

ANTITHETIC MULTILEVEL MONTE CARLO ESTIMATION FOR MULTI-DIMENSIONAL SDES WITHOUT LÉVY AREA SIMULATION

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In this paper we introduce a new multilevel Monte Carlo (MLMC) estimator for multi-dimensional SDEs driven by Brownian motions. Giles has previously shown that if we combine a numerical approximation with strong order of convergence $O(\Delta t)$ with MLMC we can reduce the computational complexity to estimate expected values of functionals of SDE solutions with a root-mean-square error of ϵ from $O(\epsilon^{-3})$ to $O(\epsilon^{-2})$. However, in general, to obtain a rate of strong convergence higher than $O(\Delta t^{1/2})$ requires simulation, or approximation, of Lévy areas. In this paper, through the construction of a suitable antithetic multilevel correction estimator, we are able to avoid the simulation of Lévy areas and still achieve an $O(\Delta t^2)$ multilevel correction variance for smooth payoffs, and almost an $O(\Delta t^{3/2})$ variance for piecewise smooth payoffs, even though there is only $O(\Delta t^{1/2})$ strong convergence. This results in an $O(\epsilon^{-2})$ complexity for estimating the value of European and Asian put and call options.

1. Introduction. In many financial engineering applications, one is interested in the expected value of a financial derivative whose payoff depends upon the solution of a stochastic differential equation (SDE). Using a simple Monte Carlo method with a numerical discretisation with first order weak convergence, to achieve a root-mean-square error of ϵ would require $O(\epsilon^{-2})$ independent paths, each with $O(\epsilon^{-1})$ time steps, giving a computational complexity which is $O(\epsilon^{-3})$, [3].

Recently, Giles [6] introduced a multilevel Monte Carlo (MLMC) estimator which enables a reduction of this computational cost to $O(\epsilon^{-2}(\log(1/\epsilon))^2)$ for Lipschitz payoffs when using the Euler–Maruyama discretisation. For other discontinuous and path-dependent payoff functions, the complexity is poorer [7]. The efficiency of the MLMC method is influenced by the strong convergence order of the discretisation, and subsequent research using MLMC with the first-order Milstein discretisation for scalar SDEs, improved the complexity significantly to $O(\epsilon^{-2})$ for digital, lookback and barrier options [5]. However, a weakness of the Milstein discretisation is that in multiple dimensions it generally requires the simulation of iterated Itô integrals known as Lévy areas, for which there is no known efficient method except in dimension 2 [4, 17, 18].

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Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, and let $w(t)$ be a D -dimensional Brownian motion defined on the probability space. We consider the numerical approximation of a general class of multi-dimensional SDEs driven by Brownian of the form

$$(1.1) \quad dx(t) = f(x(t)) dt + g(x(t)) dw(t),$$

where $x(t) \in \mathbb{R}^d$ for each $t \geq 0$, $f \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $g \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times D})$, and for simplicity we assume a fixed initial value $x_0 \in \mathbb{R}^d$.

In this paper we are primarily concerned with estimating $\mathbb{E}[P(x(T))]$, the expected value of a payoff depending on the solution at a fixed time T , defining the tensor $h_{ijk}(x)$ as

$$(1.2) \quad h_{ijk}(x) = \frac{1}{2} \sum_{l=1}^d g_{lk}(x) \frac{\partial g_{ij}}{\partial x_l}(x), \quad i = 1, \dots, d \text{ and } k, j = 1, \dots, D,$$

when using N uniform timesteps $\Delta t = T/N$, the i th component of the first order Milstein approximation $\widehat{X}_n \approx x(n\Delta t)$ has the form [13]

$$(1.3) \quad \begin{aligned} \widehat{X}_{i,n+1} &= \widehat{X}_{i,n} + f_i(\widehat{X}_n)\Delta t + \sum_{j=1}^D g_{ij}(\widehat{X}_n)\Delta w_{j,n} \\ &+ \sum_{j,k=1}^D h_{ijk}(\widehat{X}_n)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t - A_{jk,n}), \end{aligned}$$

where Ω is the correlation matrix for the driving Brownian paths, and $A_{jk,n}$ is the Lévy area defined as

$$A_{jk,n} = \int_{t_n}^{t_{n+1}} (w_j(t) - w_j(t_n)) dw_k(t) - \int_{t_n}^{t_{n+1}} (w_k(t) - w_k(t_n)) dw_j(t).$$

In some applications, the diffusion coefficient $g(x)$ has a commutativity property which gives $h_{ijk}(x) = h_{ikj}(x)$ for all i, j, k . In that case, because the Lévy areas are anti-symmetric (i.e., $A_{jk,n} = -A_{kj,n}$), it follows that $h_{ijk}(X_n)A_{jk,n} + h_{ikj}(X_n)A_{kj,n} = 0$ and therefore the terms involving the Lévy areas cancel and so it is not necessary to simulate them. However, this only happens in special cases.

Clark and Cameron [2] proved for a particular SDE that it is impossible to achieve a better order of strong convergence than the Euler–Maruyama discretisation when using just the discrete increments of the underlying Brownian motion. The analysis was extended by Müller–Gronbach [15] to general SDEs. As a consequence if we use the standard MLMC method with the Milstein scheme without simulating the Lévy areas the complexity will remain the same as for Euler–Maruyama. Nevertheless, in this paper we show that by constructing a suitable antithetic MLMC estimator one can neglect the Lévy areas and still obtain a multilevel correction estimator with a variance which decays at the same rate as

the scalar Milstein estimator. This demonstrates that a high order of the strong convergence is not necessary for our new estimator to achieve the optimal complexity $O(\epsilon^{-2})$.

We begin the paper by reviewing the multilevel Monte Carlo approach, introducing the idea of the antithetic estimator and bounding the behaviour of its variance under certain conditions. Because of its simplicity, we then consider Clark and Cameron's model problem, and prove that the antithetic path simulations do satisfy the required conditions to give an $O(\Delta t^2)$ variance convergence for a smooth payoff. This then motivates the subsequent analysis for the general class of multi-dimensional SDEs. We support our analysis by suitable numerical experiments in which we demonstrate the superiority of antithetic MLMC over the standard MLMC for both the Clark–Cameron SDE and the Heston stochastic volatility model. The [Appendix](#) contains the detailed proofs of the key theorems.

In this paper we restrict attention to financial applications with either a European payoff, dependent on the final value $x(T)$, or an Asian payoff, dependent on the average of $x(t)$ over the time interval $[0, T]$. It is proved that when the payoff is twice differentiable, with bounded derivatives, the rate of convergence of the multilevel correction variance is doubled from $O(\Delta t)$ to $O(\Delta t^2)$. If the payoff is Lipschitz, and twice differentiable almost everywhere, then the rate of convergence is reduced to $O(\Delta t^{3/2})$, but this is still sufficient to make the overall complexity $O(\epsilon^{-2})$ to achieve a root-mean-square accuracy of ϵ .

2. Multilevel Monte Carlo estimation.

2.1. MLMC estimators. In its most general form, multilevel Monte Carlo simulation uses a number of levels of resolution, $\ell = 0, 1, \dots, L$, with $\ell = 0$ being the coarsest, and $\ell = L$ being the finest. In the context of a SDEs simulation, level 0 may have just one timestep for the whole time interval $[0, T]$, whereas level L might have 2^L uniform timesteps.

If P denotes the payoff (or other output functional of interest), and P_ℓ denote its approximation on level ℓ , then the expected value $\mathbb{E}[P_L]$ on the finest level is equal to the expected value $\mathbb{E}[P_0]$ on the coarsest level plus a sum of corrections which give the difference in expectation between simulations on successive levels,

$$(2.1) \quad \mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{\ell=1}^L \mathbb{E}[P_\ell - P_{\ell-1}].$$

The idea behind MLMC is to independently estimate each of the expectations on the right-hand side of (2.1) in a way which minimises the overall variance for a given computational cost. Let Y_0 be an estimator for $\mathbb{E}[P_0]$ using N_0 samples, and let Y_ℓ , $\ell > 0$, be an estimator for $\mathbb{E}[P_\ell - P_{\ell-1}]$ using N_ℓ samples. The simplest estimator is a mean of N_ℓ independent samples, which for $\ell > 0$ is

$$(2.2) \quad Y_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} (P_\ell^i - P_{\ell-1}^i).$$

The key point here is that $P_\ell^i - P_{\ell-1}^i$ should come from two discrete approximations for the same underlying stochastic sample, so that on finer levels of resolution the difference is small (due to strong convergence) and so the variance is also small. Hence very few samples will be required on finer levels to accurately estimate the expected value.

Here we recall the theorem from [8] (which is a slight generalisation of the original theorem in [6]) which gives the complexity of MLMC estimation.

THEOREM 2.1. *Let P denote a functional of the solution of a stochastic differential equation, and let P_ℓ denote the corresponding level ℓ numerical approximation. If there exist independent estimators Y_ℓ based on N_ℓ Monte Carlo samples, and positive constants $\alpha, \beta, \gamma, c_1, c_2, c_3$ such that $\alpha \geq \frac{1}{2} \min(\beta, \gamma)$ and:*

- (i) $|\mathbb{E}[P_\ell - P]| \leq c_1 2^{-\alpha\ell}$,
- (ii) $\mathbb{E}[Y_\ell] = \begin{cases} \mathbb{E}[P_0], & \ell = 0, \\ \mathbb{E}[P_\ell - P_{\ell-1}], & \ell > 0, \end{cases}$
- (iii) $\mathbb{V}[Y_\ell] \leq c_2 N_\ell^{-1} 2^{-\beta\ell}$,
- (iv) $C_\ell \leq c_3 N_\ell 2^{\gamma\ell}$, where C_ℓ is the computational complexity of Y_ℓ ,

then there exists a positive constant c_4 such that for any $\epsilon < e^{-1}$ there are values L and N_ℓ for which the multilevel estimator

$$Y = \sum_{\ell=0}^L Y_\ell$$

has a mean-square-error with bound

$$MSE \equiv \mathbb{E}[(Y - \mathbb{E}[P])^2] < \epsilon^2$$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4 \epsilon^{-2}, & \beta > \gamma, \\ c_4 \epsilon^{-2} (\log(1/\epsilon))^2, & \beta = \gamma, \\ c_4 \epsilon^{-2 - (\gamma - \beta)/\alpha}, & 0 < \beta < \gamma. \end{cases}$$

Without the simulation of Lévy areas, the strong order of convergence of the Milstein discretisation $X(T)$ which is used is only 1/2, so that

$$\mathbb{E}[\|x(T) - X(T)\|^2] = O(\Delta t).$$

Hence, for payoffs which are a Lipschitz function of the final value, it follows that

$$\mathbb{E}[(P_\ell - P_{\ell-1})^2] = O(\Delta t)$$

and therefore the estimator given by (2.2) satisfies condition (iii) in the theorem with $\beta = 1$ when $\Delta t \propto 2^{-\ell}$. What we will show is that *without improving the*

strong order of convergence it is possible to construct an antithetic estimator for which $\beta = 2$.

To do so, we need to exploit some flexibility in the construction of the multilevel estimator. In (2.2) we have used the same estimator for the payoff P_ℓ on every level ℓ , and therefore (2.1) is a trivial identity due to the telescoping summation. However, in [5] Giles numerically showed that it can be better to use different estimators for the finer and coarser of the two levels being considered, P_ℓ^f when level ℓ is the finer level, and P_ℓ^c when level ℓ is the coarser level. In this case, we require that

$$(2.3) \quad \mathbb{E}[P_\ell^f] = \mathbb{E}[P_\ell^c] \quad \text{for } \ell = 1, \dots, L,$$

so that

$$E[P_L^f] = \mathbb{E}[P_0^f] + \sum_{\ell=1}^L \mathbb{E}[P_\ell^f - P_{\ell-1}^c].$$

The MLMC theorem is still applicable to this modified estimator. The advantage is that it gives the flexibility to construct approximations for which $P_\ell^f - P_{\ell-1}^c$ is much smaller than the original $P_\ell - P_{\ell-1}$, giving a larger value for β , the rate of variance convergence in condition (iii) in the theorem.

2.2. Antithetic MLMC estimator. Based on the well-known method of antithetic variates (see, e.g., [10]), the idea for the antithetic estimator is to exploit the flexibility of the more general MLMC estimator by defining $P_{\ell-1}^c$ to be the usual payoff $P(X^c)$ coming from a level $\ell - 1$ coarse simulation X^c , and define P_ℓ^f to be the average of the payoffs $P(X^f)$, $P(X^a)$ coming from an antithetic pair of level ℓ simulations, X^f and X^a .

