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Apparent and average accelerations of the Universe

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Abstract. In this paper we consider the relation between the volume deceleration parameter obtained within the Buchert averaging scheme and the deceleration parameter derived from supernova observation. This work was motivated by recent findings that showed that there are models which despite having $\Lambda = 0$ have volume deceleration parameter $q^{\mathrm{vol}} < 0$. This opens the possibility that back-reaction and averaging effects may be used as an interesting alternative explanation to the dark energy phenomenon.

We have calculated q^{vol} in some Lemaître–Tolman models. For those models which are chosen to be realistic and which fit the supernova data, we find that $q^{\mathrm{vol}} > 0$, while those models which we have been able to find which exhibit $q^{\mathrm{vol}} < 0$ turn out to be unrealistic. This indicates that care must be exercised in relating the deceleration parameter to observations.

Keywords: dark energy theory, supernova type Ia, cosmological constant experiments, superclusters and voids

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Contents

1. Introduction

Accelerated expansion, modeled by a positive cosmological constant, is an essential element of the current standard cosmological model of the Universe. The accelerated expansion was originally motivated by supernova observations [1] and is supported by many other types of cosmological observations. Observational data are, in modern cosmology, analyzed almost exclusively within the framework of homogeneous and isotropic Friedmann models [2]. This analysis leads to the concordance model, which provides a remarkably precise fit to cosmological observations. In this situation, if the Ehlers–Geren–Sachs theorem [3] and 'almost EGS theorem' [4] are invoked[5], then it seems that an assumption of large scale homogeneity of the Universe can be justified. This on the other hand implies that the Universe must be filled with dark energy which currently drives the acceleration of the Universe.

However, the concordance model is not the only one which can fit cosmological observations. Anti-Copernican inhomogeneous models which assume the existence of a local Gpc scale void also fit cosmological observations [6] (see [7] for a review). Moreover, on small and medium scales our Universe is not homogeneous. Therefore, one may ask whether Friedmann models can describe our Universe correctly. In particular, it is important to ask what is the best way to fit a homogeneous model to a realistic and inhomogeneous Universe. This problem, known as the fitting problem, was considered by Ellis and Stoeger [8]. In considering the fitting problem, it becomes apparent that a homogeneous model fitted to inhomogeneous data can evolve quite differently from the real Universe. The difference between the evolutions of homogeneous models and an inhomogeneous Universe is caused by back-reaction effects, due to the non-linearity of the Einstein equation. Unfortunately, in the standard approach, the back-reaction is

[5] These theorems imply that if anisotropies in the cosmic microwave background radiation are small for all fundamental observers then the Universe is locally almost spatially homogeneous and isotropic. However, as shown in [5] the almost Robertson–Walker geometry also requires smallness of the Weyl curvature.

rarely taken into account—in most cases when modeling our Universe on a local scale Newtonian mechanics is employed and on large scales the Friedmann equations (or linear perturbations of Friedmann background) are used [9]. Such an approach to cosmology is often encouraged by the 'no-go' theorem which states that the Universe can be very accurately described by the conformal Newtonian metric perturbed about a spatially flat background, even if $\delta\rho/\rho \gg 0$. In such a case the back-reaction is negligible [10, 11]. However, the results obtained by van Elst and Ellis [12] and recently by Kolb, Marra and Matarrese [13] show that the application of 'no-go' theorem is limited. Therefore, one should be aware that in the absence of an analysis of the back-reaction and other effects caused by inhomogeneities in the Universe, there remains the possibility that the observed accelerated expansion of the Universe is only apparent [14]. The direct study of the dynamical effects of inhomogeneities is difficult. Due to the non-linearity of the Einstein equations, the solution of the Einstein equations for the homogeneous matter distribution leads in principle to a different description of the Universe than an average of an inhomogeneous solution to the exact Einstein equations (even though inhomogeneities when averaged over a sufficiently large scale might tend to be zero).

Neither the analysis of the evolution of a general matter distribution nor the numerical evolution of cosmological models employing the full Einstein equations are available at the level of detail which would make them useful for this problem. There are currently several different approaches which attempt to take back-reaction effects into account. One approach is based on exact solutions—see for example [15]. Another and more popular approach is based on averaging.

In the averaging approach to back-reaction, one considers a solution to the Einstein equations for a general matter distribution and then an average of various observable quantities is taken. If a simple volume average is considered then such an attempt leads to the Buchert equations [16]. The Buchert equations are very similar to the Friedmann equations except for the back-reaction term which is in general non-vanishing, if inhomogeneities are present. For a review on back-reaction and the Buchert averaging scheme the reader is referred to [17, 18]. Within this framework and using spherically symmetric inhomogeneous models Nambu and Tanimoto [19], Paranjape and Singh [20], Kai *et al* [21], Chuang *et al* [22], provided explicit examples that one can obtain negative values of the volume deceleration parameter even if $\Lambda = 0$. Another interesting example was presented by Räsänen [17, 23] where it was shown that the total volume deceleration parameter of two isolated and locally decelerating regions can also be negative.

