APPLICABILITY AND APPLICATIONS OF THE METHOD OF FUNDAMENTAL SOLUTIONS

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ABSTRACT. In the present work, we investigate the applicability of the method fundamental solutions for solution of boundary value problems for elliptic partial differential equations and elliptic systems. More specifically, we study whether linear combinations of can approximate the solutions of the boundary value problems under consideration. In our study, the singularities of the fundamental solutions lie on a prescribed *pseudo–boundary* – the boundary of a domain which *embraces* the domain of the problem under consideration. We extend previous density results of Kupradze and Aleksidze, and of Bogomolny, to more general domains and partial differential operators, and with respect to more appropriate norms. Our domains may possess holes and their boundaries are only required to satisfy the *Segment Condition*. Our density results are with respect to the norms of the spaces $C^\ell(\overline{\Omega})$ which correspond to the classical and weak formulations of the corresponding boundary value problems. We have studied approximation by fundamental solutions of the Laplacian, $m-$harmonic and modified Helmholtz operators. In the case of elliptic systems, we obtain analogous density resutls for the Cauchy–Navier operator as well as for an operator which arises in the linear theory of thermo–elasticity. We also study alternative formulations of the method of fundamental solutions in cases when linear combinations of fundamental solutions of the equations under consideration are not dense in the solution space.

1. INTRODUCTION

Let $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha(\boldsymbol{x})\, D^\alpha$ be an elliptic partial differential operator in $\Omega \subset \mathbb{R}^n$ of order m. In Trefftz methods, the solution of the boundary value problem

$$\mathcal{L}u = 0 \quad \text{in} \quad \Omega, \tag{1.1a}$$

$$\mathcal{B}u = f \quad \text{on} \quad \partial\Omega, \tag{1.1b}$$

where Ω is an open domain in $\mathbb{R}^n$ and $\mathcal{B}u = f$ is the boundary condition[1], is approximated by linear combinations of particular solutions of (1.1a), provided that such linear combinations are dense in the set of all solutions of this equation. Erich Trefftz presented this approach in 1926 [Tre26] as a counterpart of Ritz's method. In his celebrated work [Mer52], Mergelyan showed in 1952 that holomorphic functions in bounded simply connected domains in $\mathbb{C}$ can be approximated, in the sense of the uniform norm, by polynomials, whereas in the case of multi–connected domains they can be approximated by rational functions. Mergelyan's work is a culmination of a long series of works by Runge, J. L. Walsh, Lavrent'ev and Keldysh on approximations by polynomials and rational functions. As early as 1885, Runge [Run85] proved that holomorphic functions in an open domain U of the complex plane can be approximated, uniformly in compact subsets of U, by rational functions. Peter D. Lax in 1956 [Lax56]

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[1]Throughout this work, all vectors shall be denoted in bold print.

extended Runge's Theorem to solutions of elliptic systems with real analytic coefficients. Similar extensions where obtained independently by Malgrange [Mal56].

In the *method of fundamental solutions* (MFS), the particular solutions of the partial differential equation under consideration are the fundamental solutions $\varphi(\boldsymbol{x},\boldsymbol{y})$ of the corresponding partial differential operator. They satisfy

$$\mathcal{L}_{\boldsymbol{x}}\,\varphi(\cdot,\boldsymbol{y}) \;=\; \delta_{\boldsymbol{y}} \quad \text{for every} \quad \boldsymbol{y}\in\mathbb{R}^n, \tag{1.2}$$

where the notation $\mathcal{L}_{\boldsymbol{x}}\varphi$ signifies that φ is differentiated with respect to $\boldsymbol{x}$ and $\delta_{\boldsymbol{y}}$ is the Dirac measure with unit mass at $\boldsymbol{y}$, in the sense of distributions, i.e.,

$$\int_{\mathbb{R}^n}\varphi(\boldsymbol{x},\boldsymbol{y})\,\mathcal{L}^{\star}\psi(\boldsymbol{x})\,d\boldsymbol{x} \;=\; \psi(\boldsymbol{y}),$$

for every $\psi\in C_0^\infty(\mathbb{R}^n)$, where $\mathcal{L}^\star u = \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha(a_\alpha u)$. The operator $\mathcal{L}^\star$ is known as the *adjoint* of $\mathcal{L}$. In particular, if $\mathcal{L}$ is elliptic with constant coefficients, then $\varphi(\cdot,\boldsymbol{y})$ is real analytic in $\mathbb{R}^n\smallsetminus\{\boldsymbol{y}\}$ and satisfies, in the classical sense, $\mathcal{L}_{\boldsymbol{x}}\,\varphi(\boldsymbol{x},\boldsymbol{y})=0$ for every $\boldsymbol{x}\in\mathbb{R}^n\smallsetminus\{\boldsymbol{y}\}$. (See Rudin [Rud73].) The point $\boldsymbol{y}$ is known as the *singularity* of φ. In the MFS, the singularities of the fundamental solutions lie outside the domain Ω. The fundamental solutions were first introduced by Laurent Schwartz [Sch51]. Perhaps the most important property of the fundamental solutions is that they produce solutions of the corresponding inhomogeneous equation by convolution:

If $u(\boldsymbol{x})=\int_{\mathbb{R}^n}\varphi(\boldsymbol{x},\boldsymbol{y})\,f(\boldsymbol{y})\,d\boldsymbol{y}$, *then* $\mathcal{L}u=f$ *in the sense of distributions, i.e.,*

$$\int_{\mathbb{R}^n}u\,\mathcal{L}^\star\psi\,d\boldsymbol{x} \;=\; \int_{\mathbb{R}^n}f\,\psi\,d\boldsymbol{x} \quad \text{for every} \quad \psi\in C_0^\infty(\mathbb{R}^n).$$