X^f will be defined in a way which corresponds naturally to the construction of X^c . Its antithetic “twin” X^a will be defined so that it has exactly the same distribution as X^f , conditional on X^c , which ensures that $\mathbb{E}[P(X^f)] = \mathbb{E}[P(X^a)]$ and hence (2.3) is satisfied, but at the same time

$$(X^f - X^c) \approx -(X^a - X^c)$$

and therefore

$$(P(X^f) - P(X^c)) \approx -(P(X^a) - P(X^c)),$$

so that $\frac{1}{2}(P(X^f) + P(X^a)) \approx P(X^c)$. This leads to $\frac{1}{2}(P(X^f) + P(X^a)) - P(X^c)$ having a much smaller variance than the standard estimator $P(X^f) - P(X^c)$.

We now present a lemma which motivates the rest of the paper by giving an upper bound on the convergence of the variance of $\frac{1}{2}(P(X^f) + P(X^a)) - P(X^c)$.

LEMMA 2.2. *If $P \in C^2(\mathbb{R}^d, \mathbb{R})$ and there exist constants L_1, L_2 such that for all $x \in \mathbb{R}^d$*

$$\left\| \frac{\partial P}{\partial x} \right\| \leq L_1, \quad \left\| \frac{\partial^2 P}{\partial x^2} \right\| \leq L_2,$$

then for $p \geq 2$,

$$\begin{aligned} & \mathbb{E}[(\tfrac{1}{2}(P(X^f) + P(X^a)) - P(X^c))^p] \\ & \leq 2^{p-1} L_1^p \mathbb{E}[\|\tfrac{1}{2}(X^f + X^a) - X^c\|^p] + 2^{-(p+1)} L_2^p \mathbb{E}[\|X^f - X^a\|^{2p}]. \end{aligned}$$

PROOF. If we define $\bar{X}^f \equiv \frac{1}{2}(X^f + X^a)$, then a Taylor expansion gives

$$P(X^f) = P(\bar{X}^f) + \frac{\partial P^T}{\partial x}(\bar{X}^f)(X^f - \bar{X}^f) + \frac{1}{2}(X^f - \bar{X}^f)^T \frac{\partial^2 P}{\partial x^2}(\xi_1)(X^f - \bar{X}^f)$$

for some ξ_1 on the line between \bar{X}^f and X^f . Performing a similar expansion for $P(X^a)$ and then averaging the two, the linear terms cancel, and one obtains

$$\begin{aligned} \frac{1}{2}(P(X^f) + P(X^a)) &= P(\bar{X}^f) + \frac{1}{4}(X^f - \bar{X}^f)^T \frac{\partial^2 P}{\partial x^2}(\xi_1)(X^f - \bar{X}^f) \\ &\quad + \frac{1}{4}(X^a - \bar{X}^f)^T \frac{\partial^2 P}{\partial x^2}(\xi_2)(X^a - \bar{X}^f) \\ &= P(\bar{X}^f) + \frac{1}{8}(X^f - X^a)^T \frac{\partial^2 P}{\partial x^2}(\xi_3)(X^f - X^a) \end{aligned}$$

for some ξ_3 on the line between X^a and X^f , due to the mean value theorem. We then obtain

$$\begin{aligned} \frac{1}{2}(P(X^f) + P(X^a)) - P(X^c) &= \frac{\partial P^T}{\partial x}(\xi_4)(\bar{X}^f - X^c) \\ &\quad + \frac{1}{8}(X^f - X^a)^T \frac{\partial^2 P}{\partial x^2}(\xi_3)(X^f - X^a), \end{aligned}$$

for some ξ_4 on the line between \bar{X}^f and X^c . Hence,

$$|\tfrac{1}{2}(P(X^f) + P(X^a)) - P(X^c)| \leq L_1 \|\bar{X}^f - X^c\| + \tfrac{1}{4} L_2 \|X^f - X^a\|^2$$

and the final result follows from the standard inequality

$$(2.4) \quad \left| \sum_{n=1}^N a_n \right|^p \leq N^{p-1} \sum_{n=1}^N |a_n|^p$$

and then taking the expectation. \square

In the multi-dimensional SDE applications considered in this paper, we will show that the Milstein approximation with the Lévy areas set to zero, combined

with the antithetic construction, leads to $X^f - X^a = O(\Delta t^{1/2})$ but $\bar{X}^f - X^c = O(\Delta t)$. Hence, the variance $\mathbb{V}[\frac{1}{2}(P_l^f + P_l^a) - P_{l-1}^c]$ is $O(\Delta t^2)$, which is the order obtained for scalar SDEs using the Milstein discretisation with its first order strong convergence. We first show this for the simple Clark and Cameron model problem which can be analysed in detail. We then extend the analysis to a general class of multi-dimensional SDEs.

3. Clark–Cameron example.

3.1. *Clark–Cameron analysis.* The paper of Clark and Cameron [2] addresses the question of how accurately one can approximate the solution of an SDE driven by an underlying multi-dimensional Brownian motion, using only uniformly-spaced discrete Brownian increments. Their model problem is

$$(3.1) \quad \begin{aligned} dx_1(t) &= dw_1(t), \\ dx_2(t) &= x_1(t) dw_2(t) \end{aligned}$$

with $x(0) = y(0) = 0$, and zero correlation between the two Brownian motions $w_1(t)$ and $w_2(t)$. These equations can be integrated exactly over a time interval $[t_n, t_{n+1}]$, where $t_n = n\Delta t$, to give

$$(3.2) \quad \begin{aligned} x_1(t_{n+1}) &= x_1(t_n) + \Delta w_{1,n}, \\ x_2(t_{n+1}) &= x_2(t_n) + x_1(t_n)\Delta w_{2,n} + \frac{1}{2}\Delta w_{1,n}\Delta w_{2,n} + \frac{1}{2}A_{12,n}, \end{aligned}$$

where $\Delta w_{i,n} \equiv w_i(t_{n+1}) - w_i(t_n)$, and $A_{12,n}$ is the Lévy area defined as

$$A_{12,n} = \int_{t_n}^{t_{n+1}} (w_1(t) - w_1(t_n)) dw_2(t) - \int_{t_n}^{t_{n+1}} (w_2(t) - w_2(t_n)) dw_1(t).$$

This corresponds exactly to the Milstein discretisation presented in (1.3), so for this simple model problem, the Milstein discretisation is exact.

The point of Clark and Cameron’s paper is that for a given set of discrete Brownian increments, the value for $x_1(t_n)$ is determined exactly for all n , but the value for $x_2(t_n)$ depends on the unknown Lévy areas. Since $\mathbb{E}[A_{12,n} | \Delta w_{1,n}, \Delta w_{2,n}] = 0$, the conditional expected value is given by (3.2) with the Lévy areas set to zero. In addition, it follows that for any numerical approximation $X(T)$ based solely on the set of discrete Brownian increments Δw ,

$$\begin{aligned} \mathbb{E}[(x_2(T) - X_2(T))^2] &= \mathbb{E}[\mathbb{E}[(x_2(T) - X_2(T))^2 | \Delta w]] \\ &\geq \mathbb{E}[\mathbb{V}[x_2(T) | \Delta w]] \\ &= \frac{1}{4} \sum_{n=0}^{N-1} \mathbb{V}[A_{12,n}] \\ &= \frac{1}{4} T \Delta t. \end{aligned}$$

Hence, one cannot achieve better than $O(\Delta t^{1/2})$ strong convergence, and the mean square error is minimised when the inequality in the above equation is an equality, which is when

$$(3.3) \quad X_2(T) = \mathbb{E}[x_2(T)|\Delta w],$$

which is achieved by setting the Lévy areas set to zero.

3.2. Antithetic MLMC estimator. We define a coarse path approximation X^c with timestep Δt by neglecting the Lévy area terms to give

$$(3.4) \quad \begin{aligned} X_{1,n+1}^c &= X_{1,n}^c + \Delta w_{1,n}, \\ X_{2,n+1}^c &= X_{2,n}^c + X_{1,n}^c \Delta w_{2,n} + \frac{1}{2} \Delta w_{1,n} \Delta w_{2,n}. \end{aligned}$$

This is equivalent to replacing the true Brownian path by a piecewise linear approximation as illustrated in Figure 1.

Similarly, we define the corresponding two half-timesteps of the first fine path approximation X^f by

$$\begin{aligned} X_{1,n+1/2}^f &= X_{1,n}^f + \delta w_{1,n}, \\ X_{2,n+1/2}^f &= X_{2,n}^f + X_{1,n}^f \delta w_{2,n} + \frac{1}{2} \delta w_{1,n} \delta w_{2,n}, \\ X_{1,n+1}^f &= X_{1,n+1/2}^f + \delta w_{1,n+1/2}, \\ X_{2,n+1}^f &= X_{2,n+1/2}^f + X_{1,n+1/2}^f \delta w_{2,n+1/2} + \frac{1}{2} \delta w_{1,n+1/2} \delta w_{2,n+1/2} \end{aligned}$$

in which $\delta w_n \equiv w(t_{n+1/2}) - w(t_n)$, $\delta w_{n+1/2} \equiv w(t_{n+1}) - w(t_{n+1/2})$ are the Brownian increments over the first and second halves of the coarse timestep, and so

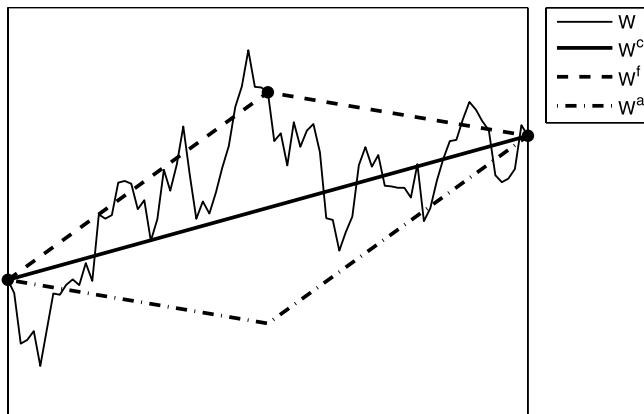


FIG. 1. Brownian path and approximations over one coarse timestep.

$\Delta w_n = \delta w_n + \delta w_{n+1/2}$. Using this relation, the equations for the two fine timesteps can be combined to give an equation for the increment over the coarse timestep,

$$(3.5) \quad \begin{aligned} X_{1,n+1}^f &= X_{1,n}^f + \Delta w_{1,n}, \\ X_{2,n+1}^f &= X_{2,n}^f + X_{1,n}^f \Delta w_{2,n} + \frac{1}{2} \Delta w_{1,n} \Delta w_{2,n} \\ &\quad + \frac{1}{2} (\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2}). \end{aligned}$$

The antithetic approximation X_n^a is defined by exactly the same discretisation except that the Brownian increments δw_n and $\delta w_{n+1/2}$ are swapped, as illustrated in Figure 1. This gives

$$\begin{aligned} X_{1,n+1/2}^a &= X_{1,n}^a + \delta w_{1,n+1/2}, \\ X_{2,n+1/2}^a &= X_{2,n}^a + X_{1,n}^a \delta w_{2,n+1/2} + \frac{1}{2} \delta w_{1,n+1/2} \delta w_{2,n+1/2}, \\ X_{1,n+1}^a &= X_{1,n+1/2}^a + \delta w_{1,n}, \\ X_{2,n+1}^a &= X_{2,n+1/2}^a + X_{1,n+1/2}^a \delta w_{2,n} + \frac{1}{2} \delta w_{1,n} \delta w_{2,n} \end{aligned}$$

and hence

$$(3.6) \quad \begin{aligned} X_{1,n+1}^a &= X_{1,n}^a + \Delta w_{1,n}, \\ X_{2,n+1}^a &= X_{2,n}^a + X_{1,n}^a \Delta w_{2,n} + \frac{1}{2} \Delta w_{1,n} \Delta w_{2,n} \\ &\quad - \frac{1}{2} (\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2}). \end{aligned}$$

Swapping δw_n and $\delta w_{n+1/2}$ does not change the distribution of the driving Brownian increments, and hence X^a has exactly the same distribution as X^f . Note also the change in sign in the last term in (3.5) compared to the corresponding term in (3.6). This is important because these two terms cancel when the two equations are averaged.