There are however important ambiguities in the application of an averaging procedure. The average itself not only depends on a choice of volume but also on a choice of time slicing. This is very crucial in cosmology. Once inhomogeneities are present the age of the Universe is not everywhere the same. Namely, the big bang in inhomogeneous models is not a single event, so the average taken over a hypersurface of constant cosmic time t is different from the average taken over a hypersurface of constant age of the Universe $t - t_{\mathrm{B}}$ [24]. Moreover, the results of the averaging procedure vary if the discrepancy between the average cosmic time and the local time is introduced (the local time is the time which is measured by local clocks; the cosmic time is the time which appears in the averaged homogeneous model). This phenomenon was studied by Wiltshire [25], and has been used in an ambitious alternative concordance model. The model proposed by Wiltshire introduces some additional assumptions which allow to some extent a comparison of averaged quantities with observations. Such a comparison shows quite good agreement with observations [26]. Thus, while serious fundamental questions remain concerning Wiltshire's approach, it is another example of an approach where one does not need dark energy to fit cosmological observations.

The averaging procedure is also gauge dependent. For example using different gauge one can obtain that the back-reaction mimics not dark energy but dark matter [27]. The averaging schemes in the literature, therefore, have been criticized, and their inherent ambiguities (and in some cases obscurity) have been discussed; cf e.g. [10]. A key point is that it is far from obvious whether the average quantities, such as the acceleration of the averaged Universe, are really the quantities which are measured in astronomical observations. In particular, an operational analysis is to a large extent lacking in the discussions of averaging. Thus, it is important to test the averaging procedures with the exact and inhomogeneous solutions of the Einstein equations. Within exact models each quantity can easily be calculated and then compared with its averaged counterpart. This paper aims to perform such an analysis within the Lemaître–Tolman model.

The structure of this paper is as follows. Buchert's averaging procedure is presented in section 2, and some background on the Lemaître–Tolman model is given in section 3. The volume and distance deceleration parameters are introduced in section 4. Finally, in section 5, we discuss the relation between the deceleration parameters, supernova observations and models of cosmic structures.

2. The Buchert scheme

If the averaging procedure is applied to the Einstein equations, then for irrotational and pressureless matter the following equations are obtained [16]:

$$3\frac{\ddot{a}}{a} = -4\pi G\langle\rho\rangle + \mathcal{Q}, \tag{1}$$

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\langle\rho\rangle - \frac{1}{2}\langle\mathcal{R}\rangle - \frac{1}{2}\mathcal{Q}, \tag{2}$$

$$\mathcal{Q} \equiv \tfrac{2}{3}\left(\langle\Theta^2\rangle - \langle\Theta\rangle^2\right) - 2\langle\sigma^2\rangle, \tag{3}$$

where $\langle\mathcal{R}\rangle$ is an average of the spatial Ricci scalar ${}^{(3)}\mathcal{R}$, Θ is the scalar of expansion, σ is the shear scalar, and $\langle\,\rangle$ is the volume average over the hypersurface of constant time: $\langle A\rangle = (\int \mathrm{d}^3x\,\sqrt{-h})^{-1}\int \mathrm{d}^3x\,\sqrt{-h}A$. The scale factor a is defined as follows:

$$a = (V/V_0)^{1/3}, \tag{4}$$

where V_0 is an initial volume.

Equations (1) and (2) are very similar to the Friedmann equations, where $Q = 0$, and ρ and $\mathcal{R}$ depend on time only. In fact, they are kinematically equivalent to a Friedmann model that has an additional scalar field source [28]. However the Buchert equation does not form a closed system. To close these equations one has to introduce some further assumptions [16]. As can be seen from (3), if the dispersion of expansion is large, Q can be large as well and one can get acceleration ($\ddot{a} > 0$) without employing the cosmological constant.

3. The Lemaître–Tolman model

The Lemaître–Tolman model[6] [30] is a spherically symmetric, pressure free and irrotational solution of the Einstein equations. Its metric is of the following form:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{R'^2(r,t)}{1+2E(r)}\,\mathrm{d}r^2 - R^2(t,r)\,\mathrm{d}\Omega^2, \tag{5}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. Because of the signature $(+,-,-,-)$, the $E(r)$ function must obey $E(r) \geq -1/2$. The prime $'$ denotes ∂_r.

The Einstein equations reduce, in the $\Lambda = 0$ case, to the following two:

$$\kappa\rho(r,t)c^2 = \frac{2M'(r)}{R^2(r,t)R'(r,t)}, \tag{6}$$

$$\frac{1}{c^2}\dot{R}^2(r,t) = 2E(r) + \frac{2M(r)}{R(r,t)}, \tag{7}$$

where $M(r)$ is another arbitrary function and $\kappa = 8\pi G/c^4$. The dot $\dot{}$ denotes ∂_t.

When $R' = 0$ and $M' \neq 0$, the density becomes infinite. This happens at shell crossings. This is a singularity additional to the big bang that occurs at $R = 0, M' \neq 0$. By setting the initial conditions appropriately the shell crossing singularity can be avoided (see [31] for detailed discussion).

Equation (7) can be solved by simple integration:

$$\int_0^R \frac{\mathrm{d}\tilde{R}}{\sqrt{2E + (2M/\tilde{R})}} = c\left[t - t_{\mathrm{B}}(r)\right], \tag{8}$$

where t_{B} appears as an integration constant and is an arbitrary function of r. This means that the big bang is not a single event as in the Friedmann models, but occurs at different times at different distances from the origin.