Malgrange [Mal56] and Ehrenpreis [Ehr56] independently established in 1955–56 the existence of fundamental solutions for partial differential operators with constant coefficients[2]. In particular, Malgrange proved the existence of bi–regular functions $\varphi(\boldsymbol{x},\boldsymbol{y})$ for which

$$\mathcal{L}_{\boldsymbol{x}}\,\varphi(\cdot,\boldsymbol{y}) \;=\; \delta_{\boldsymbol{y}} \quad\text{and}\quad \mathcal{L}^\star_{\boldsymbol{y}}\,\varphi(\boldsymbol{x},\cdot) \;=\; \delta_{\boldsymbol{x}}, \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n,$$

in the sense of distributions. Also, if $\mathcal{L}$ is self–adjoint (i.e., $\mathcal{L}=\mathcal{L}^\star$) with constant coefficients, then there exists a fundamental solution of the form $\varphi(\boldsymbol{x},\boldsymbol{y})=e(\boldsymbol{x}-\boldsymbol{y})$.

Felix Browder showed in 1962 that linear combinations of fundamental solutions of elliptic operators with singularities in an arbitrary open set outside a connected domain $\overline{\Omega}$ without holes are dense, in the sense of the uniform norm, in the space

$$\mathcal{X} \;=\; \big\{u\in C^m(\Omega)\;:\;\mathcal{L}u=0 \text{ in } \Omega\,\big\}\cap C(\overline{\Omega}).$$

Browder's proof relies on a duality argument[3]. (See [Bro62, Theorem 3].) Browder's result extends to a partial differential operator $\mathcal{L}$, with the property that its adjoint $\mathcal{L}^\star$ satisfies the *Condition of uniqueness for the Cauchy problem in the small* in Ω:

$(\mathrm{U})_s$. *If* $u\in C^m(V)$, *where* V *is an open connected subset of* Ω *with* $\mathcal{L}^\star u=0$ *and if* u *vanishes in a nonempty open subset of* V, *then* u *vanishes everywhere in* V.

[2]A simple proof can be found in [Rud73, Theorem 8.15].

[3]It is noteworthy that Runge's theorem can also be shown by a duality argument, an application of the Hahn–Banach theorem. A simple such proof can be found in [Lax02, p. 91].

Elliptic operators with real analytic coefficients satisfy the uniqueness condition $(\mathrm{U})_s$ since the solutions of the corresponding homogeneous equations are real analytic functions. (See [Hör83, Corollary 4.4.4].)

Weinstock [Wei73] extended Browder's result to approximations with respect to the norm of $C^\ell(\overline{\Omega})$, when $0 \leq \ell < m$, where m is the order of $\mathcal{L}$. In Weinstock's work, the domain Ω is required to satisfy a weaker condition, the *Segment Condition*, and $\mathcal{L}$ is assumed to be an elliptic operator with constant coefficients. A detailed survey on the extensions of Browder's work and approximations of solutions of elliptic equations, by solutions of the same equations in larger domains can be found in [Tar95].

The MFS was introduced by Kupradze and Aleksidze [KA63] in 1963 as the *method of generalized Fourier series* (метод обобщённых рядов Фурье). In a typical application to the Dirichlet problem for Laplace's equation,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

the function $\varphi(\boldsymbol{x},\boldsymbol{y}) = e_1(\boldsymbol{x}-\boldsymbol{y})$, where

$$e_1(\boldsymbol{x}) = \begin{cases} -\dfrac{\log|\boldsymbol{x}|}{2\pi}, & \text{if } n=2, \\[2ex] -\dfrac{|\boldsymbol{x}|^{2-n}}{(2-n)\,\omega_{n-1}}, & \text{if } n>2, \end{cases} \tag{1.4}$$

is a fundamental solution of the Laplacian (more precisely of $-\Delta$ which is an elliptic operator according to the definition[4]), where ω_{n-1} is the area of the surface of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Clearly, $\varphi(\boldsymbol{x},\boldsymbol{y})$ is a real analytic function of $\boldsymbol{x}$ and satisfies Laplace's equation $\Delta_{\boldsymbol{x}}\varphi = 0$ for every $\boldsymbol{x} \neq \boldsymbol{y}$. In the MFS, the solution of (1.3) is approximated by a finite linear combination of the form

$$u_N(\boldsymbol{x};\boldsymbol{c}) = \sum_{j=1}^{N} c_j\,\varphi(\boldsymbol{x},\boldsymbol{y}_j) = \sum_{j=1}^{N} c_j\,e_1(\boldsymbol{x}-\boldsymbol{y}_j), \tag{1.5}$$

where $\boldsymbol{y}_j \in \mathbb{R}^n \setminus \overline{\Omega}$ and $\boldsymbol{c} = (c_j)_{j=1}^N$. The coefficients are determined from the boundary data, which can be done in various ways. Clearly, the approximate solution u_N is harmonic in Ω.

In the original formulation of the method by Kupradze and Aleksidze [KA63], for the solution of the Dirichlet problem for Laplace's equation in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$, without holes, the singularities were located on a *pseudo–boundary*, i.e., a prescribed boundary $\partial\Omega'$ of a domain Ω', for which $\overline{\Omega} \subset \Omega'$. In the same work, the MFS was also developed for problems in linear elasticity. Kupradze also suggested the MFS for time–dependent problems [Kup64], in particular, for the solution of the heat equation. The MFS was first investigated, as a numerical technique, by Mathon and Johnston [MJ77] in 1977. In their work, the coefficients in (1.5) were chosen to minimize the L^2-distance of the approximate solution from the boundary data of (1.3). The locations of the singularities are also to be determined during the minimization process which results in a non–linear least–squares problem.