These last terms correspond to the Lévy areas for the fine path and the antithetic path, and the sign reversal is a particular instance of a more general result for time-reversed Brownian motion, [12]. If $(w_t, 0 \leq t \leq 1)$ denotes a Brownian motion on the time interval $[0, 1]$, then the time-reversed Brownian motion $(z_t, 0 \leq t \leq 1)$ defined by

$$(3.7) \quad z_t = w_1 - w_{1-t},$$

has exactly the same distribution, and it can be shown that its Lévy area is equal in magnitude and opposite in sign to that of w_t .

LEMMA 3.1. *If X_n^f, X_n^a and X_n^c are as defined above, then*

$$X_{1,n}^f = X_{1,n}^a = X_{1,n}^c, \quad \frac{1}{2} (X_{2,n}^f + X_{2,n}^a) = X_{2,n}^c \quad \forall n \leq N$$

and

$$\mathbb{E}[(X_{2,N}^f - X_{2,N}^a)^4] = \frac{3}{4} T (T + \Delta t) \Delta t^2.$$

PROOF. Comparing (3.4), (3.5) and (3.6), it is clear that $X_{1,n}^f$, $X_{1,n}^a$ and $X_{1,n}^c$ all satisfy the same difference equation and so are equal. Given this, averaging the equations for $X_{2,n}^f$ and $X_{2,n}^a$ gives the same difference equation as for $X_{2,n}^c$, and so therefore $\frac{1}{2}(X_{2,n}^f + X_{2,n}^a) = X_{2,n}^c$. Finally, summing the difference of the equations for $X_{2,n}^f$ and $X_{2,n}^a$ gives

$$X_{2,N}^f - X_{2,N}^a = \sum_{n=0}^{N-1} (\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2}).$$

Since the $\delta w_{j,n}$ are all i.i.d. normal variables with variance $\frac{1}{2} \Delta t$, it is easily shown that

$$\begin{aligned} \mathbb{E}[(\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2})^2] &= \frac{1}{2} \Delta t^2, \\ \mathbb{E}[(\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2})^4] &= \frac{3}{2} \Delta t^4 \end{aligned}$$

and it then follows that

$$\mathbb{E}[(X_{2,N}^f - X_{2,N}^a)^4] = \left(\frac{1}{2} \Delta t^2\right)^2 \frac{N(N-1)}{2} \frac{4 \times 3}{2} + \frac{3}{2} \Delta t^4 N = \frac{3}{4} T(T + \Delta t) \Delta t^2.$$

In the above derivation, when expanding $(X_{2,N}^f - X_{2,N}^a)^4$, the first contribution comes from terms of the form $(\delta w_{1,m} \delta w_{2,m+1/2} - \delta w_{2,m} \delta w_{1,m+1/2})^2 \times (\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2})^2$ for $m \neq n$, while the second contribution comes from terms of the form $(\delta w_{1,n} \delta w_{2,n+1/2} - \delta w_{2,n} \delta w_{1,n+1/2})^4$. All other terms have zero expectation. \square

Combining the above result with Lemma 2.2 for $p = 2$ gives a second order bound on the multilevel estimator variance for payoffs satisfying the required smoothness conditions. It is worth noting that an antithetic MLMC based on the simpler Euler–Maruyama discretisation, omitting the term $\Delta w_{1,n} \Delta w_{2,n}$ in (3.4), would not give similar benefits. The identity $\frac{1}{2}(X_{2,n}^f + X_{2,n}^a) = X_{2,n}^c$ no longer holds, and a similar analysis to that in the proof above gives

$$\mathbb{V}[\frac{1}{2}(X_{2,N}^f + X_{2,N}^a) - X_{2,N}^c] = \mathbb{E}[(\frac{1}{2}(X_{2,N}^f + X_{2,N}^a) - X_{2,N}^c)^2] = O(\Delta t).$$

Hence, in the simple case in which the payoff is $P = X_2(T)$, the variance of the antithetic multilevel estimator is first order, the same as the standard MLMC, and not second order.

4. General theory.

4.1. *Milstein discretisation.* In this section we extend the analysis of the Clark–Cameron example to general the multi-dimensional SDE (1.1). We make the standard assumptions that f , g and h have a uniform Lipschitz bound, and so have uniformly bounded first derivatives. In addition, we make the assumption that

f and g have uniformly bounded second derivatives. More formally, we have the following:

ASSUMPTION 4.1. Let $f \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times D})$. There exists a constant L such that for any $x \in \mathbb{R}^d$, and for all $1 \leq i \leq d$ and $1 \leq j, k, l \leq D$,

$$\begin{aligned} \left| \frac{\partial f_i}{\partial x_l}(x) \right| &\leq L, & \left| \frac{\partial g_{ij}}{\partial x_l}(x) \right| &\leq L, & \left| \frac{\partial h_{ijk}}{\partial x_l}(x) \right| &\leq L, \\ \left| \frac{\partial^2 f_i}{\partial x_k \partial x_l}(x) \right| &\leq L, & \left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(x) \right| &\leq L. \end{aligned}$$

Let us recall that the general Milstein scheme [13] has the form

$$\begin{aligned} \widehat{X}_{i,n+1} &= \widehat{X}_{i,n} + f_i(\widehat{X}_n)\Delta t + \sum_{j=1}^D g_{ij}(\widehat{X}_n)\Delta w_{j,n} \\ (4.1) \quad &+ \sum_{j,k=1}^D h_{ijk}(\widehat{X}_n)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t - A_{jk,n}). \end{aligned}$$

As in the Clark–Cameron example, we drop the Lévy areas terms, and instead use the truncated Milstein approximation

$$\begin{aligned} X_{i,n+1} &= X_{i,n} + f_i(X_n)\Delta t + \sum_{j=1}^D g_{ij}(X_n)\Delta w_{j,n} \\ (4.2) \quad &+ \sum_{j,k=1}^D h_{ijk}(X_n)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t). \end{aligned}$$

Under Assumption 4.1 it is a standard result that the moments of the general Milstein approximation \widehat{X}_n are bounded, and \widehat{X}_n strongly converges to the solution of the SDE (1.1); this remains true for the truncated Milstein approximation as stated in the following lemma.

LEMMA 4.2. For $p \geq 2$, there exists a constant K_p , independent of the time step, such that

$$\mathbb{E} \left[\max_{0 \leq n \leq N} \|X_n\|^p \right] \leq K_p$$

and

$$\mathbb{E} \left[\max_{0 \leq n \leq N} \|X_n - x(t_n)\|^p \right] \leq K_p \Delta t^{p/2}.$$

PROOF. The proof in [15] follows the standard method of analysis in references such as [13, 14]. \square

Hence, the rate of strong convergence is $O(\Delta t^{1/2})$, which is no better than the Euler–Maruyama discretisation. Nevertheless, we will show that the antithetic multilevel estimator has a variance which converges to zero at the same rate as the full Milstein approximation.

COROLLARY 4.3. *For $p \geq 2$, there exists a constant K_p , independent of the time step, such that*

$$\mathbb{E}\left[\max_{0 \leq n \leq N} |f_i(X_n)|^p\right] \leq K_p, \quad \mathbb{E}\left[\max_{0 \leq n \leq N} |g_{ij}(X_n)|^p\right] \leq K_p,$$

$$\mathbb{E}\left[\max_{0 \leq n \leq N} |h_{ijk}(X_n)|^p\right] \leq K_p$$

for all $1 \leq i \leq d$ and $1 \leq j, k \leq D$.

PROOF. The bounded first derivatives of $f(x), g(x), h(x)$ imply that they grow no faster than linearly as $\|x\| \rightarrow \infty$, and the result then follows from the bound in Lemma 4.2. \square

In order to derive appropriate bounds on the antithetic estimator we also need the following lemma.

LEMMA 4.4. *For $p \geq 2$, there exists a constant K_p , independent of the time step, such that*

$$\max_{0 \leq n \leq N} \mathbb{E}[\|X_{n+1} - X_n\|^p] \leq K_p \Delta t^{p/2}.$$

PROOF. We start from (4.2) and inequality (2.4) which gives

$$\mathbb{E}[|X_{i,n+1} - X_{i,n}|^p] \leq 3^{p-1} \left(\mathbb{E}[|f_i(X_n)\Delta t|^p] + \mathbb{E}\left[\left|\sum_{j=1}^D g_{ij}(X_n)\Delta w_{j,n}\right|^p\right] \right. \\ \left. + \mathbb{E}\left[\left|\sum_{j,k=1}^D h_{ijk}(X_n)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t)\right|^p\right] \right).$$

The first term on the right has a $O(\Delta t^p)$ bound due to the uniform bound on $\mathbb{E}[|f_i(X_n)|^p]$. For the second term we note that because $\Delta w_{j,n}$ is independent of X_n , then

$$\mathbb{E}\left[\left|\sum_{j=1}^D g_{ij}(X_n)\Delta w_{j,n}\right|^p\right] \leq D^{p-1} \sum_{j=1}^D \mathbb{E}[|g_{ij}(X_n)|^p]\mathbb{E}[|\Delta w_{j,n}|^p]$$

and we obtain a $O(\Delta t^{p/2})$ bound due to the uniform bound on $\mathbb{E}[|g_{ij}(X_n)|^p]$ and standard results for the moments of Brownian increments. The third term is handled in a similar way and has a $O(\Delta t^p)$ bound.

Together these give a $O(\Delta t^{p/2})$ bound for $\mathbb{E}[|X_{i,n+1} - X_{i,n}|^p]$ for each i , and hence also for $\mathbb{E}[\|X_{n+1} - X_n\|^p]$. \square

4.2. *Antithetic MLMC estimator.* Using the coarse timestep Δt , the coarse path approximation X_n^c , is given by the Milstein approximation without the Lévy area term,

$$X_{i,n+1}^c = X_{i,n}^c + f_i(X_n^c)\Delta t + \sum_{j=1}^D g_{ij}(X_n^c)\Delta w_{j,n} + \sum_{j,k=1}^D h_{ijk}(X_n^c)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t).$$

The first fine path approximation X_n^f uses the corresponding discretisation with timestep $\Delta t/2$,

$$(4.3) \quad X_{i,n+1/2}^f = X_{i,n}^f + f_i(X_n^f)\Delta t/2 + \sum_{j=1}^D g_{ij}(X_n^f)\delta w_{j,n} + \sum_{j,k=1}^D h_{ijk}(X_n^f)(\delta w_{j,n}\delta w_{k,n} - \Omega_{jk}\Delta t/2),$$

$$(4.4) \quad X_{i,n+1}^f = X_{i,n+1/2}^f + f_i(X_{n+1/2}^f)\Delta t/2 + \sum_{j=1}^D g_{ij}(X_{n+1/2}^f)\delta w_{n+1/2} + \sum_{j,k=1}^D h_{ijk}(X_{n+1/2}^f)(\delta w_{j,n+1/2}\delta w_{k,n+1/2} - \Omega_{jk}\Delta t/2),$$

in which

$$(4.5) \quad \delta w_n \equiv w(t_{n+1/2}) - w(t_n), \quad \delta w_{n+1/2} \equiv w(t_{n+1}) - w(t_{n+1/2})$$

are the Brownian increments over the first and second halves of the coarse timestep, and so $\Delta w_n = \delta w_n + \delta w_{n+1/2}$.

The antithetic approximation X_n^a is defined by exactly the same discretisation, except that the Brownian increments δw_n and $\delta w_{n+1/2}$ are swapped, so that

$$(4.6) \quad X_{i,n+1/2}^a = X_{i,n}^a + f_i(X_n^a)\Delta t/2 + \sum_{j=1}^D g_{ij}(X_n^a)\delta w_{n+1/2} + \sum_{j,k=1}^D h_{ijk}(X_n^a)(\delta w_{j,n+1/2}\delta w_{k,n+1/2} - \Omega_{jk}\Delta t/2),$$

$$X_{i,n+1}^a = X_{i,n+1/2}^a + f_i(X_{n+1/2}^a)\Delta t/2 + \sum_{j=1}^D g_{ij}(X_{n+1/2}^a)\delta w_{j,n} + \sum_{j,k=1}^D h_{ijk}(X_{n+1/2}^a)(\delta w_{j,n}\delta w_{k,n} - \Omega_{jk}\Delta t/2).$$

Since δw_n and $\delta w_{n+1/2}$ are independent and identically distributed, X^a has exactly the same distribution as X^f , and hence $\mathbb{E}[P(X^a)] = \mathbb{E}[P(X^f)]$. In addition, the following lemma follows directly from Lemmas 4.2 and 4.4.