The scalar of the expansion is equal to

$$\Theta = \frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}. \tag{9}$$

The shear tensor is of the following form:

$$\sigma^{\alpha}{}_{\beta} = \frac{1}{3}\left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}\right)\mathrm{diag}(0,2,-1,-1), \tag{10}$$

and thus $\sigma^2 \equiv (1/2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} = (1/3)(\dot{R}'/R' - \dot{R}/R)^2$.

The spatial Ricci scalar in the Lemaître–Tolman is equal to

$${}^{(3)}\mathcal{R} = -\frac{4}{R^2}\left(E + \frac{E'R}{R'}\right). \tag{11}$$

[6] The pressure free and irrotational solution of the Einstein equations for spherically symmetric space–time is often called the Tolman, Tolman–Bondi, or Lemaître–Tolman–Bondi model. However, it is more justified to refer to this solution as the Lemaître–Tolman model (cf [29]).

4. The apparent and average accelerations

The deceleration parameter within the Friedmann models is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}, \tag{12}$$

where a is the scale factor. By analogy we can define the deceleration parameter which is based on the averaging scheme. Substituting (4) into (12) and using (1) and (2) we get

$$q^{\rm vol} = -\frac{-4\pi G\langle\rho\rangle + \mathcal{Q}}{8\pi G\langle\rho\rangle - 1/2\langle\mathcal{R}\rangle - 1/2\mathcal{Q}}. \tag{13}$$

We refer to this deceleration parameter as the *volume deceleration parameter*, $q^{\rm vol}$, since it is positive when the second derivative of volume is negative and negative when the second derivative of volume is positive (and of sufficiently large value).

On the other hand one can introduce a deceleration parameter defined relative to the distance. Within homogeneous models the distance to a given redshift is larger for accelerating models than for decelerating ones. Taylor expanding the luminosity distance in the Friedmann model we obtain

$$\begin{aligned} D_{\rm L} &= \left.\frac{{\rm d}D_{\rm L}}{{\rm d}z}\right|_{z=0} z + \frac{1}{2}\left.\frac{{\rm d}^2 D_{\rm L}}{{\rm d}z^2}\right|_{z=0} z^2 + \mathcal{O}(z^3) \\ &= \frac{c}{H_0}z + \frac{c}{2H_0}(1-q)z^2 + \mathcal{O}(z^3). \end{aligned} \tag{14}$$

Employing a similar procedure in the case of the Lemaître–Tolman model we get

$$D_{\rm L} = \frac{cR'}{\dot{R}'}z + \frac{c}{2}\frac{R'}{\dot{R}'}\left(1 + \frac{R'\ddot{R}'}{\dot{R}'^2} + \frac{cR''}{R'\dot{R}'} - \frac{c\dot{R}''}{\dot{R}'^2}\right)z^2 + \mathcal{O}(z^3). \tag{15}$$

Thus by comparing (15) with (14), the Hubble and the deceleration parameters in the Lemaître–Tolman model can be defined as

$$H_0^{\rm dis} = \frac{\dot{R}'}{R'}, \qquad q_0^{\rm dis} = -\frac{R'\ddot{R}'}{\dot{R}'^2} - \frac{cR''}{R'\dot{R}'} + \frac{c\dot{R}''}{\dot{R}'^2}. \tag{16}$$

The above quantities are defined at the origin ($r = 0$). However, following Partovi and Mashhoon [32] we can extend the above quantities to any r. Then, the coefficients of Taylor expansion are

$$\begin{aligned} \frac{{\rm d}D_{\rm L}}{{\rm d}z} &= 2R + \dot{R}\frac{{\rm d}t}{{\rm d}z} + R'\frac{{\rm d}r}{{\rm d}z} \\ \frac{{\rm d}^2 D_{\rm L}}{{\rm d}z^2} &= 2R + 4\dot{R}\frac{{\rm d}t}{{\rm d}z} + 4R'\frac{{\rm d}r}{{\rm d}z} + \ddot{R}\left(\frac{{\rm d}t}{{\rm d}z}\right)^2 + 2\dot{R}'\frac{{\rm d}t}{{\rm d}z}\frac{{\rm d}r}{{\rm d}z} + R''\left(\frac{{\rm d}r}{{\rm d}z}\right)^2 + \dot{R}\frac{{\rm d}^2 t}{{\rm d}z^2} + R'\frac{{\rm d}^2 r}{{\rm d}z^2}, \end{aligned} \tag{17}$$

and we obtain

$$H^{\rm dis} = \left(\frac{{\rm d}D_{\rm L}}{{\rm d}z}\right)^{-1}, \qquad q^{\rm dis} = 1 - H^{\rm dis}\frac{{\rm d}^2 D_{\rm L}}{{\rm d}z^2}. \tag{18}$$

We refer to this deceleration parameter as the *distance deceleration parameter*. Although of physical importance is the luminosity distance and its ability of fitting the supernova

