorphically $K(B)$ onto P. P is a convex, w^*-compact subset of $K(A)$ and for all $a \in A$*

$$\hat{a}(P) = \overline{\operatorname{co}}\ \hat{a}(P^e) = V_B(\varphi(a)).$$

PROOF. First we prove $\varphi'(K(B)) = P$. If $\tau \in K(B)$, then $(\tau \circ \varphi)(1) = 1$ and $|(\tau \circ \varphi)(a)| \leqslant \|\varphi(a)\|$, $a \in A$, thus $\varphi'(\tau) \in P$. Conversely, if $\psi \in P$, then, because $\ker \varphi \subset \ker \psi$, the linear functional

$$\tau(\varphi(a)) = \psi(a), \qquad a \in A,$$

is well defined on $\varphi(A)$. Now, for $a \in A$: $|\tau(\varphi(a))| = |\psi(a)| \leqslant \|\varphi(a)\|$, hence we may extend τ to B and $\tau(1) = \tau(\varphi(1)) = \psi(1) = 1$, thus $\tau \in K(B)$. Since $K(B)$ is convex, w^*-compact in B' and φ' is linear, continuous and one-to-one, the first two assertions follow. Now, if $a \in A$, then

$$\hat{a}(P) = \{\psi(a)\colon \psi \in P\} = \{\tau(\varphi(a))\colon \tau \in K(B)\} = V_B(\varphi(a))$$

and the equality $\hat{a}(P) = \overline{\operatorname{co}}\ \hat{a}(P^e)$ follows from the Krein-Milman theorem and from the linearity and w^*-continuity of $\hat{a}$. Q.E.D.

The next proposition answers the question stated above.

PROPOSITION 2. *For a commutative Banach algebra A with unit* 1 *the following conditions are equivalent*:

(a) $\overline{\text{co}}\,\text{Spec}\,A = K(A)$ (*the closure in the w^*-topology of A'*);

(b) $K(A)^e \subset \text{Spec}\,A$;

(c) *for every $a \in A$, $r(a) = v(a)$.*

PROOF. (a) $\Rightarrow$ (b) follows from a result of [**8**, p. 9] and (b) $\Rightarrow$ (a)–from the Krein-Milman theorem. Let F: $A \to C(\text{Spec}\,A)$ be the Gelfand transform of A. The image $\hat{A}$ of A under F is a function algebra on $\text{Spec}\,A$. Since $\|F(a)\| = r(a) \leqslant \|a\|$, $a \in A$ and $F(1) = 1$, we may identify $K(\hat{A})$, by Proposition 1, with the convex, w^*-compact subset $\{\psi \in K(A)\colon |\psi(a)| \leqslant \|F(a)\| = r(a),\ a \in A\}$ of $K(A)$ and we identify $\text{Spec}\,A$ with a subset of $\text{Spec}\,\hat{A}$ by point-evaluation functionals. Since $\hat{A}$ is a function algebra on $\text{Spec}\,A$, $K(\hat{A}) = \overline{\text{co}}\,\text{Spec}\,A$, by [**8**, Lemma 6.1]. Hence we have a sequence of equivalences:

$K(A) = \overline{\text{co}}\,\text{Spec}\,A \Leftrightarrow K(A) = K(\hat{A}) \Leftrightarrow$ for all $\psi \in K(A)$, $a \in A$,

$|\psi(a)| \leqslant r(a) \Leftrightarrow v(a) \leqslant r(a)$, for all $a \in A$,

which, together with (V2) completes the proof of (a) $\Leftrightarrow$ (c). Q.E.D.

Note, that condition (c) is discussed in [**1**, Theorem 7, p. 40].