LEMMA 4.5. *Let X^f and X^a be as defined above. Then for $p \geq 2$, there exists a constant K_p , independent of the time step, such that*

$$\begin{aligned} \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f\|^p\right] &\leq K_p, & \max_{0 \leq n < N} \mathbb{E}[\|X_{n+1/2}^f - X_n^f\|^p] &\leq K_p \Delta t^{p/2}, \\ \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^a\|^p\right] &\leq K_p, & \max_{0 \leq n < N} \mathbb{E}[\|X_{n+1/2}^a - X_n^a\|^p] &\leq K_p \Delta t^{p/2}. \end{aligned}$$

4.3. *Numerical analysis.* The analysis is presented as a sequence of lemmas and theorems, with the proofs deferred to the [Appendix](#). The outline is as follows:

- Lemma 4.6 bounds $\|X_n^f - X_n^a\|$ over a coarse timestep;
- Lemma 4.7 gives a representation of the discrete equation for X_n^f over a coarse timestep, and Corollary 4.8 gives the corresponding representation for X_n^a ;
- Lemma 4.9 gives a representation of the discrete equation describing the evolution of the average $\bar{X}_n^f = \frac{1}{2}(X_n^f + X_n^a)$ over a coarse timestep;
- Theorem 4.10 bounds $\|\bar{X}_n^f - X_n^c\|$ over a coarse timestep.

LEMMA 4.6. *For all integers $p \geq 2$, there exists a constant K_p such that*

$$\mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f - X_n^a\|^p\right] \leq K_p \Delta t^{p/2}.$$

LEMMA 4.7. *Difference equation (4.4) for X_n^f can be expressed as*

$$\begin{aligned} X_{i,n+1}^f &= X_{i,n}^f + f_i(X_n^f) \Delta t + \sum_{j=1}^D g_{ij}(X_n^f) \Delta w_{j,n} \\ &+ \sum_{j,k=1}^D h_{ijk}(X_n^f) (\Delta w_{j,n} \Delta w_{k,n} - \Omega_{jk} \Delta t) \\ &- \sum_{j,k=1}^D h_{ijk}(X_n^f) (\delta w_{j,n} \delta w_{k,n+1/2} - \delta w_{k,n} \delta w_{j,n+1/2}) \\ &+ M_{i,n}^f + N_{i,n}^f, \end{aligned}$$

where $\mathbb{E}[M_n^f | \mathcal{F}_n] = 0$, and for any integer $p \geq 2$ there exists a constant K_p such that

$$\max_{0 \leq n \leq N} \mathbb{E}[\|M_n^f\|^p] \leq K_p \Delta t^{3p/2}, \quad \max_{0 \leq n \leq N} \mathbb{E}[\|N_n^f\|^p] \leq K_p \Delta t^{2p}.$$

COROLLARY 4.8. *Difference equation (4.6) for X_n^a can be expressed as*

$$\begin{aligned} X_{i,n+1}^a &= X_{i,n}^a + f_i(X_n^a)\Delta t + \sum_{j=1}^D g_{ij}(X_n^a)\Delta w_{j,n} \\ &+ \sum_{j,k=1}^D h_{ijk}(X_n^a)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t) \\ &+ \sum_{j,k=1}^D h_{ijk}(X_n^a)(\delta w_{j,n}\delta w_{k,n+1/2} - \delta w_{k,n}\delta w_{j,n+1/2}) \\ &+ M_{i,n}^a + N_{i,n}^a, \end{aligned}$$

where $\mathbb{E}[M_n^a|\mathcal{F}_n] = 0$, and for any integer $p \geq 2$ there exists a constant K_p such that

$$\max_{0 \leq n \leq N} \mathbb{E}[\|M_n^a\|^p] \leq K_p \Delta t^{3p/2}, \quad \max_{0 \leq n \leq N} \mathbb{E}[\|N_n^a\|^p] \leq K_p \Delta t^{2p}.$$

LEMMA 4.9. *The difference equation for $\bar{X}_n^f \equiv \frac{1}{2}(X_n^f + X_n^a)$ can be expressed as*

$$\begin{aligned} \bar{X}_{i,n+1}^f &= \bar{X}_{i,n}^f + f_i(\bar{X}_n^f)\Delta t + \sum_{j=1}^D g_{ij}(\bar{X}_n^f)\Delta w_{j,n} \\ &+ \sum_{j,k=1}^D h_{ijk}(\bar{X}_n^f)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t) \\ &+ M_{i,n} + N_{i,n}, \end{aligned}$$

where $\mathbb{E}[M_n|\mathcal{F}_n] = 0$, and for any integer $p \geq 2$ there exists a constant K_p such that

$$\max_{0 \leq n \leq N} \mathbb{E}[\|M_n\|^p] \leq K_p \Delta t^{3p/2}, \quad \max_{0 \leq n \leq N} \mathbb{E}[\|N_n\|^p] \leq K_p \Delta t^{2p}.$$

THEOREM 4.10. *For all $p \geq 2$, there exists a constant K_p such that*

$$\mathbb{E}\left[\max_{0 \leq n \leq N} \|\bar{X}_n^f - X_n^c\|^p\right] \leq K_p \Delta t^p.$$

4.4. *Piecewise linear interpolation analysis.* The piecewise linear interpolant $X^c(t)$ for the coarse path is defined within the coarse timestep interval $[t_k, t_{k+1}]$ as

$$X^c(t) \equiv (1 - \lambda)X_k^c + \lambda X_{k+1}^c, \quad \lambda \equiv \frac{t - t_k}{t_{k+1} - t_k}.$$

Likewise, the piecewise linear interpolants $X^f(t)$ and $X^a(t)$ are defined on the fine timestep $[t_k, t_{k+1/2}]$ as

$$X^f(t) \equiv (1 - \lambda)X_k^f + \lambda X_{k+1/2}^f, \quad X^a(t) \equiv (1 - \lambda)X_k^a + \lambda X_{k+1/2}^a,$$

$$\lambda \equiv \frac{t - t_k}{t_{k+1/2} - t_k}$$

and there is a corresponding definition for the fine timestep $[t_{k+1/2}, t_{k+1}]$.

The proofs of the next two lemmas are in the [Appendix](#), and the theorem then follows directly.

LEMMA 4.11. *For all integers $p \geq 2$, there exists a constant K_p such that*

$$\max_{0 \leq n < N} \mathbb{E}[\|X_{n+1/2}^f - X_{n+1/2}^a\|^p] \leq K_p \Delta t^{p/2}.$$

LEMMA 4.12. *For all $p \geq 2$, there exists a constant K_p such that*

$$\max_{0 \leq n < N} \mathbb{E}[\|\bar{X}_{n+1/2}^f - X^c(t_{n+1/2})\|^p] \leq K_p \Delta t^p,$$

where $X^c(t_{n+1/2}) = \frac{1}{2}(X_n^c + X_{n+1}^c)$ is the midpoint value of the coarse path interpolant.

THEOREM 4.13. *For all $p \geq 2$, there exists a constant K_p such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}[\|X^f(t) - X^a(t)\|^p] \leq K_p \Delta t^{p/2},$$

$$\sup_{0 \leq t \leq T} \mathbb{E}[\|\bar{X}^f(t) - X^c(t)\|^p] \leq K_p \Delta t^p,$$

where $\bar{X}^f(t)$ is the average of the piecewise linear interpolants $X^f(t)$ and $X^a(t)$.

5. European and Asian payoffs.

5.1. *European options.* In the case of payoff which is a smooth function of the final state $x(T)$, taking $p = 2$ in Lemma 2.2, $p = 4$ in Lemma 4.6 and $p = 2$ in Theorem 4.10, immediately gives the result that the multilevel variance

$$\mathbb{V}[\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(X_N^c)]$$

has an $O(\Delta t^2)$ upper bound. This matches the convergence rate for the multilevel method for scalar SDEs using the standard first order Milstein discretisation, and is much better than the $O(\Delta t)$ convergence obtained with the Euler–Maruyama discretisation.

However, very few financial payoff functions are twice differentiable on the entire domain \mathbb{R}^d . A more typical 2D example is a call option based on the minimum of two assets,

$$P(x(T)) \equiv \max(0, \min(x_1(T), x_2(T)) - K),$$

which is piecewise linear, with a discontinuity in the gradient along the three lines (s, K) , (K, s) and (s, s) for $s \geq K$.

To handle such payoffs, we introduce a new assumption which bounds the probability of the solution of the SDE having a value at time T close to such lines with discontinuous gradients, and then formulate a theorem to show that the multi-level variance which results from using the antithetic estimator has an upper bound which is almost $O(\Delta t^{3/2})$.

ASSUMPTION 5.1. The payoff function $P \in C(\mathbb{R}^d, \mathbb{R})$ has a uniform Lipschitz bound, so that there exists a constant L such that

$$|P(x) - P(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^d$$

and the first and second derivatives exist, are continuous and have uniform bound L at all points $x \notin K$, where K is a set of zero measure, and there exists a constant c such that the probability of the SDE solution $x(T)$, being within a neighbourhood of the set K , has the bound

$$\mathbb{P}\left(\min_{y \in K} \|x(T) - y\| \leq \varepsilon\right) \leq c\varepsilon \quad \forall \varepsilon > 0.$$

In a 1D context, Assumption 5.1 corresponds to an assumption of a locally bounded density for $x(T)$.

THEOREM 5.2. *If the SDE satisfies the conditions of Assumption 4.1, and the payoff satisfies Assumption 5.1, then*

$$\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(X_N^c)\right)^2\right] = o(\Delta t^{3/2-\delta})$$

for any $\delta > 0$.

PROOF. We start by noting that

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(X_N^c)\right)^2\right] \\ &\leq 2\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^2\right] + 2\mathbb{E}\left[\left(\frac{1}{2}(P(\bar{X}_N^f) - P(X_N^c))\right)^2\right]. \end{aligned}$$

The second term on the right-hand side has an $O(\Delta t^2)$ bound due to the uniform Lipschitz bound for the payoff, together with the result from Theorem 4.10 for $p = 2$.

The objective now is to prove that the first term has a $o(\Delta t^{3/2-\delta})$ bound for any $\delta > 0$. The analysis follows the approach used in [7]. To prove this for a particular value of δ , we define $\varepsilon = \Delta t^{1/2-\delta/2}$, and consider the three events

$$\begin{aligned} A &\equiv \left\{ \min_{y \in K} \|x(T) - y\| \leq \varepsilon \right\}, \\ B &\equiv \left\{ \|x(T) - X_N^f\| \geq \frac{1}{2}\varepsilon \right\}, \\ C &\equiv \left\{ \|X_N^f - X_N^a\| \geq \frac{1}{2}\varepsilon \right\}. \end{aligned}$$

Using $\mathbf{1}_A$ to indicate the indicator function for event A , and A^c to denote the complement of A , we have

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^2\right] \\ &= \mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^2 \mathbf{1}_{A \cup B \cup C}\right] \\ &\quad + \mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^2 \mathbf{1}_{A^c \cap B^c \cap C^c}\right]. \end{aligned}$$

Looking at the first of the two terms on the right-hand side, then Hölder’s inequality gives

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^2 \mathbf{1}_{A \cup B \cup C}\right] \\ &\leq \mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^{2p}\right]^{1/p} (\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C))^{1/q} \end{aligned}$$

for any $p, q \geq 1$, with $p^{-1} + q^{-1} = 1$. The Markov inequality gives

$$\mathbb{P}(B) \leq \mathbb{E}[\|x(T) - X_N^f\|^m] / \left(\frac{1}{2}\varepsilon\right)^m$$

for any $m \geq 1$. Using the strong convergence property from Lemma 4.2, and the definition of ε , we can take m to be sufficiently large so that

$$\frac{1}{2}m - \frac{1-\delta}{2}m > \frac{1-\delta}{2}$$

and hence there exists a constant c_1 such that $\mathbb{P}(B) \leq c_1\varepsilon$. Using Lemma 4.6, one can obtain a similar bound $\mathbb{P}(C) \leq c_2\varepsilon$, and then q can be chosen sufficiently close to 1 so that

$$(\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C))^{1/q} \leq (1 + c_1 + c_2)^{1/q} \Delta t^{(1/2-\delta/2)/q} = o(\Delta t^{1/2-\delta}).$$

Since

$$\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f) = \frac{1}{2}(P(X_N^f) - P(\bar{X}_N^f)) + \frac{1}{2}(P(X_N^a) - P(\bar{X}_N^f)),$$

the uniform Lipschitz bound gives

$$\mathbb{E}\left[\left(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f)\right)^{2p}\right]^{1/p} \leq L^2 \mathbb{E}[\|X_N^f - X_N^a\|^{2p}]^{1/p} \leq c_3 \Delta t$$

for some constant c_3 due to Lemma 4.6, and hence

$$\mathbb{E}[(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f))^2 \mathbf{1}_{A \cup B \cup C}] = o(\Delta t^{3/2-\delta}).$$

Lastly, we consider the second term

$$\mathbb{E}[(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f))^2 \mathbf{1}_{A^c \cap B^c \cap C^c}].$$

Given a path sample $\omega \in (B^c \cap C^c)$, if the straight line between X_N^f and X_N^a contains a point $y \in K$, then $\|y - X_N^f\|$ and $\|x(T) - X_N^f\|$ are both less than $\varepsilon/2$, and hence $\|x(T) - y\| < \varepsilon$.