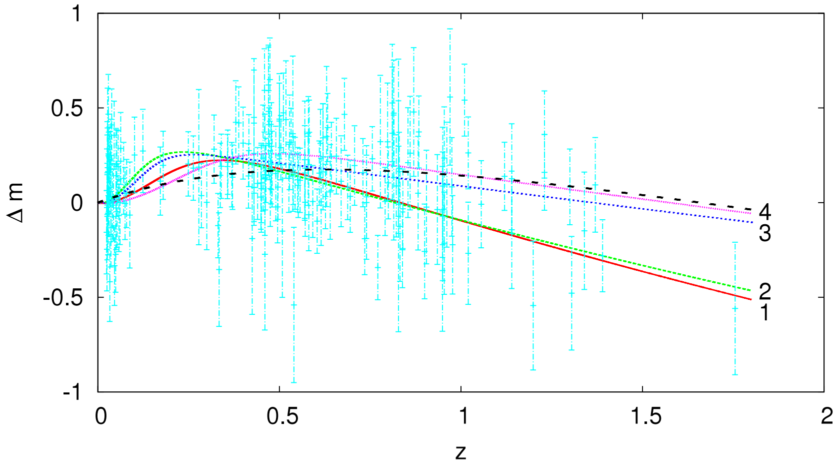

Figure 1. The residual Hubble diagram for models 1–4. The black dashed line presents Δm for the ΛCDM model.

data, $q^{\rm dis}$ is of great usefulness. It allows us, without solving the geodesic equations, to easily check whether a model being considered can be used to fit supernova data. As we will see in the next section, models which fit supernova data have at least in some regions $q^{\rm dis} < 0$.

5. Connection between the deceleration parameter and observations

Let us first focus on supernova observations. There is already a considerable literature on inhomogeneous models which are able to fit the supernova observations without the cosmological constant [6]. We shall examine four such models in this section. For each of these models we shall calculate the volume and distance deceleration parameters and compare them with each other. The four models to be considered present a very good fit to supernova data. The supernova data consists of 182 supernovae from the Riess gold sample [33]. The χ^2 test values for models 1–4 are respectively 183.6, 184.3, 164.7, and 178.5 (for comparison the χ^2 of fitting the ΛCDM model is 165.3). The residual Hubble diagram for these models is presented in figure 1. The deceleration parameters for models 1–4 are presented in figure 2. The left panel presents the distance deceleration parameter (as defined by (18)—where $\mathrm{d}t/\mathrm{d}z$ and $\mathrm{d}r/\mathrm{d}z$ were calculated for the radial geodesic). The distance deceleration parameter is positive at the origin, but soon becomes negative. Moreover, a very similar shape is obtained if instead of $q^{\rm dis}$ (as defined by (18)), $q_0^{\rm dis}$ (as defined by (16)) is used. Thus, $q^{\rm dis}$ (or even $q_0^{\rm dis}$, if treated as a function of r) can be regarded as a useful test for checking whether a given model is able to fit supernova data. However, the most significant feature is that the volume deceleration parameter, which is presented in the right panel of figure 2, is strictly positive. Thus, the ability of reproducing the supernova data does not require that the volume deceleration parameter is negative. This raises the question of whether the average acceleration has any relation to the observed acceleration of the Universe; and if yes, are models with average acceleration also able to fit supernova data?

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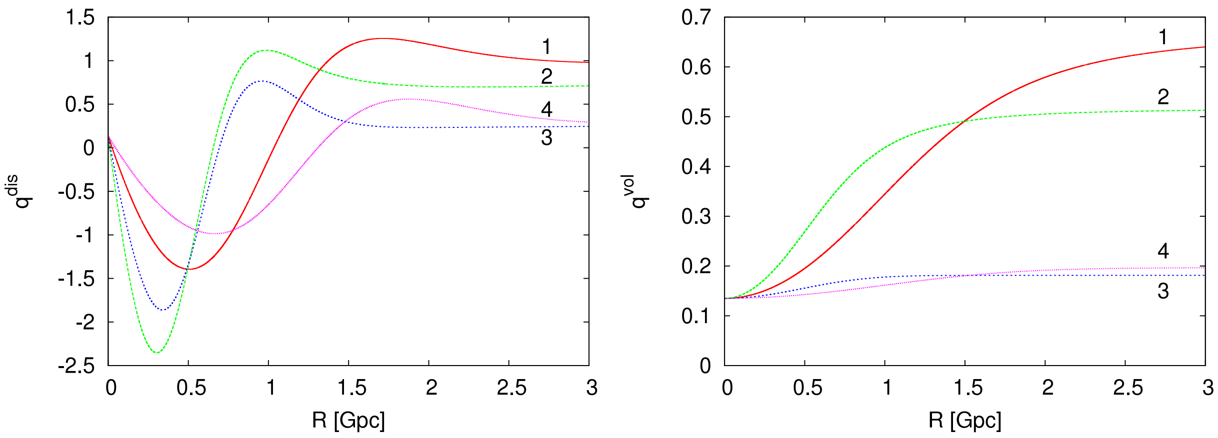

Figure 2. The distance deceleration parameter (left panel) and the volume deceleration parameter (right panel) for models 1–4.