3. The main theorem. Let X be a compact, Hausdorff space, $A \subset C(X)$–a function algebra on X, φ: $A \to L(H)$–a contractive representation. B denotes the norm-closure of $\varphi(A)$. Recall, that a subset $Y \subset X$ is called a *set of antisymmetry* for A if every function $a \in A$ real on Y is constant on Y. The algebra A is called *antisymmetric* if X is a set of antisymmetry for A [**3**, p. 136]. We adopt the notation introduced at the beginning of §2, to our φ and we always identify X with the subset of $\text{Spec}\,A$ consisting of point-evaluation functionals τ_λ, $\lambda \in X$. If $a \in A$, then, obviously, $\hat{a}(\tau_\lambda) = a(\lambda)$, $\lambda \in X$. Preserving these notations, we get the following

THEOREM. *Suppose that*

$$\textit{the set } P^e \textit{ of all extreme points of } P \textit{ is a subset of } X. \tag{$*$}$$

Then $\varphi(A)$ is antisymmetric if and only if P^e is a set of antisymmetry for A.

PROOF. Assume that P^e is a set of antisymmetry for A and take $a \in A$ such that $\varphi(a) = \varphi(a)^*$. By (V1) and (V3), $\varphi(a)$ is a hermitian element of B, and, by Proposition 1, $\hat{a}(P) = V_B(\varphi(a)) \subset \mathbf{R}$. Hence $a(P^e) \subset \mathbf{R}$ and, by the assumption, $a(P^e) = \{c\}$ for some real c. By Proposition 1, $\hat{a}(P) = \overline{\text{co}}\,\hat{a}(P^e) = \{c\}$, hence

$$V_B(\varphi(a)) = \{c\} \quad \text{and} \quad V_B(\varphi(a) - c) = \{0\}.$$

This implies $v(\varphi(a) - c) = 0$ and, by (V2), $\varphi(a) = c$. Conversely, take $a \in A$ such that $a(P^e) \subset \mathbf{R}$ and assume the antisymmetry of $\varphi(A)$. Then, again by

Proposition 1,

$$V_B(\varphi(a)) = \hat{a}(P) = \overline{\mathrm{co}}\ \hat{a}(P^e) \subset \mathbf{R},$$

hence, by (V3), $\varphi(a) = \varphi(a)^*$. The antisymmetry of $\varphi(A)$ implies that $\varphi(a) = c$ for some real c, thus $\hat{a}(P) = V_B(\varphi(a)) = \{c\}$ and, in particular, $a(P^e) = \{c\}$. Q.E.D.

As an immediate corollary we get:

COROLLARY 1. *If φ: $A \to L(H)$ is isometric, then $\varphi(A)$ is antisymmetric if and only if A is antisymmetric.*

PROOF. By the Krein-Milman theorem and Theorem 2.2.8 in **[3]** (or Proposition 6.2 of **[8]**), for all $a \in A$:

$$V_A(a) = \overline{\mathrm{co}}\ a(X) = \overline{\mathrm{co}}\ a(\mathrm{Ch}(A)),$$

because $\mathrm{Ch}(A) = K(A)^e$. If φ is an isometry, then $P = K(A)$, hence $P^e = K(A)^e \subset X$, which proves that $(*)$ of the Theorem is satisfied. The application of the Theorem finishes the proof. Q.E.D.

Let us prove now several equivalent conditions for the isometry of φ. φ': $B' \to A'$ is the map introduced at the beginning of §1.

PROPOSITION 3. *The following conditions are equivalent*:
(a) φ *is isometric*,
(b) $K(A) = P$,
(c) $\mathrm{Spec}\, A = \varphi'(\mathrm{Spec}\, B)$,
(d) $X \subset \varphi'(\mathrm{Spec}\, B)$,
(e) $X \subset P$.

PROOF. The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are clear.

(b) $\Rightarrow$ (c). If $\psi \in \mathrm{Spec}\, A \subset K(A)$, then, by (b) and Proposition 1, there is $\tau \in K(B)$ such that $\psi = \tau \circ \varphi$. Since ψ is multiplicative, so is τ, hence $\tau \in \mathrm{Spec}\, B$.