[4]**Definition**. *Let* $\mathcal{L} = \sum_{|\alpha|\leq m} a_\alpha(\boldsymbol{x})D^\alpha$ *be a linear partial differential operator in* $\Omega \subset \mathbb{R}^n$ *of order m. The expression* $\sigma(\mathcal{L})(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{|\alpha|=m} a_\alpha(\boldsymbol{x})(\mathrm{i}\boldsymbol{\xi})^\alpha$ *is called principal symbol of* $\mathcal{L}$, *where* $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$, $\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$ *and* $(\mathrm{i}\boldsymbol{\xi})^\alpha = \mathrm{i}^{|\alpha|}\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. *The operator* $\mathcal{L}$ *is said to be (uniformly) elliptic in* Ω *if there is a constant* $c>0$, *such that*

$$\sigma(\mathcal{L})(\boldsymbol{x},\boldsymbol{\xi}) \geq c|\boldsymbol{\xi}|^m,$$

for every $\boldsymbol{\xi}\in\mathbb{R}^n$ *and* $\boldsymbol{x}\in\Omega$.

During the last four decades several formulations of the MFS have appeared in the literature. The two most popular are the following:

- In the first, the locations of the singularities are on a fixed surface. This formulation leads to a linear system.
- In the second they are determined as part of the solution of the discrete problem. This formulation leads to a non–linear least–squares problem.

In the first formulation, the coefficients can be obtained by collocation of the boundary data. In the case of (1.3), this is done by choosing M points $\{\boldsymbol{y}_j\}_{j=1}^M$ on $\partial\Omega'$ – the singularities – and N collocation points $\{\boldsymbol{x}_k\}_{k=1}^N$ on $\partial\Omega$, and require that the approximate solution $u_{M,N}$ satisfies

$$u_{M,N}(\boldsymbol{x}_k;\boldsymbol{c}) \;=\; \sum_{j=1}^{M} c_j\,\varphi(\boldsymbol{x}_k,\boldsymbol{y}_j) \;=\; f(\boldsymbol{x}_k), \quad k=1,\ldots,N. \tag{1.6}$$

This is an $M\times N$ linear system. If $M=N$, the coefficients can be determined uniquely, provided the matrix $G=\big(\varphi(\boldsymbol{x}_k,\boldsymbol{y}_j)\big)_{k,j=1}^N$, is non–singular. If $M>N$, the system (1.6) is over–determined. In such a case, the coefficients can be chosen by linear least–squares, i.e., by minimizing the quantity

$$F(\boldsymbol{c}) \;=\; \frac{1}{N}\sum_{k=1}^{N}\Big(u_{M,N}(\boldsymbol{x}_k;\boldsymbol{c})-f(\boldsymbol{x}_k)\Big)^2 \approx\; \frac{1}{|\partial\Omega|}\|u_{M,N}(\cdot,\boldsymbol{c})-f\|^2_{L^2(\partial\Omega)},$$

where $|\partial\Omega|$ is the area of $\partial\Omega$. A description and analysis of the linear least–squares MFS can be found in [GC97, Koł01, Ram02, SK04a]. A weighted least–squares algorithm is developed in [Smy], in which the error on the boundary is minimized with respect to a suitable discrete Sobolev norm, stronger than the $L^\infty-$norm. In particular, the coefficients are chosen to minimize a quantity approximating the distance $\|u_{M,N}(\cdot,\boldsymbol{c})-f\|_{H^s(\partial\Omega)}$, for suitable values of $s>0$.

There is considerable literature concerning error estimates, stability and convergence analyses of the MFS for boundary value problems in specific domains with specific distributions of the singularities on the pseudo–boundary and of the collocation points on the boundary. In the case of the Dirichlet problem for Laplace's equation in D_ϱ, the disk of radius ϱ, and with the circumference of a concentric disk as a pseudo–boundary, it is shown that the error in the MFS approximation

$$\varepsilon_N \;=\; \sup_{\boldsymbol{x}\in\overline{D}_\varrho}|u_N(\boldsymbol{x})-u(\boldsymbol{x})|,$$

tends to zero as N, the number of singularities and collocation points, tends to infinity, provided that both singularities and collocation points are uniformly distributed on $\partial\Omega$ and $\partial\Omega'$ and the boundary data have absolutely convergent Fourier series. The rate of convergence increases as the smoothness of the data improves. In particular, if the boundary data are analytic, then the convergence is exponential, whereas if they belong to $C^\ell(\partial D_\varrho)$, then the error is $\mathcal{O}\big(N^{-\ell+1}\big)$. (See [Kat89, KO88, Smy].) In [Kat90], this result is generalized to regions in the plane whose boundaries are analytic Jordan curves, while in [TSK], the same result is generalized to annular regions. Convergence of the MFS for the Helmholtz equation in the exterior of a disk was established in [UC03].

As been reported by several authors, in the above formulation there are two contradictory facts:

A. The approximation improves as the distance between the pseudo–boundary and the boundary increases. In particular, it is shown that the MFS approximation of the Dirichlet problem in a disk D_ϱ

and pseudo–boundary a concentric circumference of radius $R > \varrho$, converges exponentially to the solution with N and ϱ/R. More precisely, it is shown that [SK04b]

$$\sup_{x\in\overline{D}_\varrho} \left|u_N(\boldsymbol{x}) - u(\boldsymbol{x})\right| = \mathcal{O}\left(\left(\frac{R}{\varrho}\right)^{N/2}\right),$$

provided that there exists a harmonic extension of u in the entire plane.

B. The condition number $\kappa = \|G\|\cdot\|G^{-1}\|$ of the coefficient matrix $G = \big(\varphi(\boldsymbol{x}_k, \boldsymbol{y}_\ell)\big)$ grows exponentially with N and R/ϱ. In fact, Kitagawa [Kit88] (see also [SK04b]) has shown that

$$\kappa \sim \frac{\log R}{2} N \left(\frac{R}{\varrho}\right)^{N/2}.$$

The poor conditioning of the MFS is widely reported in the literature $\big($see, for example, [BR99, GC97]$\big)$. In particular, Aleksidze [Ale66] demonstrated that the matrices which orthonormalize the fundamental solutions, in the sense of the Hilbert space $L^2(\Omega)$, become extremely ill–conditioned, as the number of fundamental solutions increases.