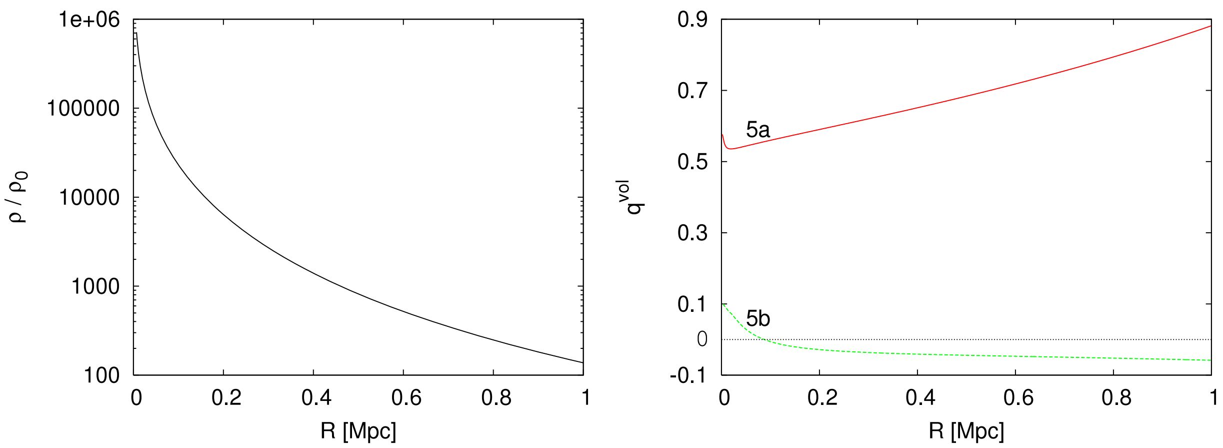

Figure 3. The current density distribution (left panel) and deceleration parameter (right panel) for model 5.

Let us now focus on models of cosmic structures. It was recently shown that using a perturbative approach, back-reaction cannot explain the apparent acceleration [34]. However, because of the large density fluctuations within cosmic structures, results obtained in terms of the perturbation framework might be questionable. Moreover, in view of the fact that there are known examples of exact inhomogeneous models with negative volume deceleration parameter and $\Lambda = 0$, it is worthwhile to check whether realistically evolving models of cosmic structures can have negative values of the deceleration parameter. First, let us consider a model of galaxy clusters with the Navarro–Frenk–White density distribution [35] (left panel of figure 3). Although the NFW profile describes virialized systems[7], the use of this profile will prove to be very instructive. The average deceleration parameter $q^{\rm vol}$ for model 5 is presented in the right panel of figure 3.

[7] The Lemaître–Tolman model which evolves from a smooth density profile at last scattering to a high value profile like the NFW profile is always characterized by a collapse central region; within this models are at the current instant collapsing. Thus such systems cannot be considered as virialized systems.

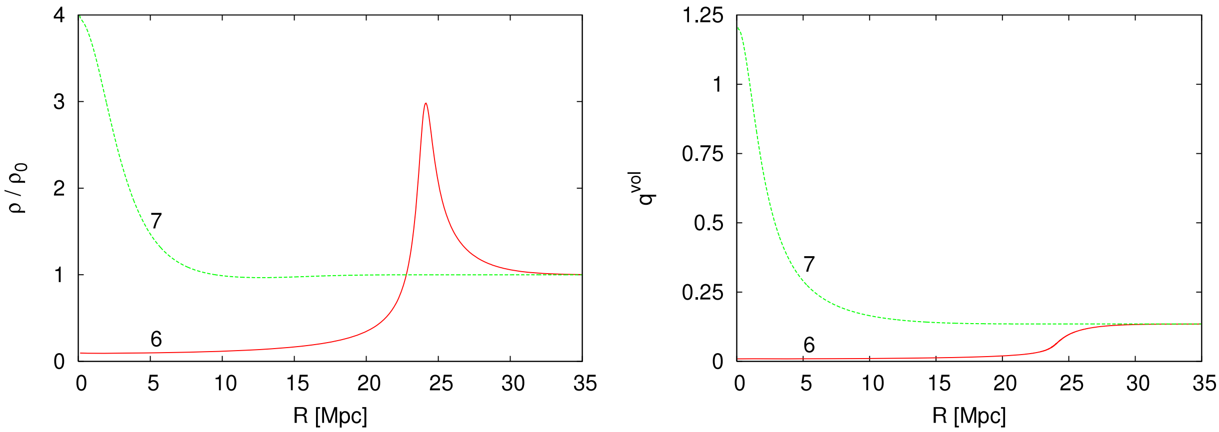

Figure 4. The current density distribution (left panel) and volume deceleration parameter (right panel) for models of cosmic structures (models 6, 7).

As can be seen, in this case the deceleration parameter is positive (curve 5a). However, it is possible to modify this model so that $q^{\rm vol}$ becomes negative—curve 5b in the right panel of figure 3. This was obtained by choosing the E function which is of large positive value (for details see the appendix). However, after such a modification this model becomes unrealistic. Specifically, the age of the Universe in this model becomes unrealistically small. The bang time function $t_{\rm B}$ in this model is of large amplitude, around 11.44×10^9 y. This means that the actual age of the Universe in this model is approximately a few hundred thousand years.

Now let us examine the volume deceleration parameter within models of cosmic voids and superclusters. Figure 4 presents density distribution of realistically evolving cosmic structures (void—curve 6, supercluster—curve 7). It can be seen from the right panel of figure 4 that the volume deceleration parameter within these models is positive. As above, we can modify our models in such a way that the volume deceleration parameter is negative, but again this leads to a very large amplitude of $t_{\rm B}$. For example, in model 8, whose density and the volume deceleration parameter are presented in figure 5, the[8] volume deceleration parameter is negative. However, the bang time function in model 8 is of amplitude $\approx 11 \times 10^9$ y, which leads to an unrealistically small age of the Universe.