Thus, for a path sample $\omega \in (A^c \cap B^c \cap C^c)$, the straight line between X_N^f and X_N^a does not contain any points in K . It is therefore possible to perform a second order truncated Taylor expansion as in the proof of Lemma 2.2, and deduce that there exists a constant c_4 such that

$$\mathbb{E}[(\frac{1}{2}(P(X_N^f) + P(X_N^a)) - P(\bar{X}_N^f))^2 \mathbf{1}_{A^c \cap B^c \cap C^c}] \leq c_4 \mathbb{E}[\|X_N^f - X_N^a\|^4],$$

which has an $O(\Delta t^2)$ bound due to Lemma 4.6. \square

5.2. Asian payoffs. For an Asian option, the payoff depends on the average

$$x_{\text{ave}} \equiv T^{-1} \int_0^T x(t) dt.$$

This can be approximated by integrating the appropriate piecewise linear interpolant which gives

$$\begin{aligned} X_{\text{ave}}^c &\equiv T^{-1} \int_0^T X^c(t) dt = N^{-1} \sum_{n=0}^{N-1} \frac{1}{2}(X_n^c + X_{n+1}^c), \\ X_{\text{ave}}^f &\equiv T^{-1} \int_0^T X^f(t) dt = N^{-1} \sum_{n=0}^{N-1} \frac{1}{4}(X_n^f + 2X_{n+1/2}^f + X_{n+1}^f), \\ X_{\text{ave}}^a &\equiv T^{-1} \int_0^T X^a(t) dt = N^{-1} \sum_{n=0}^{N-1} \frac{1}{4}(X_n^a + 2X_{n+1/2}^a + X_{n+1}^a). \end{aligned}$$

Due to Hölder’s inequality,

$$\begin{aligned} \mathbb{E}[\|X_{\text{ave}}^f - X_{\text{ave}}^a\|^p] &\leq T^{-1} \int_0^T \mathbb{E}[\|X^f(t) - X^a(t)\|^p] dt \\ &\leq \sup_{[0,T]} \mathbb{E}[\|X^f(t) - X^a(t)\|^p] \end{aligned}$$

and similarly,

$$\mathbb{E}\left[\left\|\frac{1}{2}(X_{\text{ave}}^f + X_{\text{ave}}^a) - X_{\text{ave}}^c\right\|^p\right] \leq \sup_{[0,T]} \mathbb{E}[\|\bar{X}^f(t) - X^c(t)\|^p].$$

Hence, if the Asian payoff is a smooth function of the average, then taking $p = 2$ in Lemma 2.2, $p = 4$ in Corollary 4.11 and $p = 2$ in Corollary 4.12, again gives a second order bound for the multilevel correction variance.

This analysis can be extended to include payoffs which are a smooth function of a number of intermediate variables, each of which is a linear functional of the path $x(t)$ of the form

$$\int_0^T g^T(t)x(t)\mu(dt)$$

for some vector function $g(t)$ and measure $\mu(dt)$. This includes weighted averages of $x(t)$ at a number of discrete times, as well as continuously-weighted averages over the whole time interval.

As with the European options, the analysis can also be extended to payoffs which are Lipschitz functions of the average, and have first and second derivatives which exist and are continuous and uniformly bounded, except for a set of points K of zero measure.

ASSUMPTION 5.3. The payoff $P \in C(\mathbb{R}^d, \mathbb{R})$ has a uniform Lipschitz bound, so that there exists a constant L such that

$$|P(x) - P(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^d$$

and the first and second derivatives exist, are continuous and have uniform bound L at all points $x \notin K$, where K is a set of zero measure, and there exists a constant c such that the probability of x_{ave} being within a neighbourhood of the set K has the bound

$$\mathbb{P}\left(\min_{y \in K} \|x_{\text{ave}} - y\| \leq \varepsilon\right) \leq c\varepsilon \quad \forall \varepsilon > 0.$$

THEOREM 5.4. If the SDE satisfies the conditions of Assumption 4.1, and the payoff satisfies Assumption 5.3, then

$$\mathbb{E}\left[\left(\frac{1}{2}(P(X_{\text{ave}}^f) + P(X_{\text{ave}}^a)) - P(X_{\text{ave}}^c)\right)^2\right] = o(\Delta t^{3/2-\delta})$$

for any $\delta > 0$.

5.3. *Nonasymptotic result.* The analysis above concerns the asymptotic behaviour of the multilevel variance as $\Delta t \rightarrow 0$. However, it is also worth noting that since X^f and X^a have exactly the same distribution, conditional on the coarse path Brownian increments ΔW^c , then $P^f - P^c$ and $P^a - P^c$ are identically distributed, and hence

$$\begin{aligned} \mathbb{V}\left[\frac{1}{2}(P^f + P^a) - P^c\right] &= \mathbb{V}\left[\frac{1}{2}(P^f - P^c) + \frac{1}{2}(P^a - P^c)\right] \\ (5.1) \qquad \qquad \qquad &= \frac{1}{2}(1 + \rho)\mathbb{V}[P^f - P^c], \end{aligned}$$

where ρ is the correlation between the $P^f - P^c$ and $P^a - P^c$. Thus, regardless of the size of the timestep, the variance of the antithetic estimator cannot be larger than the variance of the standard estimator, and could be significantly smaller if ρ is negative. What the asymptotic analysis shows is that $\rho \rightarrow -1$ as $\Delta t \rightarrow 0$.

6. Numerical experiments. In this section we present numerical tests in which we compare classical Monte Carlo (MC), standard MLMC and antithetic MLMC estimators. We consider the Clark–Cameron SDEs and Heston’s stochastic volatility model with both smooth and non-smooth payoffs. We will see that in all cases the antithetic MLMC variance is significantly smaller than the standard MLMC variance on all levels of approximation.

6.1. *Clark–Cameron SDEs.* The first set of results in Figure 2 is for the Clark–Cameron SDEs with initial conditions $x_1(0) = x_2(0) = 0$, final time $T = 1$, and smooth payoff $P = \cos(x_1(T))$.

The top left plot shows the behaviour of the variance as a function of the level of approximation, so that $\Delta t = 2^{-\ell}$. These values were estimated using 10^6 samples,

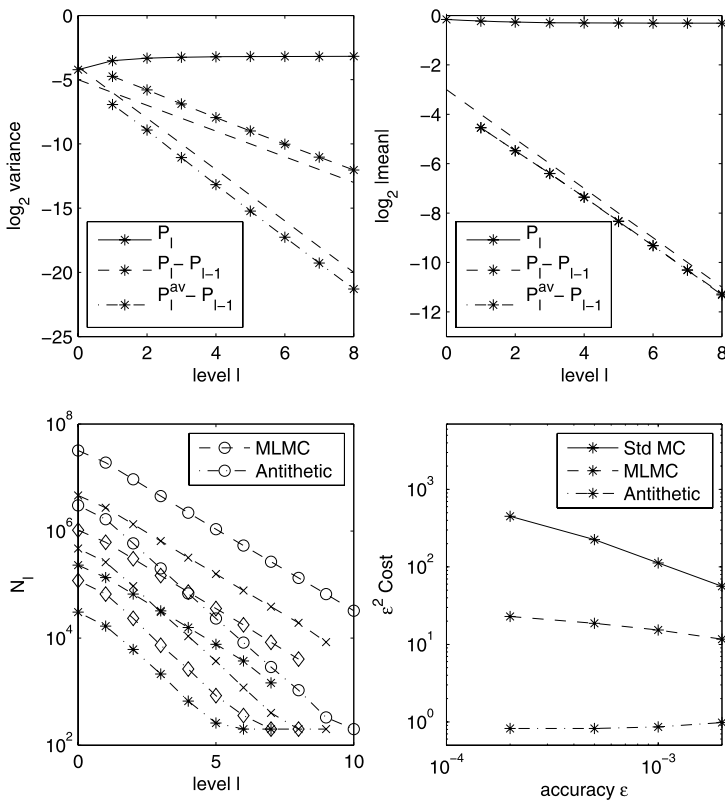


FIG. 2. Clark–Cameron SDEs with smooth payoff $P = \cos(x_1(T))$.

so the sampling error is very small. The solid line is the variance of the standard Monte Carlo estimator which varies very little with level. The dashed line is the usual MLMC estimator $P_\ell^f - P_{\ell-1}^c$, and the accompanying reference line with slope -1 confirms its expected first order convergence. The dot-dash line is for the antithetic estimator $\frac{1}{2}(P_\ell^f + P_\ell^a) - P_{\ell-1}^c$, and its accompanying reference line with slope -2 confirms its second order convergence. Note also that even on level $\ell = 1$ in which the multilevel estimator comes from the difference between simulations with 2 timesteps (on level 1) and 1 timestep (on level 0), the antithetic estimator has a variance which is roughly a factor 4 smaller than the standard MLMC estimator.

The top right plot shows the mean value for the multilevel correction. As expected the standard MLMC and antithetic MLMC estimator have exactly the same expected value, and it converges at first order as indicated by the reference line with slope -1 .

The bottom right plot shows the dependence of the computational complexity C (defined as the total number of random numbers generated) as a function of the desired accuracy ϵ . Because of Theorem 2.1 the plot is of $\epsilon^2 C$ versus ϵ , because we expect to see that $\epsilon^2 C$ is only weakly dependent on ϵ for the standard MLMC and independent of ϵ for the antithetic MLMC. For the standard Monte Carlo method, theory predicts that $\epsilon^2 C$ should be proportional to the number of timesteps on the finest level, which in turn is roughly proportional to ϵ^{-1} due to the first order weak convergence order. We see that computational complexity of the antithetic MLMC is much lower than for the standard MLMC.

Further insight into the complexity cost is provided by the bottom left plot. Each point in the bottom right complexity plot corresponds to a line in the bottom left plot, showing the number of samples taken on each level of the multilevel approximation. Lines with the same plotting symbol correspond to the same desired accuracy ϵ , with the upper line being for the MLMC estimator, and the lower line being for the antithetic estimator.

There are several points to note in this plot. The first is that for a given accuracy, the number of samples on each level decays rapidly as ℓ increases. This follows the prescription given in [6] in which the optimal number of samples on each level is proportional to $\sqrt{V_\ell/C_\ell}$ where V_ℓ is the multilevel variance and C_ℓ is the cost of a single sample on level ℓ . The constant of proportionality is chosen so that the overall variance $\sum_{\ell=0}^L N_\ell^{-1} V_\ell$ is less than $\epsilon^2/2$. Because the antithetic variance converges to zero more rapidly, the slope of the antithetic lines is slightly greater than the slope of the standard MLMC lines.

The next point to note is that the lines with circular symbols (which correspond to the tightest accuracy specification $\epsilon = 10^{-4}$) extend to level $\ell = 10$, while the other lines terminate at lower levels. This is again following the prescription in [6] in which the mean square error is brought below ϵ^2 by ensuring that the square of the bias is also below $\epsilon^2/2$, like the total variance. Using a simple heuristic to estimate the remaining discretisation bias, because of the first order weak convergence, fewer approximation levels are required when ϵ is larger.