The poor conditioning of the MFS can be alleviated, either by preconditioning of the system matrix G (see, for example, [Sun05] and references therein) or by iterative refinement (see [CGGC02]). This ill–conditioning can be removed in special cases where an accurate diagonalization of the system matrix is possible (see [Smy]).

Despite these hurdles, the MFS remains a popular meshless technique for the solution of elliptic boundary value problems in which the fundamental solution of the underlying partial differential equation is known, for the following reasons:

- its simplicity and the ease with which it can be implemented;
- unlike the boundary element method, it does not require an elaborate discretization of the boundary;
- it does not involve potentially troublesome and costly integrations over the boundary;
- it requires little data preparation;
- the evaluation of the approximate solution at a point in the interior of the domain can be carried out directly unlike the boundary element method for which a quadrature rule is needed;
- the derivatives of the MFS approximation can also be evaluated directly.

For further details see [FK98]. Comprehensive reference lists of applications of the MFS can be found in [DEW00, FKM03, GC99]. Finally, a rigorous mathematical foundation of the MFS for the numerical solution of a variety of boundary value problems in mathematical physics can be found in [Ale91].

In this work, we investigate the applicability of the MFS in various elliptic problems. More specifically, we provide answers to the question:

Let $\mathcal{L}$ be an elliptic operator and φ a fundamental solution of $\mathcal{L}$. Is the space $\mathcal{X}$ of linear combinations of the form (1.5)*, where the $\boldsymbol{y}_j$'s lie on a prescribed pseudo–boundary, dense in the space $\mathcal{Y}$ of all solutions of the equation $\mathcal{L}u = 0$ in Ω?*

We extend the results of Kupradze and Aleksidze [KA63], and of Bogomolny [Bog85]. In particular, Kupradze and Aleksidze proved that if Ω and Ω' are sufficiently smooth bounded domains in $\mathbb{R}^3$, without holes, with $\overline{\Omega} \subset \Omega'$, then linear combinations of the fundamental solutions of the Laplacian in $\mathbb{R}^3$ with singularities on $\partial\Omega'$ are dense in $L^2(\partial\Omega)$ with respect to the L^2-norm. They also obtained similar density results for the Cauchy–Navier system in linear elasticity. Bogomolny obtained density of $\mathcal{X}$ for harmonic and biharmonic problems with respect to the norm of the Sobolev space $H^s(\Omega)$, when Ω is a bounded domain without holes possessing a smooth boundary. In his proof, Bogomolny used the duality argument (an application of the Hahn–Banach Theorem) of the proof[5] of Theorem 3 in Browder [Bro62].

In our work, we study approximation by fundamental solutions with respect to more pertinent norms, namely the norms of the spaces $C^\ell(\overline{\Omega})$. Such norms correspond to the classical formulations of elliptic boundary value problems. Our domains may possess holes[6]. More specifically, the complements of our domains are not required to be connected. We obtain our density results assuming a rather weak boundary regularity requirement, namely the *Segment Condition*. In this work, we are mainly concerned with the applicability of the MFS to problems of physical interest. We restrict our attention to partial differential operators – scalar and systems – with constant coefficients and known fundamental solutions.

The paper is organized as follows:

Section 2. We define the notion of the *embracing* pseudo–boundary, and provide a list of boundary regularity properties. Next, we briefly describe the function spaces which are used, and particularly their duals. Finally, we state Lemma 1, which is the main tool in the proof of our density results.

Section 3. We investigate the applicability of the MFS for harmonic problems in bounded domains. We establish a $C^\ell-$density result for $n-$dimensional problems, where $n \geq 3$. In the two dimensional case, linear combinations of fundamental solutions with singularities on a prescribed pseudo–boundary are not always dense in the space of harmonic functions, and alternative formulations of the method are proposed. In particular, it is proved (when $n = 2$) that the required density result holds if the pseudo–boundary is a subset of a unit disk; this allows the use of rescaled fundamental solutions (i.e., $\varphi_R(\boldsymbol{x}) = e_1(\boldsymbol{x}/R)$, where R is sufficiently large) in the MFS approximation. We also describe how to obtain density results with respect to the $W^{k,p}-$norm, when $k \in \mathbb{N}$ and $p \in (1, \infty)$.

Section 4. We investigate similar questions for the biharmonic and more generally the $m-$harmonic $\big($i.e., $\mathcal{L} = (-\Delta)^m\big)$ operators. We provide an example where linear combinations of fundamental solutions of the biharmonic operator, with singularities located on a given pseudo–boundary, do not approximate all biharmonic functions. Density of such linear combinations is achieved when we include, in the linear combinations of the fundamental solutions of Δ^2, linear combinations of the fundamental

[5]As noted before (see for example [Tar95]), the formulation of Theorem 3 in [Bro62] contains mistakes. Its proof corresponds to the following result:

Browder's Theorem. *Let $\mathcal{L}$ be a linear operator with coefficients in $C^\infty(G)$, where G is a domain in $\mathbb{R}^n$ without holes. Assume that both $\mathcal{L}$ and $\mathcal{L}^\star$ satisfy condition $(\mathrm{U})_s$ and there exists a bi-regular fundamental solution e of $\mathcal{L}$ satisfying $\mathcal{L}_{\boldsymbol{x}} e(\cdot, \boldsymbol{y}) = \delta_{\boldsymbol{y}}$ and $\mathcal{L}^\star_{\boldsymbol{y}} e(\boldsymbol{x}, \cdot) = \delta_{\boldsymbol{x}}$, for every $\boldsymbol{x}, \boldsymbol{y} \in G$. Let Ω be an open subset of G, satisfying the Cone Condition, such that $\overline{\Omega} \subset G$ and $G \smallsetminus \overline{\Omega}$ does not contain any closed connected components. Let V be an open subset of G, such that $\overline{\Omega} \cap \overline{V} = \varnothing$. Then every solution u of $\mathcal{L}u = 0$ in Ω, which lies in $C^\infty(\Omega) \cap C(\overline{\Omega})$ can be approximated, with respect to the uniform norm, by finite linear combinations of functions of the form $e(\cdot, \boldsymbol{y})$, where $\boldsymbol{y} \in V$.*