6. Conclusions

In this paper we have studied the relation between the volume deceleration parameter obtained within the Buchert averaging scheme and the deceleration parameter derived from the observations of supernovae. This work was motivated by recent results showing that there are models for which, despite Λ being zero, the average expansion rate is accelerating, i.e. $\ddot{a} > 0$ (where a is defined by relation (4)). This opens the possibility that back-reaction and averaging effects may be used as an interesting alternative explanation to the dark energy phenomenon.

[8] Employing a model of qualitatively similar features to model 8, Hossain [36] showed that the observer situated at the origin in order to successfully employ the Friedmann model has to assume the existence of dark energy.

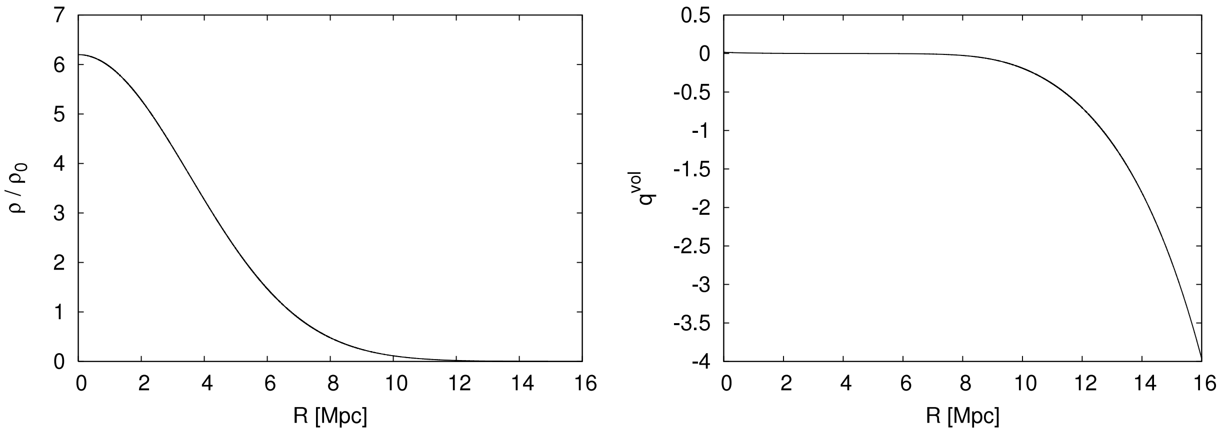

Figure 5. The current density distribution (left panel) and deceleration parameter (right panel) for model 8.

We have compared the quantities obtained within the exact and inhomogeneous models with their average counterparts. We focused on the supernova observations and models of cosmic structures. For this purpose the Lemaître–Tolman model was employed. It was shown numerically that the averaging of models which fit the supernova observations does not lead to volume acceleration ($\ddot{a} < 0$ for these averaged models and hence $q^{\rm vol} > 0$). It was also shown that realistically evolving models of cosmic structures have also $q^{\rm vol} > 0$. It was possible to modify these models in such a way that after the averaging, $q^{\rm vol} < 0$. This was obtained by choosing the E function of positive amplitude—as was recently proved by Sussman [37] this is a necessary condition for obtaining $q^{\rm vol} < 0$. However, in models with realistic density distributions, in such case, $E \gg 1 \gg M/R \approx 10^{-7}$–$10^{-6}$; hence, as seen from (8), $t_{\rm B} \approx t$ (recall that $c \times 10^{10} y \approx 3$ Gpc). Thus, within such models, the age of the Universe is unrealistically small.

Our analysis has been performed for the limited class of Lemaître–Tolman models, which due to their spherical symmetry are arguably too simple to give a full understanding of averaging and back-reaction problems. However, we conclude that, within this class, the volume deceleration parameter $q^{\rm vol}$ is not a quantity which can be directly related to observations.

It is possible that the volume deceleration parameter $q^{\rm vol}$ becomes negative only after averaging over scales which are larger than 100 Mpc. On such large scales the structure of the Universe becomes too complicated to be fully described by spherically symmetric models. However, it is intriguing that models which fit the supernova observations and for which the distance deceleration parameter, $q^{\rm dis}$, is negative still have $q^{\rm vol} > 0$. This suggests that the volume deceleration $q^{\rm vol}$ does not have a clear interpretation in terms of observable quantities. It does not, of course, mean that averaging and back-reaction effects cannot potentially be employed to explain the phenomenon of dark energy. However, our work here indicates that such a potential solution of the dark energy problem should be based upon methods different to those related to the volume deceleration parameter. Rather than showing that $q^{\rm vol} < 0$, the averaging approach should explain observations—reproduce correct values of distances to supernovae, the correct shape of the CMB power spectrum, etc. An interesting, quasi-Friedmannian approach was recently suggested in [38]. In this approach, back-reaction is modeled in terms of the morphon field [28]. In such a case a Universe is described by a homogeneous model with the spatial curvature being just a function of time. As shown in [38], such an approach leads to agreement with supernova and CMB data without the need for dark energy, but requires $q^{\rm vol} < 0$.