The final observation to be made is that the antithetic line lies well below the standard MLMC line for the same accuracy ϵ . This is what produces the overall computational savings shown in the bottom right plot. However, on level 0 the two are using exactly the same estimator, so why does the antithetic estimator use fewer samples than the standard MLMC on level 0? The answer is that both have a variance budget of $\epsilon^2/2$ to be spread over all of the levels in the way which minimises the total computational cost [6]. In the standard MLMC case, this budget is spread fairly evenly over the different levels, but in the antithetic case most of the budget is allocated to level 0 (because the estimator variance decays so rapidly on the higher levels) and so fewer samples are required on level 0.

The next set of results in Figure 3 are for the same Clark–Cameron SDE but with the Lipschitz payoff

$$P = \max(x_1(T), 0).$$

The same comments as before apply to the plots in this figure. The only difference is that the lower of the two reference lines in the top left plot has slope -1.5 ,

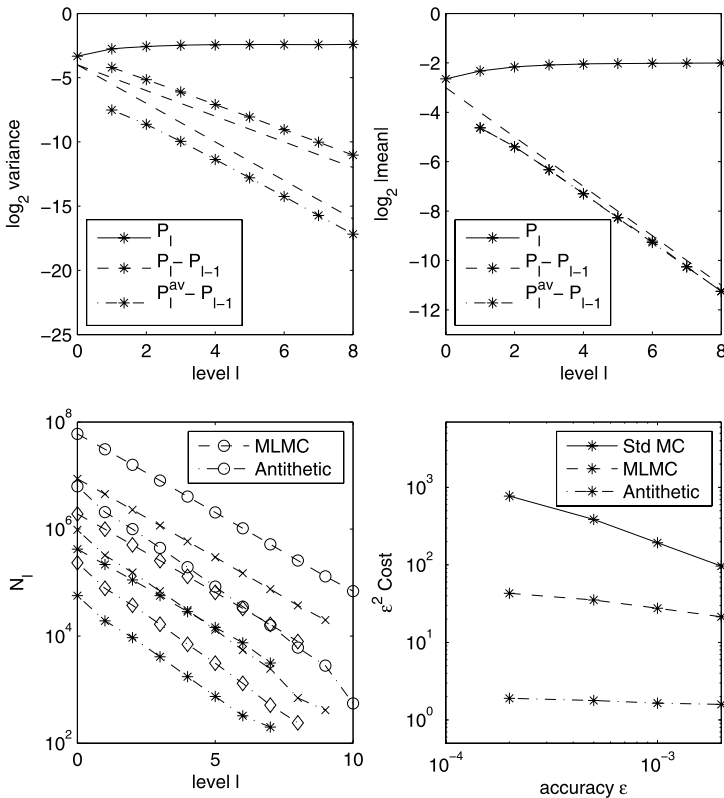


FIG. 3. Clark–Cameron SDEs with $P = \max(x_1(T), 0)$.

confirming that the multilevel variance is $O(\Delta t^{3/2})$ rather than $O(\Delta t^2)$ because of the discontinuity in the first derivative of the payoff function. Apart from that, the results are very similar with the antithetic estimator have a much lower variance on all grid levels, and overall giving a much reduced computational cost.

6.2. *Heston stochastic volatility model.* The Heston model [11], which is an asset price model with stochastic volatility, is one of the most popular SDEs in finance

$$\begin{aligned} ds(t) &= rs(t) dt + \sqrt{v(t)}s(t) dw_1(t), & s(0) > 0, \\ dv(t) &= \kappa(\theta - v(t)) dt + \sigma\sqrt{v(t)} dw_2(t), & v(0) > 0, \end{aligned}$$

where $\mathbb{E}[w_1(t)w_2(t)] = 0$, $r > 0$ and $2\kappa\theta \geq \sigma^2$, ensuring that the zero boundary is not attainable for the volatility process. Due to the nonlinearity of the diffusion coefficient in the price process $s(t)$ we work with log-Heston model

$$\begin{aligned} d\log(s(t)) &= (r - \frac{1}{2}v(t)) dt + \sqrt{v(t)} dw_1(t), \\ dv(t) &= \kappa(\theta - v(t)) dt + \sigma\sqrt{v(t)} dw_2(t). \end{aligned}$$

Although the coefficients of the volatility process $\{v(t)\}_{t \geq 0}$ are not Lipschitz continuous, and hence the assumptions imposed in the current paper are not satisfied, the numerical tests show that the antithetic MLMC performs very well. To approximate the volatility process we use a drift implicit Milstein scheme that preserves the positivity of the original SDE, and has a good strong convergence property recently established by Neuenkirch and Szpruch in [16]. Hence, the Milstein scheme for Heston’s stochastic volatility model with the Lévy area term set to zero is given by

$$\begin{aligned} \log(S_{n+1}) &= \log(S_n) + (r - \frac{1}{2}V_n)\Delta t + \sqrt{V_n}\Delta w_{1,n} + \frac{1}{4}\sigma\Delta w_{1,n}\Delta w_{2,n}, \\ V_{n+1} &= V_n + \kappa(\theta - V_{n+1})\Delta t + \sigma\sqrt{V_n}\Delta w_{2,n} + \frac{1}{2}\sigma^4(\Delta w_{2,n}^2 - \Delta t). \end{aligned}$$

For the simulation studies we choose $s_0 = v_0 = 1$, $r = 0.05$, $T = 1$ and $\kappa = 0.5$, $\theta = 0.9$, $\sigma = 0.05$ in order to ensure the Feller boundary condition for the volatility process.

Figure 4 presents our results for the smooth payoff $P = x(T)$. The four plots have a similar structure to the results of the Clark–Cameron application. The two reference lines in the top left plot again have slopes -1 and -2 , confirming that the antithetic MLMC variance is $O(\Delta t^2)$, whereas the standard MLMC variance is $O(\Delta t)$. The top right plot shows that the weak discretisation error is again first order.

The bottom right plot shows that computational savings of the antithetic MLMC compared to the standard MLMC are not as great as for the Clark–Cameron application. The reason for this can be seen in the bottom left plot. The multilevel variance on levels 1 and above is much smaller than the variance on level 0, where

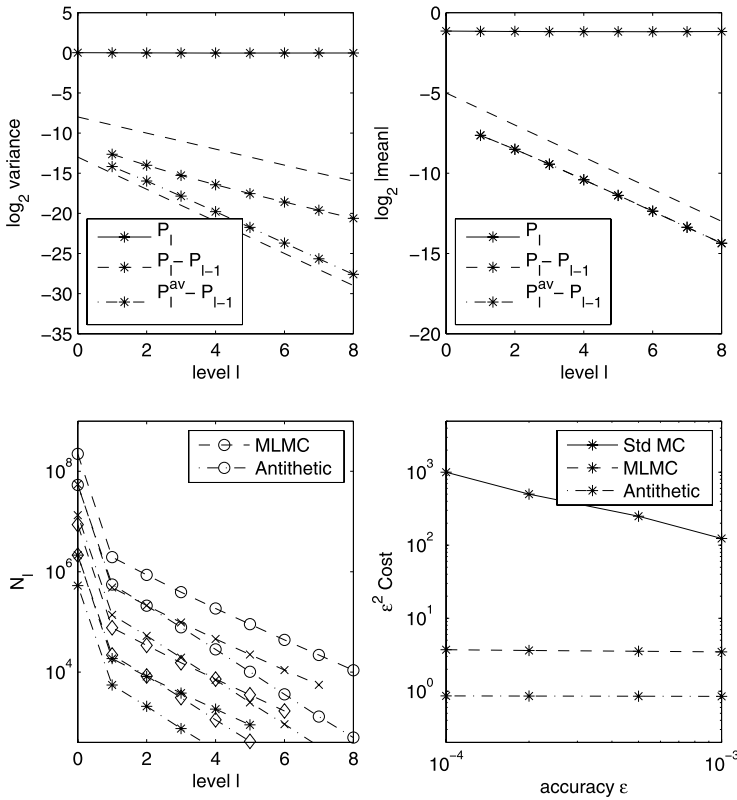


FIG. 4. *Heston SDEs with $P = x(T)$.*

both methods use the same estimator. Hence, in both cases much of the computational effort is expended on the coarsest level and so the benefits of the antithetic treatment are reduced.

The final results in Figure 5 are for the same Heston SDEs but with the call option payoff $P = \max(s(T) - 1, 0)$. The steeper of the two reference lines in the top left plot has a slope of -2 , not the -1.5 used for the Clark–Cameron case for the non-smooth payoff. This indicates that the antithetic variance is $O(\Delta t^2)$, not the $O(\Delta t^{3/2})$ predicted by the analysis. It is possible that there is indeed an $O(\Delta t^{3/2})$ component to the error, but that the corresponding coefficient is so small that it does not become apparent until much smaller values of Δt . Other than this, the results are very similar to the previous case.

7. Conclusions. In this paper we have constructed a new antithetic multi-level Monte Carlo estimator for multi-dimensional SDEs, with a variance which is $O(\Delta t^2)$ when the payoff function is smooth, and almost an $O(\Delta t^{3/2})$ when it is Lipschitz and piecewise smooth. The algorithm is very easy to implement; all that

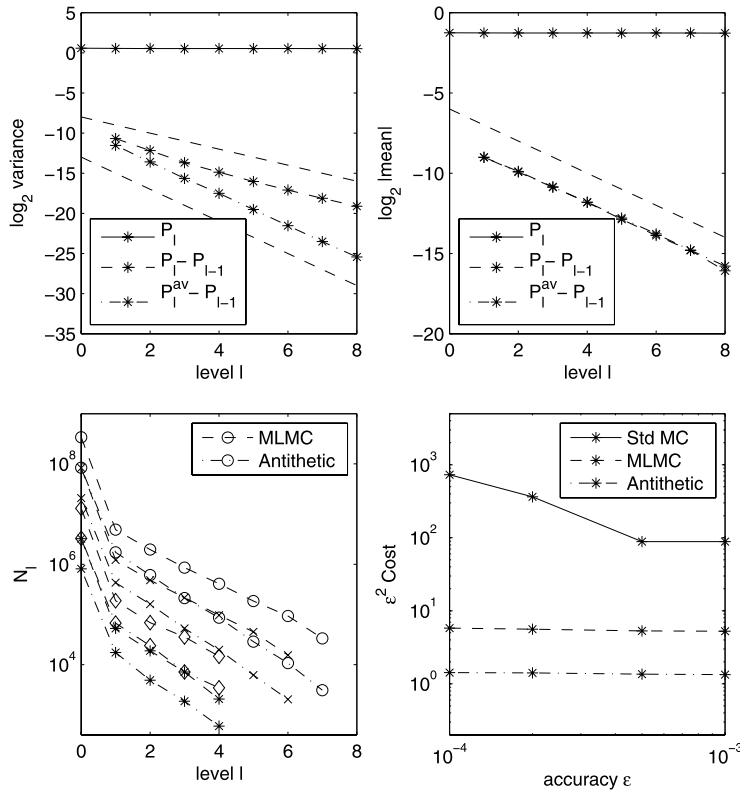


FIG. 5. Heston SDEs with $P = \max(s(T) - 1, 0)$.

is required is to calculate a second fine path for which the odd and even Brownian increments are swapped.

In the European and Asian payoff cases considered in this paper, it reduces the computational complexity for an ϵ root-mean-square error to $O(\epsilon^{-2})$, compared to $O(\epsilon^{-2}(\log(1/\epsilon))^2)$ for the multilevel method using the Euler–Maruyama discretisation, and $O(\epsilon^{-3})$ for the standard Monte Carlo method. Furthermore, by ensuring that the dominant computational effort is on the coarsest levels (since $\beta > 1$), it is now feasible to obtain further improvements using quasi-Monte Carlo techniques [9].

In a future paper, we will extend the analysis to cover digital and barrier options. The improvements from an extended version of the antithetic treatment are then more substantial, improving the complexity from $O(\epsilon^{-5/2})$ to approximately $O(\epsilon^{-2})$.

APPENDIX: PROOF OF MAIN RESULTS

A.1. Proof of Lemma 4.6. Conditional on the Brownian increments Δw for the coarse path X^c , the Brownian increments for X^f and X^a have exactly the same

distribution, and therefore $X_n^a - X_n^c$ has exactly the same distribution as $X_n^f - X_n^c$. Hence we obtain, using inequality (2.4),

$$\begin{aligned} & \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f - X_n^a\|^p\right] \\ & \leq 2^{p-1} \left(\mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f - X_n^c\|^p\right] + \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^a - X_n^c\|^p\right] \right) \\ & = 2^p \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f - X_n^c\|^p\right] \\ & \leq 2^{2p-1} \left(\mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^f - x(t_n)\|^p\right] + \mathbb{E}\left[\max_{0 \leq n \leq N} \|X_n^c - x(t_n)\|^p\right] \right). \end{aligned}$$

The desired result then follows from the strong convergence property in Lemma 4.2.