[6]Very often in the literature the domains without (resp., with) holes are called simply (resp., multiply) connected. While this is correct for two–dimensional domains, simply connected domains in $\mathbb{R}^d$, $d > 2$, may possess holes. In this work, we say that a domain does not possess holes if and only if its complement is connected. It is noteworthy that, in the context of Algebraic Topology, a bounded domain in $\mathbb{R}^n$ has connected complement if and only if its cohomology group $H^{n-1}(\Omega)$ is isomorphic to $\mathbb{Z}$, due to the *Alexander duality* (see [Spa94]).

solutions of $-\Delta$. Analogous density results hold for the solutions of the $m-$harmonic equation. We also study an alternative MFS formulation which exploits Almansi's representation.

Section 5. Similar density results are obtained for the fundamental solution of the modified Helmholtz operator: $\mathcal{L} = \Delta - \kappa^2$, where $\kappa > 0$. We also investigate the applicability of the MFS for operators of the form: $\mathcal{L} = \prod_{j=1}^{m} \big(\Delta - \kappa_j^2\big)^{\nu_j}$, where $\{\kappa_j\}_{j=1}^{m}$ are distinct nonnegative reals and $\{\nu_j\}_{j=1}^{m}$, positive integers.

Section 6. We describe how the MFS is formulated for homogeneous systems of partial differential equations and provide similar density results for the Cauchy–Navier system of elastostatics in three space dimensions. Analogous density results are provided for a 4×4 system which describes a linear model in the static theory of thermo–elasticity in three space dimensions.

Section 7. We provide a summary of this work and concluding remarks and ideas for possible extensions.

Appendix. In order to avoid overloading the main text, certain proofs are given in the Appendix.

2. A FEW WORDS ON FUNCTION SPACES AND BOUNDARIES

2.1. **Boundary regularity properties.** In boundary value problems, the regularity of the solution often depends on the regularity of the boundary. In this work, we assume that our domain Ω is bounded and may possess holes. We next provide a list of boundary regularity properties:

Definition 1. (***The Segment Condition***) *Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω satisfies the* Segment Condition *if every $\boldsymbol{x} \in \partial\Omega$ has a neighborhood $U_{\boldsymbol{x}}$ and a nonzero vector $\boldsymbol{\xi}_{\boldsymbol{x}}$ such that, if $\boldsymbol{y} \in U_{\boldsymbol{x}} \cap \overline{\Omega}$ then $\boldsymbol{y} + t\,\boldsymbol{\xi}_{\boldsymbol{x}} \in \Omega$ for every $t \in (0,1)$.*

Note that the Segment Condition allows the boundaries to have corners and cusps. However, if a domain satisfies this condition it cannot lie on both sides of any part of its boundary. In fact, domains satisfying the Segment Condition coincide with the interior of their closure. It is not hard to prove that, if a domain satisfies the Segment condition, then every connected component of its complement has nonempty interior. Nevertheless, many Sobolev Imbedding and Extension Theorems require the *Cone Condition*:

Definition 2. (***The Cone Condition***) *Let Ω be an open subset of $\mathbb{R}^n$. We say that its boundary $\partial\Omega$ satisfies the* Cone Condition *if there is a finite open cover $\{U_j\}_{j=1}^{J}$ of $\partial\Omega$ and an $h > 0$, such that, for every $x \in \Omega \cup U_j$, there is a unit vector $\boldsymbol{\xi}_j \in \mathbb{R}^n$ such that the cone*

$$C_h(\boldsymbol{\xi}_j) = \big\{\boldsymbol{y} = \boldsymbol{x} + r\boldsymbol{\xi} \,:\, r \in (0,h) \text{ and } |\boldsymbol{\xi} - \boldsymbol{\xi}_j| < h\big\}$$

is a subset of Ω.

In order to define normal derivatives on the boundary, and thus define classical solutions of Neumann problems, stronger boundary regularity is required:

Definition 3. (***The $C^\ell-$Regularity Condition***) *Let Ω be a bounded open subset of $\mathbb{R}^n$. We say that its boundary $\partial\Omega$ satisfies the* Uniform $C^\ell-$Regularity Condition *if there exists a finite open cover $\{U_j\}_{j=1}^{J}$ of $\partial\Omega$ and a corresponding set of $C^\ell-$diffeomorphisms $\{\psi_j\}_{j=1}^{J}$ from U_j to the unit ball $B_1 = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| < 1\big\}$ such that $\psi_j[U_j \cap \Omega] = \big\{\boldsymbol{x} \in B_1 : x_n > 0\big\}$, for each $j = 1, \dots, J$.*

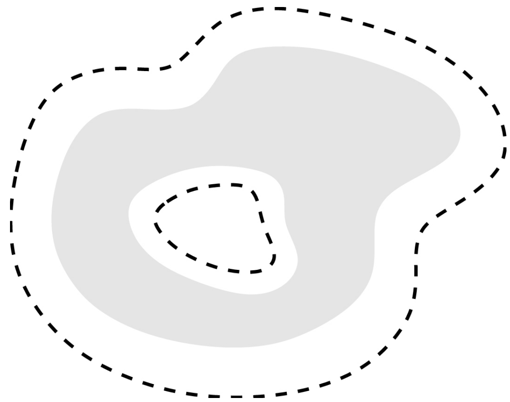

FIGURE 1. The grey region is the domain Ω; the broken lines corresponds to the embracing pseudo–boundary $\partial\Omega'$.