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Appendix. Model specification

There are three arbitrary functions of the radial coordinate in the Lemaître–Tolman case. However only two functions are independent and the third one is specified by the choice of the radial coordinate. The models considered in this paper are defined as follows:

(i) Models 1 and 2
The radial coordinate is chosen as the present day value of the areal distance $r := R_0$. Models 1 and 2 are specified by the present day density distribution and the bang time function. The density distribution is parameterized by

$$\rho(t_0, r) = \rho_{\rm b}\left[1 + \delta_\rho - \delta_\rho \exp\left(-\frac{r^2}{\sigma^2}\right)\right], \tag{A.1}$$

where $\rho_{\rm b} = \Omega_m \times (3H_0^2)/(8\pi G)$, $\Omega_m = 0.27$, $H_0 = 70$ km s^{-1} Mpc^{-1}. In model 1 $\rho_\delta = 1.9$, $\sigma = 0.9$ Mpc, and in model 2 $\rho_\delta = 1.5$, $\sigma = 0.5$ Mpc. In these models the big bang is assumed to occur simultaneously at every point, i.e. $t_{\rm B} = 0$. The functions M and E are then calculated using equations (6) and (8) respectively. The background density $\rho_{\rm b}$ in all models (1–8) is chosen as the density of a Friedmann model ($\Omega_m = 0.27$, $H_0 = 70$ km s^{-1}) and the time instants are calculated using the following formula [9]:

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{{\rm d}\tilde{z}}{(1+\tilde{z})\sqrt{\Omega_{\rm mat}(1+\tilde{z})^3 + \Omega_K(1+\tilde{z})^2}}, \tag{A.2}$$

where $\Omega_K = 1 - \Omega_m$. The last scattering instant ($t_{\rm LS}$) is set to take place when $z = 1089$ and the current instant (t_0) when $z = 0$—$t_{\rm LS} = 4.98 \times 10^5$ y, and $t_0 = 11.4421 \times 10^9$ y.

(ii) Models 3 and 4
As above, the radial coordinate is chosen as the present day value of the areal distance $r := R_0$. These two models are defined by the current expansion rate, and the assumption that $\rho(t_0, r) = \rho_{\rm b}$. The expansion rate is parametrized using

$$H_{\rm T}(t_0, r) = \frac{\dot{R}}{R} = H_0\left[1 - \delta_H + \delta_H \exp\left(-\frac{r^2}{\sigma^2}\right)\right], \tag{A.3}$$

where $H_0\delta_H = 9.6$ km s^{-1} Mpc^{-1}, $\sigma = 0.6$ Mpc, and $H_0\delta_H = 12$ km s^{-1} Mpc^{-1}, $\sigma = 1.2$ Mpc for models 3 and 4 respectively. In these models, density is assumed to be homogeneous at the current epoch. The function M is then calculated using the above relation and equation (7). It should be noted that the $H_{\rm T}$ is one of several generalizations of the Hubble constant; in the Friedmann model it is $H_0 = \dot{a}/a$. Besides the transverse Hubble parameter, $H_{\rm T}$, one can also define the radial Hubble parameter, $H_{\rm R}$ (see equation (16)), and the volume Hubble parameter defined as $H_V = (1/3)\Theta = H_{\rm R} + 2H_{\rm T}$.

(iii) Model 5a
The radial coordinate is chosen as the present day value of the areal distance, i.e. $r := R_0$. The model is defined by density distributions given at the present instant and at last scattering. The density distribution at the current instant is parametrized by

$$\rho(t_0, r) = \rho_{\rm b} \frac{\delta}{(r/r_{\rm s})(1 + r/r_{\rm s})^2}, \tag{A.4}$$

where $\delta = 28170$ and $r_{\rm s} = 191$ kpc. This is a Navarro, Frenk, and White galaxy cluster profile [35]. As can be seen, this profile is singular at the origin but this problem can be overcome by matching the NFW profile with a singularity free profile, as $f(r) = -ar^2 + b$.
The density profile at last scattering is assumed to be homogeneous; thus the areal distance at last scattering is

$$R(t_{\rm LS}, r) = \left(\frac{M}{\kappa \rho_{\rm LS} c^2}\right)^{1/3}. \tag{A.5}$$

The function $M(r)$ is then calculated from equation (6). Function E can be calculated by subtracting solutions of (8) for $t_{\rm LS}$ and t_0 (for details see [39]). The function E is presented in the left panel of figure A.1.

(iv) Model 5b
The radial coordinate is chosen as the present day value of the areal distance, $r := R_0$. The model is defined by density distribution given by (A.4) and E of the following form:

$$E(r) = 10^3 \sin\left(10^{-3} r \text{ Mpc}^{-1}\right). \tag{A.6}$$

This profile is presented in the left panel of figure A.1 and the bang time function $t_{\rm B}$ in the right panel.