(e) $\Rightarrow$ (a). By Proposition 1 and (V2) we have for all $a \in A$:

$$\|a\| \leqslant \sup\{|\psi(a)| : \psi \in P\} = v(\varphi(a)) \leqslant \|\varphi(a)\| \leqslant \|a\|. \quad \text{Q.E.D.}$$

Now we describe precisely the set P^e.

PROPOSITION 4. (a) $P \cap \mathrm{Ch}(A) \subset P^e$.
(b) *If* $P^e \subset X$, *then* $P^e \subset \mathrm{Ch}(A)$.
(c) *If* $P^e \subset \varphi'(\mathrm{Spec}\, B)$ *and* $X = \mathrm{Spec}\, A$, *then* $P^e = \mathrm{Ch}(A) \cap \varphi'(\mathrm{Spec}\, B)$.

PROOF. (c) follows from (a) and (b), since $\varphi'(\mathrm{Spec}\, B) \subset \mathrm{Spec}\, A = X$. To prove (a) take $\psi \in P \cap \mathrm{Ch}(A)$ such that $\psi = \alpha\psi_1 + (1-\alpha)\psi_2$, $0 \leqslant \alpha \leqslant 1$, $\psi_1, \psi_2 \in P$. Since $P \subset K(A)$ and $\psi \in \mathrm{Ch}(A) = K(A)^e$, we have $\alpha = 0$ or 1, hence $\psi \in P^e$. To prove (b) we show, that if $\psi \in P^e$, $\psi = \tau_\lambda$ with some $\lambda \in X$, then the only representing measure μ on X for ψ is the point mass δ_λ.

Then, applying Theorem 2.2.8 of [**3**] we will finish the proof. Take a probability measure μ on X, representing ψ. We may transport μ to a measure μ' on P, as in [**8**, p. 37], such that μ' represents ψ on P (see [**8**] for the definitions). But $\psi \in P^e$, hence, by [**8**, Proposition 1.4], the only measure, which represents ψ on P is $\delta_\psi = \delta_\lambda$, what, coming back to X, proves, that δ_λ is the only representing measure for $\psi = \tau_\lambda$ on X. Q.E.D.

4. Applications. In this section we want to apply previous results to some special representation of function algebras, which satisfy the condition $(*)$ of the Theorem; and in particular, to subnormal representations and to the von Neumann calculus for subnormal contractions. To begin with we prove a simple remark for the sake of completeness.

REMARK. *Let A be a Banach algebra with unit* 1. *Let $a_n \in A$ be a sequence of elements of A, which converge to a in norm. Then for every neighbourhood U of $V_A(a)$ there is n_0 such that for all $n \geqslant n_0$: $V_A(a_n) \subset U$.*

PROOF. Since $V_A(a)$ is compact, we may consider

$$\operatorname{dist}(\lambda, V_A(a)) = \inf\{|\lambda - z| : z \in V_A(a)\}$$

for $\lambda \in \mathbf{C}$. Note first, that if $b \in A$, then for all $\psi \in K(A)$:

$$\operatorname{dist}(\psi(b), V_A(a)) \leqslant |\psi(b) - \psi(a)| \leqslant v(a - b) \leqslant \|a - b\|.$$

Take now arbitrary $\varepsilon > 0$ and define $U_\varepsilon = \{\lambda \in \mathbf{C}\colon \operatorname{dist}(\lambda, V_A(a)) < \varepsilon\}$. Choose n_0 such that $\|a_n - a\| < \varepsilon$ for $n \geqslant n_0$. Then for all $\psi \in K(A)$:

$$\operatorname{dist}(\psi(a_n), V_A(a)) \leqslant \|a_n - a\| < \varepsilon,$$

thus $V_A(a_n) \subset U_\varepsilon$. Q.E.D.