A.2. Proof of Lemma 4.7 and Corollary 4.8. Combining the two equations in (4.3), and using the identity

$$\Delta w_{j,n} \Delta w_{k,n} = (\delta w_{j,n} + \delta w_{j,n+1/2})(\delta w_{k,n} + \delta w_{k,n+1/2})$$

together with the definition of h_{ijk} in (1.2) gives, after considerable re-arrangement,

$$\begin{aligned} X_{i,n+1}^f &= X_{i,n}^f + f_i(X_n^f) \Delta t + \sum_{j=1}^D g_{ij}(X_n^f) \Delta w_{j,n} \\ & \quad + \sum_{j,k=1}^D h_{ijk}(X_n^f) (\Delta w_{j,n} \Delta w_{k,n} - \Omega_{jk} \Delta t) \\ & \quad - \sum_{j,k=1}^D h_{ijk}(X_n^f) (\delta w_{j,n} \delta w_{k,n+1/2} - \delta w_{k,n} \delta w_{j,n+1/2}) \\ & \quad + R_{i,n} + M_{i,n}^{(2)} + M_{i,n}^{(3)}, \end{aligned}$$

where

$$\begin{aligned} R_{i,n} &= (f_i(X_{n+1/2}^f) - f_i(X_n^f)) \Delta t / 2, \\ M_{i,n}^{(2)} &= \sum_{j=1}^D \left(g_{ij}(X_{n+1/2}^f) - g_{ij}(X_n^f) - 2 \sum_{k=1}^D h_{ijk}(X_n^f) \delta w_{k,n} \right) \delta w_{j,n+1/2}, \\ M_{i,n}^{(3)} &= \sum_{j,k=1}^D (h_{ijk}(X_{n+1/2}^f) - h_{ijk}(X_n^f)) (\delta w_{j,n+1/2} \delta w_{k,n+1/2} - \Omega_{jk} \Delta t / 2). \end{aligned}$$

Considering R_n , a Taylor expansion gives

$$\begin{aligned} & f_i(X_{n+1/2}^f) - f_i(X_n^f) \\ &= \sum_{j=1}^d \frac{\partial f_i}{\partial x_j}(X_n^f)(X_{j,n+1/2}^f - X_{j,n}^f) \\ & \quad + \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\xi_1)(X_{j,n+1/2}^f - X_{j,n}^f)(X_{k,n+1/2}^f - X_{k,n}^f) \end{aligned}$$

for some ξ_1 which lies on the line between X_n^f and $X_{n+1/2}^f$. Hence, R_n can be split into two parts, $R_n = M_n^{(1)} + N_n$, where

$$M_{i,n}^{(1)} = \sum_{j=1}^d \sum_{k=1}^D \frac{\partial f_i}{\partial x_j}(X_n^f) g_{jk}(X_n^f) \delta w_{k,n} \Delta t / 2,$$

and

$$\begin{aligned} N_{i,n} &= \sum_{j=1}^d \frac{\partial f_i}{\partial x_j}(X_n^f) \left(f_j(X_n^f) \Delta t / 2 \right. \\ & \quad \left. + \sum_{k,l=1}^D h_{jkl}(X_n^f) (\delta w_{k,n} \delta w_{l,n} - \Omega_{kl} \Delta t / 2) \right) \Delta t / 2 \\ & \quad + \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\xi_1)(X_{j,n+1/2}^f - X_{j,n}^f)(X_{k,n+1/2}^f - X_{k,n}^f) \Delta t / 2. \end{aligned}$$

Considering $M_n^{(2)}$, a Taylor expansion gives

$$\begin{aligned} & g_{ij}(X_{n+1/2}^f) - g_{ij}(X_n^f) \\ &= \sum_{k=1}^d \frac{\partial g_{ij}}{\partial x_k}(X_n^f)(X_{k,n+1/2}^f - X_{k,n}^f) \\ & \quad + \frac{1}{2} \sum_{k,l=1}^d \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(\xi_2)(X_{k,n+1/2}^f - X_{k,n}^f)(X_{l,n+1/2}^f - X_{l,n}^f) \end{aligned}$$

for some ξ_2 on the line between X_n^f and $X_{n+1/2}^f$, and therefore

$$\begin{aligned} M_{i,n}^{(2)} &= \sum_{j=1}^d \sum_{k=1}^D \frac{\partial g_{ij}}{\partial x_k}(X_n^f) \left(f_k(X_n^f) \Delta t / 2 \right. \\ & \quad \left. + \sum_{l,m=1}^D h_{klm}(X_n^f) (\delta w_{l,n} \delta w_{m,n} - \Omega_{lm} \Delta t / 2) \right) \delta w_{j,n+1/2} \end{aligned}$$

$$+ \frac{1}{2} \sum_{j=1}^D \sum_{k,l=1}^d \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(\xi_2)(X_{k,n+1/2}^f - X_{k,n}^f)(X_{l,n+1/2}^f - X_{l,n}^f)\delta w_{j,n+1/2}.$$

Finally, considering $M_n^{(3)}$ we have

$$\begin{aligned} M_{i,n}^{(3)} &= \sum_{j,k=1}^D (h_{ijk}(X_{n+1/2}^f) - h_{ijk}(X_n^f))(\delta w_{j,n+1/2}\delta w_{k,n+1/2} - \Omega_{jk}\Delta t/2) \\ &= \sum_{j,k=1}^D \sum_{l=1}^d \frac{\partial h_{ijk}}{\partial x_l}(\xi_3)(X_{l,n+1/2}^f - X_{l,n}^f)(\delta w_{j,n+1/2}\delta w_{k,n+1/2} - \Omega_{jk}\Delta t/2) \end{aligned}$$

for some ξ_3 on the line between X_n^f and $X_{n+1/2}^f$.

Setting $M_n^f \equiv M_n^{(1)} + M_n^{(2)} + M_n^{(3)}$, it is clear that $\mathbb{E}[M_n^f | \mathcal{F}_n] = 0$ since δw_n is independent of X_n^f , and $\delta w_{n+1/2}$ is independent of X_n^f and $X_{n+1/2}^f$.

All that remains is to bound the magnitude of $\mathbb{E}[\|M_n^f\|^p]$ and $\mathbb{E}[\|N_n^f\|^p]$. Looking at two of the terms in $M_{i,n}^{(2)}$, for example, the uniform bound on the first derivatives of g , together with the fact that $\delta w_{n+1/2}$ is independent of both X_n^f and δw_n leads to

$$\begin{aligned} &\mathbb{E} \left[\left| \frac{\partial g_{ij}}{\partial x_k}(X_n^f)h_{klm}(X_n^f)\delta w_{l,n}\delta w_{m,n}\delta w_{j,n+1/2} \right|^p \right] \\ &\leq L^p \mathbb{E}[|h_{klm}(X_n^f)|^p] \mathbb{E}[\|\delta w_n\|^{2p}] \mathbb{E}[\|\delta w_{n+1/2}\|^p] \end{aligned}$$

and the uniform bound on the second derivatives of g , together with the fact that $\delta w_{n+1/2}$ is independent of both X_n^f and $X_{n+1/2}^f$ leads to

$$\begin{aligned} &\mathbb{E} \left[\left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(\xi_2)(X_{k,n+1/2}^f - X_{k,n}^f)(X_{l,n+1/2}^f - X_{l,n}^f)\delta w_{j,n+1/2} \right|^p \right] \\ &\leq L^p \mathbb{E}[\|X_{n+1/2}^f - X_n^f\|^{2p}] \mathbb{E}[\|\delta w_{n+1/2}\|^p]. \end{aligned}$$

Combining the uniform bound on $\mathbb{E}[|h_{ijk}(X_n^f)|^{2p}]$ from Corollary 4.3 with the bounds from Lemma 4.4, and standard results for the moments of Brownian increments, gives the required $O(\Delta t^{3p/2})$ bound for each of the two terms considered.

Deriving similar bounds for the other terms in M^f and N^f , and combining them using (2.4), eventually gives the desired bounds for both $\mathbb{E}[\|M_n^f\|^p]$ and $\mathbb{E}[\|N_n^f\|^p]$.

The proof is almost exactly the same for Corollary 4.8. The sign change in the second line of the equation in the statement of the corollary is due to the swapping of the Brownian increments for the first and second halves of the timestep.

A.3. Proof of Lemma 4.9. Recalling that $\bar{X}^f = \frac{1}{2}(X^f + X^a)$, taking the average of the results from Lemma 4.7 and Corollary 4.8 gives

$$\begin{aligned} \bar{X}_{i,n+1}^f &= \bar{X}_{i,n}^f + f_i(\bar{X}_n^f)\Delta t + \sum_{j=1}^D g_{ij}(\bar{X}_n^f)\Delta w_{j,n} \\ &\quad + \sum_{j,k=1}^D h_{ijk}(\bar{X}_n^f)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t) \\ &\quad + \frac{1}{2}(M_{i,n}^f + N_{i,n}^f + M_{i,n}^a + N_{i,n}^a) + M_{i,n}^{(1)} + M_{i,n}^{(2)} + M_{i,n}^{(3)} + N_{i,n}^{(1)}, \end{aligned}$$

where

$$\begin{aligned} N_{i,n}^{(1)} &= (\frac{1}{2}(f_i(X_n^f) + f_i(X_n^a)) - f_i(\bar{X}_n^f))\Delta t, \\ M_{i,n}^{(1)} &= \sum_{j=1}^D \left(\frac{1}{2}(g_{ij}(X_n^f) + g_{ij}(X_n^a)) - g_{ij}(\bar{X}_n^f)\right)\Delta w_{j,n}, \\ M_{i,n}^{(2)} &= \sum_{j,k=1}^D \left(\frac{1}{2}(h_{ijk}(X_n^f) + h_{ijk}(X_n^a)) - h_{ijk}(\bar{X}_n^f)\right)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t), \\ M_{i,n}^{(3)} &= \sum_{j,k=1}^D \frac{1}{2}(h_{ijk}(X_n^f) - h_{ijk}(X_n^a))(\delta w_{j,n}\delta w_{k,n+1/2} - \delta w_{k,n}\delta w_{j,n+1/2}). \end{aligned}$$

Setting

$$M_n = \frac{1}{2}(M_n^f + M_n^a) + M_n^{(1)} + M_n^{(2)} + M_n^{(3)}, \quad N_n = \frac{1}{2}(N_n^f + N_n^a) + N_n^{(1)},$$

it is clear that $\mathbb{E}[M_n|\mathcal{F}_n] = 0$, and all that remains is to bound the magnitude of $\mathbb{E}[\|M_n\|^p]$ and $\mathbb{E}[\|N_n\|^p]$. By performing second order Taylor series expansions for $f(x)$ and $g(x)$, and first order expansions for $h(x)$, all about \bar{X}_n^f , we obtain

$$\begin{aligned} N_{i,n}^{(1)} &= \frac{1}{16} \sum_{j,k=1}^d \left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\xi_1) + \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\xi_2) \right) (X_{j,n}^f - X_{j,n}^a)(X_{k,n}^f - X_{k,n}^a)\Delta t, \\ M_{i,n}^{(1)} &= \frac{1}{16} \sum_{j=1}^D \sum_{k,l=1}^d \left(\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(\xi_3) + \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(\xi_4) \right) \\ &\quad \times (X_{k,n}^f - X_{k,n}^a)(X_{l,n}^f - X_{l,n}^a)\Delta w_{j,n}, \\ M_{i,n}^{(2)} &= \frac{1}{4} \sum_{j,k=1}^D \sum_{l=1}^d \left(\frac{\partial h_{ijk}}{\partial x_l}(\xi_5) - \frac{\partial h_{ijk}}{\partial x_l}(\xi_6) \right) \\ &\quad \times (X_{l,n}^f - X_{l,n}^a)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t), \end{aligned}$$

$$M_{i,n}^{(3)} = \frac{1}{4} \sum_{j,k=1}^D \sum_{l=1}^d \left(\frac{\partial h_{ijk}}{\partial x_l}(\xi_7) + \frac{\partial h_{ijk}}{\partial x_l}(\xi_8) \right) \times (X_{l,n}^f - X_{l,n}^a)(\delta w_{j,n} \delta w_{k,n+1/2} - \delta w_{k,n} \delta w_{j,n+1/2})$$

for some $\xi_1, \xi_3, \xi_5, \xi_7$ between \bar{X}_n^f and X_n^f , and $\xi_2, \xi_4, \xi_6, \xi_8$ between \bar{X}_n^f and X_n^a .