(v) Models 6 and 7
The radial coordinate is chosen as the value of the areal distance at last scattering instant, $r := R_{\rm LS}$. Models 6 and 7 are defined by the assumption that $t_{\rm B} = 0$ and the density distribution, which at last scattering is of the following form:

$$\rho(t_{\rm LS}, r) = \rho_{\rm b}\left(1 - \delta \exp\left(-a\ell^2 r^2\right) + \gamma \exp\left[-\left(\frac{\ell r - c}{d}\right)^2\right]\right), \tag{A.7}$$

where $\ell = 1$ kpc^{-1}; $\delta = 1.2 \times 10^{-3}$ and 2×10^{-3} for models 6 and 7 respectively; $\gamma = 14.62 \times 10^{-4}$ and 8.03×10^{-4} for models 6 and 7 respectively; $a = 0.01$ and 0.04

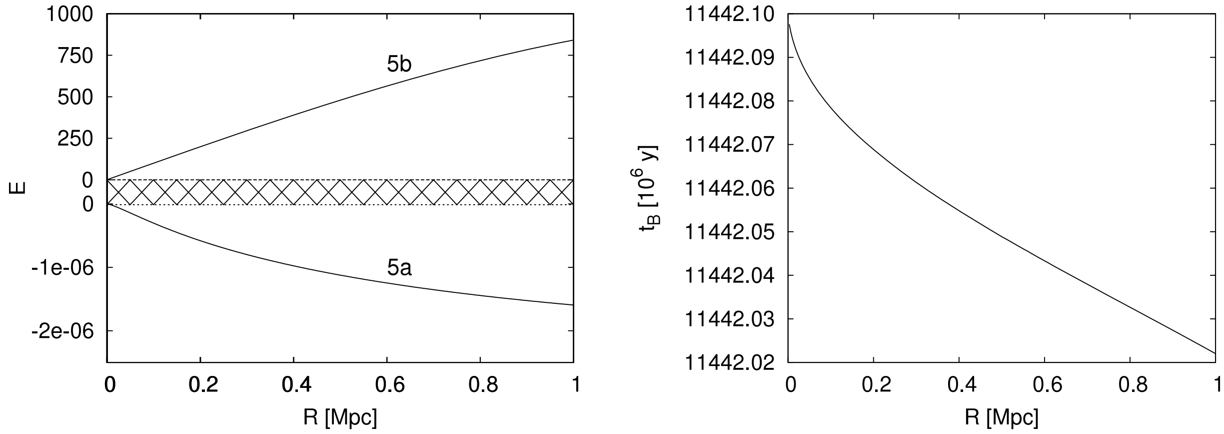

Figure A.1. The left panel presents the function $E(r)$ for models 5a and 5b. Please note that the y-scale in the upper part of the left panel is different than in the lower part. Right panel: bang time function for model 5b.

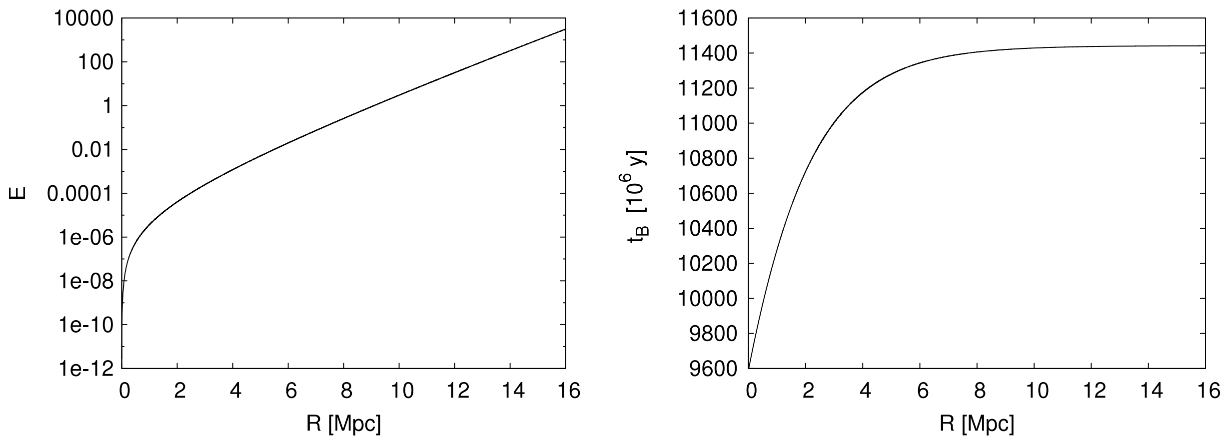

Figure A.2. The functions $E(r)$ (left panel) and $t_{\rm B}(r)$ (right panel) for model 8.

for models 6 and 7 respectively; $c = 18$ and 12 for models 6 and 7 respectively; and $d = 6$ and 5 for models 6 and 7 respectively. The bang time function for both of these models is $t_{\rm B} = 0$. The mass function $M(r)$ is calculated from equation (6), and the function $E(r)$ is calculated from equation (8).

(vi) Model 8

The radial coordinate is chosen as a present day value of the areal distance: $r := R_0$. The density distribution is of the following form:

$$\rho(t_0, r) = 6.2\rho_{\rm b} \exp\left(-4 \times 10^{-8}(\ell r)^2\right), \tag{A.8}$$

and the function E is

$$E(r) = \left(\frac{H_0}{c} r\right)^2 \exp\left(10^{-3}\ell r\right), \tag{A.9}$$

which except for $[\exp(10^{-3}\ell r)]$ is the same as the $E(r)$ profile in the empty Universe. This profile is presented in the left panel of figure A.2 and the bang time function t_B in the right panel.

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