An operator $T \in L(H)$ is called *convexoid* [**5**, p. 114] if $\overline{W(T)} = \operatorname{co} \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T and $\operatorname{co} \sigma(T)$ is the convex hull of $\sigma(T)$, which is compact. If T is convexoid, then $r(T) = v(T)$.

PROPOSITION 5. *The set $\mathcal{C}$ of all convexoid operators in $L(H)$ is norm-closed.*

PROOF. $L(H)^{-1}$ denotes the set of all invertible operators in $L(H)$. It has been proved by Hildebrandt, [**6**, Satz 4], that for every $T \in L(H)$

$$\operatorname{co} \sigma(T) = \bigcap \left\{ \overline{W(STS^{-1})} : S \in L(H)^{-1} \right\}. \tag{H}$$

Take now a sequence $T_n \in \mathcal{C}$, such that $T_n \to T$ in norm. It is enough to prove that $W(T) \subset \operatorname{co} \sigma(T)$, because the opposite inclusion always holds [**5**, Problem 169]. Suppose the converse. Then, by (H), there is $S \in L(H)^{-1}$ and $x \in H$, $\|x\| = 1$ such that $(Tx, x) \notin \overline{W(STS^{-1})}$. Choose a neighbourhood U of $\overline{W(STS^{-1})}$ such that $\operatorname{dist}((Tx, x), \overline{U}) = \delta > 0$. Since $ST_nS^{-1} \to STS^{-1}$ in norm, there is n_0 such that $\overline{W(ST_nS^{-1})} \subset U$ for $n \geqslant n_0$, by the previous Remark and (V3). Since $T_n \in \mathcal{C}$, then, by (H),

$$(T_n x, x) \in \overline{W(ST_nS^{-1})}$$

for all n. But, by the choice of U:

$$\|T - T_n\| \geqslant |(Tx, x) - (T_n x, x)| \geqslant \delta > 0 \quad \text{for } n \geqslant n_0,$$

a contradiction. Q.E.D.

Now we are able to give some applications of the results of §§2, 3. Consider a function algebra $A \subset C(X)$ and assume $X = \operatorname{Spec} A$, in other words, A acts on its maximal ideal space. Let φ: $A \to L(H)$ be a contractive representation of A and let $B = \overline{\varphi(A)}$. Let φ': $B' \to A'$ be as in §2.

COROLLARY 2. *If all the operators* $\varphi(a)$, $a \in A$ *are convexoid, then the set* $\operatorname{Ch}(A) \cap \varphi'(\operatorname{Spec} B)$ *is a subset of* X *and* $\varphi(A)$ *is antisymmetric if and only if* $\operatorname{Ch}(A) \cap \varphi'(\operatorname{Spec} B)$ *is a set of antisymmetry for* A.

PROOF. Let P be as in §2. We show that $P^e \subset X$. By Proposition 5 and the assumptions, all the operators $S \in B$ are convexoid, hence $r(S) = v(S)$, $S \in B$. By Proposition 2 applied to the algebra B, $K(B)^e \subset \operatorname{Spec} B$ and, since φ' preserves extreme points,

$$P^e = \varphi'\big(K(B)^e\big) \subset \varphi'(\operatorname{Spec} B) \subset \operatorname{Spec} A = X.$$

Now, the first assertion follows from Proposition 4(c), and the second one–from the Theorem. Q.E.D.

An operator $T \in L(H)$ is called *subnormal* [**5**, p. 100] if there is a Hilbert space $K \supset H$ and a normal operator $N \in L(K)$ such that H is invariant for N and $T = N_{|H}$. An algebra homomorphism φ: $A \to L(H)$ of a function algebra $A \subset C(X)$ is called *subnormal* if there is a Hilbert space $K \supset H$ and an involution-preserving homomorphism $\bar{\varphi}$: $C(X) \to L(K)$ such that all the operators $\bar{\varphi}(a)$, $a \in A$ leave H invariant and $\varphi(a) = \bar{\varphi}(a)_{|H}$, $a \in A$.