Using the same arguments as in the final part of the proof of Lemma 4.7, together with the bounds on $\mathbb{E}[\|M_n^f\|^p]$, $\mathbb{E}[\|M_n^a\|^p]$, $\mathbb{E}[\|N_n^f\|^p]$ and $\mathbb{E}[\|N_n^a\|^p]$, leads to the required bounds for the moments of M_n and N_n .

A.4. Proof of Theorem 4.10. If we define $S_n = \mathbb{E}[\max_{m \leq n} \|\bar{X}_m^f - X_m^c\|^p]$, then inequality (2.4) gives

$$(A.1) \quad S_n \leq d^{p-1} \sum_{i=1}^d \mathbb{E}[\max_{m \leq n} |\bar{X}_{i,m}^f - X_{i,m}^c|^p].$$

Taking the difference between the equation in Lemma 4.9 and equation (4.2), and summing over the first m timesteps, we obtain

$$\begin{aligned} \bar{X}_{i,m}^f - X_{i,m}^c &= \sum_{l=0}^{m-1} (f_i(\bar{X}_{i,l}^f) - f_i(X_{i,l}^c)) \Delta t \\ &\quad + \sum_{l=0}^{m-1} \sum_{j=1}^D (g_{ij}(\bar{X}_{i,l}^f) - g_{ij}(X_{i,l}^c)) \Delta w_{j,l} \\ &\quad + \sum_{l=0}^{m-1} \sum_{j,k=1}^D (h_{ijk}(\bar{X}_{i,l}^f) - h_{ijk}(X_{i,l}^c)) (\Delta w_{j,l} \Delta w_{k,l} - \Omega_{jk} \Delta t) \\ &\quad + \sum_{l=0}^{m-1} M_{i,l} + \sum_{l=0}^{m-1} N_{i,l} \end{aligned}$$

and using inequality (2.4) again gives

$$\begin{aligned} &\mathbb{E}[\max_{m \leq n} |\bar{X}_{i,m}^f - X_{i,m}^c|^p] \\ &\leq 5^{p-1} \left(\mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} (f_i(\bar{X}_{i,l}^f) - f_i(X_{i,l}^c)) \Delta t \right|^p \right] \right. \\ (A.2) \quad &\quad \left. + \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} \sum_{j=1}^D (g_{ij}(\bar{X}_{i,l}^f) - g_{ij}(X_{i,l}^c)) \Delta w_{j,l} \right|^p \right] \right) \end{aligned}$$

$$\begin{aligned}
 &+ \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} \sum_{j,k=1}^D (h_{ijk}(\bar{X}_{i,l}^f) - h_{ijk}(X_{i,l}^c)) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \times (\Delta w_{j,l} \Delta w_{k,l} - \Omega_{jk} \Delta t) \right|^p \right] \\
 &+ \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} M_{i,l} \right|^p \right] + \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} N_{i,l} \right|^p \right].
 \end{aligned}$$

We now need to bound each of the five expectations on the right-hand side of (A.2). The last is the easiest, since

$$\left| \sum_{l=0}^{m-1} N_{i,l} \right|^p \leq m^{p-1} \sum_{l=0}^{m-1} |N_{i,l}|^p \leq n^{p-1} \sum_{l=0}^{n-1} |N_{i,l}|^p$$

and therefore

$$\mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} N_{i,l} \right|^p \right] \leq n^{p-1} \sum_{l=0}^{n-1} \mathbb{E}[|N_{i,l}|^p] \leq c_1 (n \Delta t)^p \Delta t^p$$

for some constant c_1 (which like other such constants in this proof will depend on p, L and T but not on Δt) due to Lemma 4.9.

Similarly, there exists a constant c_2 such that

$$\begin{aligned}
 \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} (f_i(\bar{X}_{i,l}^f) - f_i(X_{i,l}^c)) \Delta t \right|^p \right] &\leq n^{p-1} \sum_{l=0}^{m-1} \mathbb{E}[|f_i(\bar{X}_{i,l}^f) - f_i(X_{i,l}^c)|^p] \Delta t^p \\
 &\leq c_2 (n \Delta t)^{p-1} \sum_{m=0}^{n-1} S_m \Delta t
 \end{aligned}$$

with the second step being due to the uniform bound on the first derivatives of f .

The other three expectations in (A.2) involve martingales, and so we can use the discrete Burkholder–Davis–Gundy inequality [1]. Starting again with the easiest, there are constants c_3, c_4 such that

$$\begin{aligned}
 \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} M_{i,l} \right|^p \right] &\leq c_3 \mathbb{E} \left[\left(\sum_{m=0}^{n-1} (M_{i,m})^2 \right)^{p/2} \right] \\
 &\leq c_3 n^{p/2-1} \sum_{m=0}^{n-1} \mathbb{E}[|M_{i,m}|^p] \leq c_4 (n \Delta t)^{p/2} \Delta t^p
 \end{aligned}$$

with the final step being due to Lemma 4.9.

Similarly, there exists a constant c_5 such that

$$\begin{aligned} & \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} \sum_{j=1}^D (g_{ij}(\bar{X}_{i,l}^f) - g_{ij}(X_{i,l}^c)) \Delta w_{j,l} \right|^p \right] \\ & \leq c_5 n^{p/2-1} D^{p-1} \sum_{m=0}^{n-1} \sum_{j=1}^D \mathbb{E} [|(g_{ij}(\bar{X}_{i,m}^f) - g_{ij}(X_{i,m}^c)) \Delta w_{j,m}|^p]. \end{aligned}$$

Since $\Delta w_{j,m}$ is independent of both $\bar{X}_{i,m}^f$ and $X_{i,m}^c$, it follows that

$$\mathbb{E} [|(g_{ij}(\bar{X}_{i,m}^f) - g_{ij}(X_{i,m}^c)) \Delta w_{j,m}|^p] = \mathbb{E} [|g_{ij}(\bar{X}_{i,m}^f) - g_{ij}(X_{i,m}^c)|^p] \mathbb{E} [|\Delta w_{j,m}|^p].$$

Hence, because of the uniformly bounded first derivatives of g , and standard results for the moments of Brownian increments, there exists a constant c_6 such that

$$\mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} \sum_{j=1}^D (g_{ij}(\bar{X}_{i,l}^f) - g_{ij}(X_{i,l}^c)) \Delta w_{j,l} \right|^p \right] \leq c_6 (n \Delta t)^{p/2-1} \sum_{m=0}^{n-1} S_m \Delta t.$$

Finally, following the same approach, there exists a constant c_7 such that

$$\begin{aligned} & \mathbb{E} \left[\max_{m \leq n} \left| \sum_{l=0}^{m-1} \sum_{j,k=1}^D (h_{ijk}(\bar{X}_{i,l}^f) - h_{ijk}(X_{i,l}^c)) (\Delta w_{j,l} \Delta w_{k,l} - \Omega_{jk} \Delta t) \right|^p \right] \\ & \leq c_5 (n \Delta t)^{p/2-1} \Delta t^{p/2} \sum_{m=0}^{n-1} S_m \Delta t. \end{aligned}$$

Since $n \Delta t \leq T$ in all of the above inequalities, combining the above bounds for each term in (A.2), and inserting these into (A.1), there then exists a constant c_8 such that

$$S_n \leq c_8 \left(\Delta t^p + \sum_{m=0}^{n-1} S_m \Delta t \right).$$

The desired result is then obtained from a discrete Grönwall inequality.

A.5. Proof of Lemma 4.11. The identity $X_{n+1/2}^f - X_{n+1/2}^a = (X_{n+1/2}^f - X_n^f) + (X_n^f - X_n^a) + (X_n^a - X_{n+1/2}^a)$ gives

$$\begin{aligned} & \|X_{n+1/2}^f - X_{n+1/2}^a\|^p \\ & \leq 3^{p-1} (\|X_{n+1/2}^f - X_n^f\|^p + \|X_n^f - X_n^a\|^p + \|X_n^a - X_{n+1/2}^a\|^p). \end{aligned}$$

It then follows from Lemmas 4.5 and 4.6 that there exists a constant K_p , independent of both Δt and n , for which

$$\mathbb{E} [\|X_{n+1/2}^f - X_{n+1/2}^a\|^p] \leq K_p \Delta t^{p/2}.$$

A.6. Proof of Lemma 4.12. Averaging the discrete equations for $X_{n+1/2}^f$ and $X_{n+1/2}^a$, and using the identities $\delta w_n = \frac{1}{2}\Delta w_n + \frac{1}{2}(\delta w_n - \delta w_{n+1/2})$ and $\delta w_{n+1/2} = \frac{1}{2}\Delta w_n - \frac{1}{2}(\delta w_n - \delta w_{n+1/2})$, gives

$$(A.3) \quad \bar{X}_{i,n+1/2}^f = \bar{X}_{i,n}^f + \frac{1}{2}f_i(\bar{X}_n^f)\Delta t + \frac{1}{2}\sum_{j=1}^D g_{ij}(\bar{X}_n^f)\Delta w_{j,n} + N_{i,n},$$

where

$$\begin{aligned} N_{i,n} = & \frac{1}{2}\left(\frac{1}{2}(f_i(X_n^f) + f_i(X_n^a)) - f_i(\bar{X}_n^f)\right)\Delta t \\ & + \frac{1}{2}\sum_{j=1}^D\left(\frac{1}{2}(g_{ij}(X_n^f) + g_{ij}(X_n^a)) - g_{ij}(\bar{X}_n^f)\right)\Delta w_{j,n} \\ & + \frac{1}{4}\sum_{j=1}^D(g_{ij}(X_n^f) - g_{ij}(X_n^a))(\delta w_{j,n} - \delta w_{j,n+1/2}) \\ & + \frac{1}{2}\sum_{j,k=1}^D\left(h_{ijk}(X_n^f)\left(\delta w_{j,n}\delta w_{k,n} - \frac{1}{2}\Omega_{jk}\Delta t\right)\right. \\ & \quad \left.+ h_{ijk}(X_n^a)\left(\delta w_{j,n+1/2}\delta w_{k,n+1/2} - \frac{1}{2}\Omega_{jk}\Delta t\right)\right). \end{aligned}$$

Following the same method of analysis as in the proof of Lemma 4.7 it can be proved that $\mathbb{E}[|N_{i,n}|^p]$ has an $O(\Delta t^p)$ bound.

Next, defining $X_{n+1/2}^c$ to be the linear interpolant value $\frac{1}{2}(X_n^c + X_{n+1}^c)$, then the equation for X_{n+1}^c yields

$$(A.4) \quad \begin{aligned} X_{i,n+1/2}^c = & X_{i,n}^c + \frac{1}{2}f_i(X_n^c)\Delta t + \frac{1}{2}\sum_{j=1}^d g_{ij}(X_n^c)\Delta w_{j,n} \\ & + \frac{1}{2}\sum_{j,k=1}^d h_{ijk}(X_n^c)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t). \end{aligned}$$

Subtracting (A.4) from (A.3) gives

$$\begin{aligned} \bar{X}_{i,n+1/2}^f - X_{i,n+1/2}^c = & \bar{X}_{i,n}^f - X_{i,n}^c + \frac{1}{2}(f_i(\bar{X}_n^f) - f_i(X_n^c))\Delta t \\ & + \frac{1}{2}\sum_{j=1}^d(g_{ij}(\bar{X}_n^f) - g_{ij}(X_n^c))\Delta w_{j,n} \\ & + N_{i,n} + \frac{1}{2}\sum_{j,k=1}^d h_{ijk}(X_n^c)(\Delta w_{j,n}\Delta w_{k,n} - \Omega_{jk}\Delta t). \end{aligned}$$

Using the bounds on $\mathbb{E}[\|\bar{X}_n^f - X_n^c\|^p]$, the bounded first derivatives of $f(x)$ and $g(x)$, the uniform bound on $\mathbb{E}[|h_{ijk}(X_n^c)|^p]$ and standard results for Brownian increments, we can conclude that there exists a constant K_p , independent of both Δt and n , such that

$$\mathbb{E}[\|\bar{X}_{i,n+1/2}^f - X_{i,n+1/2}^c\|^p] \leq K_p \Delta t^p.$$

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