If the diffeomorphisms in the previous definition are $C^\infty-$diffeomorphisms, then the boundary is said to satisfy the *Uniform $C^\infty-$Regularity Condition*, whereas if they are $C^\omega-$diffeomorphisms (i.e., real analytic), then the boundary is called analytic. For a detailed list of boundary regularity conditions, see [AF03].

2.2. **The embracing pseudo–boundary.** In this work, the singularities of the fundamental solutions will be located on a prescribed *pseudo–boundary*, i.e., the boundary $\partial\Omega'$ of a domain Ω' *embracing* Ω (see Figure 1).

Definition 4. *Let Ω, Ω' be open connected subsets of $\mathbb{R}^n$. We say that Ω'* embraces *Ω if $\overline{\Omega} \subset \Omega'$, and for every connected component V of $\mathbb{R}^n \smallsetminus \overline{\Omega}$, there is an open connected component V' of $\mathbb{R}^n \smallsetminus \overline{\Omega}'$ such that $\overline{V}' \subset V$.*

For example, the annulus $A_{r_1,r_2} = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$ embraces the annulus A_{ϱ_1,ϱ_2} provided that $0 < r_1 < \varrho_1 < \varrho_2 < r_2$. On the other hand, a disk cannot embrace an annulus. Note that, if Ω does not have holes and $U \neq \varnothing$ is open, such that $\overline{\Omega} \cap \overline{U} = \varnothing$, then $\mathbb{R}^n \smallsetminus \overline{U}$ embraces Ω. In fact, if Ω is bounded and embraced by Ω', then Ω can have only finitely many holes.

An alternative definition of the embracing pseudo–boundary is the following.

Definition 4$'$. *Let Ω, Ω' be open connected subsets of $\mathbb{R}^n$. We say that Ω'* embraces *Ω if $\overline{\Omega} \subset \Omega'$ and $\Omega' \smallsetminus \Omega$ does not contain any closed connected components.*

If Ω and Ω' satisfy the Segment Condition, then Definition 4$'$ implies Definition 4. For a proof see Appendix A.

2.3. **Our Function Spaces and their Duals.**

2.3.1. *The spaces $C^\ell(\overline{\Omega})$, test functions and distributions.* If Ω is an open domain in $\mathbb{R}^n$, then the space $C^\ell(\Omega)$, where ℓ is a nonnegative integer, contains all functions u which, together with all their partial derivatives $D^\alpha u$ of orders $|\alpha| \leq \ell$, are continuous in Ω and $C^\infty(\Omega) = \bigcap_{\ell \in \mathbb{N}} C^\ell(\Omega)$. The space $C^\ell(\overline{\Omega})$ consists of all functions $u \in C^\ell(\Omega)$ for which $D^\alpha u$ is uniformly continuous and bounded in Ω for all $|\alpha| \leq \ell$. In fact, $C^\ell(\overline{\Omega})$ is a Banach space with norm

$$|u|_\ell = \max_{|\alpha| \leq \ell} \sup_{x \in \overline{\Omega}} \big|D^\alpha u(x)\big|. \tag{2.1}$$

By $\mathscr{D}(\Omega)$ or $C_0^\infty(\Omega)$ we denote the space of infinitely many times differentiable functions with compact support in Ω. The elements of $\mathscr{D}(\Omega)$ are known as the *test functions*. If Ω is bounded, then the elements of the dual of $C(\overline{\Omega})$ are represented by the signed Borel measures on $\overline{\Omega}$. In particular, according to the *Riesz–Kakutani Theorem* (see [Lax02]), for every $T \in \big(C(\overline{\Omega})\big)'$ there exists a unique signed Borel measure μ on $\overline{\Omega}$ such that

$$T(u) \;=\; \int_{\overline{\Omega}} u\, d\mu, \qquad \text{for every} \quad u \in C(\overline{\Omega}).$$

The set $\mathfrak{M}(\overline{\Omega})$ of signed Borel measures on $\overline{\Omega}$ is a Banach space with norm $\|\mu\|$, the *total variation of* μ. The elements of the dual of $C^\ell(\overline{\Omega})$ can also be represented by signed Borel measures. For every $\nu \in \big(C^\ell(\overline{\Omega})\big)'$, there exist $\{\nu_\alpha\}_{|\alpha|\le\ell} \subset \mathfrak{M}(\overline{\Omega})$ such that

$$\nu(u) \;=\; \sum_{|\alpha|\le\ell} \int_{\overline{\Omega}} D^\alpha u\, d\nu_\alpha \qquad \text{for every} \qquad u \in C^\ell(\overline{\Omega}). \tag{2.2}$$

This representation can be achieved by the isometric imbedding $\mathcal{P} : C^\ell(\overline{\Omega}) \to C\big(\overline{\Omega}^{(\ell)}\big)$, where $\overline{\Omega}^{(\ell)}$ is the union of $N = \begin{pmatrix} n+\ell \\ \ell \end{pmatrix}$ (the number of $\alpha \in \mathbb{N}^n$ with $|\alpha| \le \ell$) mutually disjoint copies $\{\overline{\Omega}_\alpha\}_{|\alpha|\le\ell}$ of $\overline{\Omega}$ and $\mathcal{P}u \;=\; (D^\alpha u)_{|\alpha|\le\ell}$. The dual of $C\big(\overline{\Omega}^{(\ell)}\big)$ is representable by a sum of signed Borel measures $\{\nu_\alpha\}_{|\alpha|\le m}$, with $\operatorname{supp}\nu_\alpha \subset \overline{\Omega}_\alpha$, and since $C = \mathcal{P}\big[C^\ell(\overline{\Omega})\big]$ is a closed subspace of $C^\ell(\overline{\Omega})$, every bounded linear functional on C can be extended to a bounded linear functional ν on $C^\ell(\overline{\Omega})$, due to the Hahn–Banach theorem, and thus ν can be expressed in the form (2.2).