COROLLARY 3. *Let* φ: $A \to L(H)$ *be a subnormal, contractive representation of a function algebra* $A \subset C(X)$ *such that* $X = \operatorname{Spec} A$. *Then* $\varphi(A)$ *is antisymmetric if and only if* $\operatorname{Ch}(A) \cap \varphi'(\operatorname{Spec} B)$ *is a set of antisymmetry for* A (*where* φ', B *are as above*).

PROOF. Clearly, all the operators $\varphi(a)$, $a \in A$ are subnormal and every subnormal operator is convexoid [**2**, Lemma 5]. Q.E.D.

Finally we give applications of the previous result to the von Neumann functional calculus and spectral sets. If $X \subset \mathbf{C}$ is a compact set, then $P(X) \subset C(X)$ stands for the $C(X)$-closure of the restrictions of all polynomials to X. In what follows we assume that $\mathbf{C} \setminus X$ is connected. Then $\operatorname{Spec} P(X) = X$. If X is a spectral set for an operator $T \in L(H)$ (see [**7**], [**9**] for the definitions) then there is a contractive representation φ_T: $P(X) \to L(H)$ such that $\varphi_T(z) = T$, where z denotes the identity function on X. This representation has been constructed by von Neumann in [**7**]. Let $B = \overline{\varphi_T(P(X))}$ (the norm closure).

COROLLARY 4. *Let* X *be a spectral set for a subnormal operator* $T \in L(H)$ *such that* $\mathbf{C} \setminus X$ *is connected. The algebra* $\varphi_T(P(X))$ *is antisymmetric if and only if* $\operatorname{Ch}(P(X)) \cap \operatorname{sp}_B T$ *is a set of antisymmetry for* $P(X)$.

PROOF. We show $\mathrm{sp}_B\, T = \varphi_T'(\mathrm{Spec}\, B)$, where φ_T': $B \to P(X)'$ is as in §2. Here $\varphi_T'(\mathrm{Spec}\, B)$ is identified with the set

$$\{\lambda \in X: \tau_\lambda = \tau \circ \varphi, \tau \in \mathrm{Spec}\, B\}.$$

If $\lambda \in \varphi_T'(\mathrm{Spec}\, B)$, then $\tau_\lambda = \tau \circ \varphi$ with some $\tau \in \mathrm{Spec}\, B$, hence $\lambda = \tau_\lambda(z) = \tau(\varphi_T(z)) = \tau(T) \in \mathrm{sp}_B\, T$. Conversely, if $\lambda \in \mathrm{sp}_B\, T$, then $\lambda = \tau(T)$ for some $\tau \in \mathrm{Spec}\, B$, hence $\tau_\lambda(z) = \tau(\varphi(z))$. Since $\tau, \tau_\lambda, \varphi$ are multiplicative and continuous, $\tau_\lambda = \tau \circ \varphi$, thus $\lambda \in \varphi_T'(\mathrm{Spec}\, B)$. Since φ_T is a subnormal, contractive representation of $P(X)$, we apply Corollary 3. Q.E.D.

Note, that $\mathrm{Ch}(P(X)) \cap \mathrm{sp}_B\, T$ is always nonempty and $\mathrm{Ch}(P(X)) \cap \mathrm{sp}_B\, T$ contains only one point if and only if $\dim \varphi_T(P(X)) = 1$, by Proposition 4(c).

If $T \in L(H)$ is a contraction ($\|T\| \leqslant 1$), then the closed unit disc D is a spectral set for T (the von Neumann inequality; see [**7**, 4.3], [**9**, Theorem A, p. 437]). Γ denotes the unit circle.