The space of test functions $\mathscr{D}(\Omega)$ is not equipped with a norm; it is a locally convex topological vector space. In particular, $\mathscr{D}(\Omega)$ is a Fréchet space, i.e., its topology is induced by a complete invariant metric. (See [Rud73, Chapter 6].) A sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ converges to zero, with respect to the topology of $\mathscr{D}(\Omega)$, if there exists a compact subset K of Ω such that $\operatorname{supp}\psi_n \subset K$, for all $n \in N$, and $D^\alpha\psi_n \to 0$, uniformly in K, for every multi–index $\alpha \in \mathbb{N}^n$. The space $\mathscr{D}(\Omega)$ possesses a topological dual $\mathscr{D}'(\Omega)$, the elements of which are the continuous linear functionals on $\mathscr{D}(\Omega)$ known as the *distributions* on Ω. Any function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution through the pairing

$$T_f(\psi) \;=\; \langle \psi, f \rangle \;=\; \int_\Omega \psi\, f\, d\boldsymbol{x}, \qquad \psi \in \mathscr{D}(\Omega).$$

If f is sufficiently smooth, then $T_{D^\alpha f}(\psi) \;=\; \int_\Omega \psi\, D^\alpha f\, d\boldsymbol{x} \;=\; (-1)^{|\alpha|} \int_\Omega D^\alpha\psi\, f\, d\boldsymbol{x}$. Nevertheless, the expression $(-1)^{|\alpha|} \int_\Omega D^\alpha\psi\, f\, d\boldsymbol{x}$ defines a distribution, even if f is not smooth. In fact, if T is a distribution, then so is $D^\alpha T$ defined accordingly as $D^\alpha T(\psi) = (-1)^{|\alpha|}\, T(D^\alpha\psi)$. Similarly, if $\mathcal{L}$ is a linear partial differential operator with constant coefficients, then $\mathcal{L}\,T$ is defined by $\mathcal{L}\,T(\psi) = T(\mathcal{L}^\star\psi)$, and it is also a distribution. The signed Borel measures, as elements of the dual of $C(\overline{\Omega})$, can be thought of as distributions, i.e., $\mathfrak{M}(\overline{\Omega}) \subset \mathscr{D}'(\mathbb{R}^n)$, and in particular, the functional ν in (2.2) is a distribution in $\mathbb{R}^n$ (in general $\nu \in \mathscr{D}'(\Omega_1)$, provided that $\overline{\Omega} \subset \Omega_1$), and can be alternatively written as $T_\nu \;=\; \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha d\nu_\alpha$. We say that a distribution $T \in \mathscr{D}'(\Omega)$ has compact support if there exists a compact set $K \subset \Omega$, such that, if $\psi\big|_K = 0$, then $T(\psi) = 0$. If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$, then their convolution $u{*}\psi$ is defined as in the case in which u is a locally integrable function, namely,

$$\big(u{*}\psi\big)(\boldsymbol{x}) \;=\; \int_{\mathbb{R}^n} \psi(\boldsymbol{x}{-}\boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \;=\; u\big(\tau_{\boldsymbol{x}}\check{\psi}\big), \tag{2.3}$$

where $\tau_{\boldsymbol{x}}\psi(\boldsymbol{y}) \;=\; \psi(\boldsymbol{x}+\boldsymbol{y})$ and $\check{\psi}(\boldsymbol{x}) \;=\; \psi(-\boldsymbol{x})$. Clearly, $u{*}\psi$ defines a function in $C^\infty(\mathbb{R}^n)$, and if u has compact support, then $u{*}\psi \in \mathscr{D}(\mathbb{R}^n)$. Finally, if $u, v \in \mathscr{D}'(\mathbb{R}^n)$, one of which (say v) has compact support, then their convolution is defined as expected $\big(u{*}v\big)(\psi) = u(\check{v}{*}\psi)$. In fact, $u{*}v \in \mathscr{D}'(\mathbb{R}^n)$. It is noteworthy that, if e is a fundamental solution of the partial differential operator $\mathcal{L}$ and $T \in \mathscr{D}'(\Omega)$ with compact support, then we have

$$\mathcal{L}(e{*}T) = (\mathcal{L}\,e){*}T = \delta{*}T = T,$$

in the sense of distributions, where $\delta(\psi) = \psi(0)$.

2.3.2. *The Sobolev spaces.* If Ω is an open domain in $\mathbb{R}^n$ and $p \in [1,\infty)$, then $L^p(\Omega)$ is defined to be the space of all measurable functions u on Ω for which $\int_\Omega |u(\boldsymbol{x})|^p\,d\boldsymbol{x} < \infty$. The space $L^p(\Omega)$ is a Banach space with norm $\|u\|_{0,p} = \big(\int_\Omega |u(\boldsymbol{x})|^p\,d\boldsymbol{x}\big)^{1/p}$. The dual of $L^p(\Omega)$ is identified with $L^q(\Omega)$, such that $1/p + 1/q = 1$, i.e., for every $l \in \big(L^p(\Omega)\big)'$, there exists a unique $v \in L^q(\Omega)$, such that $l(u) = \int_\Omega uv\,d\boldsymbol{x}$. The space $L^p_{\mathrm{loc}}(\Omega)$ consists of all measurable functions on Ω which belong to $L^p(K)$, for every compact subset K of Ω. If $u, v \in L^1_{\mathrm{loc}}(\Omega)$, we say that v *is a weak $\alpha-$derivative* of u if

$$\int_\Omega u(\boldsymbol{x})\,D^\alpha\psi(\boldsymbol{x})\,d\boldsymbol{x} = (-1)^{|\alpha|}\int_\Omega v(\boldsymbol{x})\,\psi(\boldsymbol{x})\,d\boldsymbol{x},$$

for every $\psi \in C_0^\infty(\Omega)$. The weak $\alpha-$derivative is uniquely defined almost everywhere (if it exists) and we use for it the same notation as for the classical derivative of u. Let m be a positive integer and $p \in [1,\infty)$. The space $W^{m,p}(\Omega)$ consists of all u in $L^p(\Omega)$ for which all the weak derivatives $D^\alpha u$, $|\alpha| \le m$, belong to $L^p(\Omega)$. The space $W^{m,p}(\Omega)$ is a Banach space with norm

$$\|u\|_{m,p} = \bigg(\sum_{|\alpha|\le m}\int_\Omega \big|D^\alpha u(\boldsymbol{x})\big|^p\,d\boldsymbol{x}\bigg)^{1/p}.$$