COROLLARY 5. *If $T \in L(H)$ is a subnormal contraction, which is not any scalar multiple of I, then the following conditions are equivalent*:

(a) *$\varphi_T(P(D))$ is antisymmetric*,

(b) *$\mathrm{sp}_B\, T$ contains Γ*,

(c) *φ_T is an isometry*.

PROOF. If T is not any multiple of I, then $\dim \varphi_T(P(D)) > 1$, hence, by the above remark, $\mathrm{sp}_B\, T \cap \Gamma$ is a closed, non one-point subset of Γ (because $\mathrm{Ch}(P(D)) = \Gamma$, see [**3**, p. 84]). One can show, that no proper, non one-point closed subset of Γ is a set of antisymmetry for $P(D)$. Hence $\mathrm{sp}_B\, T \cap \Gamma$ is a set of antisymmetry for $P(D)$ if and only if $\Gamma \subset \mathrm{sp}_B\, T$. Applying Corollary 4, we have just proved (a) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (c) If P is as in §1, then, by Proposition 4(c) $P^e = \mathrm{sp}_B\, T \cap \Gamma = \mathrm{Ch}(P(D))$; by the Krein-Milman theorem, $P = K(P(D))$ and by Proposition 3, φ_T is isometric. (c) $\Rightarrow$ (a) follows from the antisymmetry of $P(D)$ and Corollary 1. Q.E.D.

EXAMPLE 1. Let $T \in L(H)$ be a nonunitary isometry. $C^*(T)$ denotes the C^*-algebra generated by T and I. By [**4**, p. 86], for every $\lambda \in \Gamma$ there is a linear, multiplicative functional ψ_λ on $C^*(T)$ such that $\psi_\lambda(T) = \lambda$. The algebra $B = \overline{\varphi_T(P(D))}$ is a Banach subalgebra of $C^*(T)$ and the restrictions of ψ_λ to B are multiplicative on B, hence $\psi_\lambda \in \mathrm{Spec}\, B$. Now $\mathrm{sp}_B\, T = \{\tau(T)$: $\tau \in \mathrm{Spec}\, B\} \supset \{\psi_\lambda(T)$: $\lambda \in \Gamma\} = \Gamma$ and since every isometry is subnormal, Corollary 5 implies, that $\varphi_T(P(D))$ is antisymmetric.

EXAMPLE 2. Suppose that $T_1 \in L(H_1)$ is a nonunitary isometry and $T_2 \in L(H_2)$ is a subnormal contraction. Define $T = T_1 \oplus T_2$. The algebra $\varphi_T(P(D))$ is antisymmetric. Indeed, let B, B_1 be the norm-closures of $\varphi_T(P(D))$ in $L(H_1 \oplus H_2)$ and $\varphi_{T_1}(P(D))$ in $L(H_1)$, respectively. Since for all polynomials p: $p(T) = p(T_1) \oplus p(T_2)$, we have $\|p(T_1)\| \leqslant \|p(T)\|$, thus for all $f \in P(D)$: $\|\varphi_{T_1}(f)\| \leqslant \|\varphi_T(f)\|$. Hence the map φ: $B \to B_1$ given by $\varphi(\varphi_T(f)) = \varphi_{T_1}(f)$ is well defined. Since $\varphi(I) = I_1$ (the identity in H_1), φ is a contractive representation of B. Thus $\varphi'\colon B_1' \to B'$ (as in §2) maps Spec B into a subset of Spec B_1. This implies $\mathrm{sp}_{B_1} T \subset \mathrm{sp}_B T$ and, by Example 1, $\Gamma \subset \mathrm{sp}_{B_1} T$. Since T is a subnormal contraction, Corollary 5 finishes the proof.

EXAMPLE 3. If $T \in L(H)$ is a subnormal partial isometry, then, by [**5**, Problem 161], T may be written as an orthogonal sum $T = T_1 \oplus 0$, where T_1 is an isometry. If T_1 is not unitary, then Example 2 implies that $\varphi_T(P(D))$ is an antisymmetric algebra.

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