$W^{m,p}(\Omega)$ is a *Sobolev space*. Clearly, $C_0^\infty(\Omega) \subset W^{m,p}(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, which is in general, a proper subspace of $W^{m,p}(\Omega)$, is denoted by $W_0^{m,p}(\Omega)$. The local Sobolev spaces $W^{m,p}_{\mathrm{loc}}(\Omega)$ contain all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ such that $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$, for every $|\alpha| \le m$.

2.3.3. *The duals of the Sobolev spaces.* The elements of the dual of $W^{m,p}(\Omega)$, $p \in [1,\infty)$, can be represented by distributions through the pairing $\langle u, v\rangle = \int_\Omega u\,v\,d\boldsymbol{x}$. This is achieved by the isometric imbedding $\mathcal{P} : W^{m,p}(\Omega) \to L^p(\Omega^{(m)})$, where $\Omega^{(m)}$ is the union of N (the number of $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$) mutually disjoint copies $\{\Omega_\alpha\}_{|\alpha|\le\ell}$ of Ω and $\mathcal{P}u = (D^\alpha u)_{|\alpha|\le\ell}$. From Hahn–Banach theorem, for every $T \in \big(W^{m,p}(\Omega)\big)'$, there exists a $v_\alpha \in L^q(\Omega)$, $|\alpha| \le m$, $1/p + 1/q = 1$, such that

$$T(u) = \sum_{|\alpha|\le m}\int_\Omega D^\alpha u\,v_\alpha\,d\boldsymbol{x} \qquad \text{for all} \quad u \in W^{m,p}(\Omega).$$

Since $\mathscr{D}(\Omega) \subset W^{m,p}(\Omega)$, the functional T is also a linear functional on $\mathscr{D}(\Omega)$, and, in fact, a distribution with the alternative form

$$T = \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha v_\alpha. \tag{2.4}$$

In particular, since $\mathscr{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega)$, every element T of $\big(W_0^{m,p}(\Omega)\big)'$, can be represented by (2.4). The elements of the dual of $W_0^{m,p}(\Omega)$, which is denoted by $W^{-m,q}(\Omega)$, $1/p + 1/q = 1$, are derivatives of functions in $L^q(\Omega)$ of order up to m. Conversely, every distribution with the representation (2.4) belongs to $W^{-m,q}(\Omega)$ $\big($see [AF03, p. 63]$\big)$. The norm of $W^{-m,q}(\Omega)$ (the negative norm of Lax) is defined through the pairing with $W_0^{m,p}(\Omega)$:

$$\|T\|_{-m,q} = \sup_{\substack{u\in W_0^{m,p}(\Omega)\\ \|u\|_{m,p}=1}}\big|T(u)\big|.$$

Our density results are based on the following lemma:

Lemma 1. *Let* $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ *be an elliptic operator with constant coefficients in* $\mathbb{R}^n$ *and* $e = e(\boldsymbol{x})$ *be a fundamental solution of* $\mathcal{L}$. *Also, let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *satisfying the Segment Condition and* $\nu \in \big(C^\ell(\overline{\Omega})\big)'$. *If* $\vartheta = e * \nu$ *is the convolution of the distributions* e *and* ν *and* $\operatorname{supp}\vartheta \subset \overline{\Omega}$, *then there exists a sequence* $\{\psi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$, *such that* $\{\mathcal{L}\psi_k\}_{k\in\mathbb{N}}$ *converges to* ν *in the weak* sense of* $\big(C^\ell(\overline{\Omega})\big)'$, *i.e., for every* $u \in C^\ell(\overline{\Omega})$

$$\lim_{k\to\infty} \int_\Omega u(\boldsymbol{x})\, \mathcal{L}\psi_k(\boldsymbol{x})\, d\boldsymbol{x} \;=\; \langle u, \nu\rangle.$$

Proof. See Appendix B.

3. Harmonic problems

3.1. **A density result.** Our first density result corresponds to the classical solutions of Dirichlet and Neumann problems for Laplace's equation in bounded domains.

Theorem 1. *Let* Ω, Ω' *be open domains in* $\mathbb{R}^n$ *with* Ω *bounded and satisfying the Segment Condition and* Ω' *embracing* Ω *and let* ℓ *be a nonnegative integer. Then the space* $\mathcal{X}$ *of finite linear combinations of the form* $\sum_{j=1}^N c_j\, e_1(\boldsymbol{x}-\boldsymbol{y}_j)$, *where* e_1 *is given by* (1.4) *is dense in*

$$\mathcal{Y}_\ell \;=\; \big\{u \in C^2(\Omega) \,:\, \Delta u = 0 \text{ in } \Omega\big\} \cap C^\ell(\overline{\Omega}), \tag{3.1}$$

with respect to the norm of the space $C^\ell(\overline{\Omega})$ *if* $n \ge 3$. *If* $n = 2$ *then the linear sum* $\mathcal{X} \oplus \{c \,|\, c \in \mathbb{R}\}$ *is dense in* $\mathcal{Y}_\ell$ *also with respect to the same norm.*

Proof. We follow the ideas developed in [Bog85] and [Bro62]. Both sets $\mathcal{X}$ and $\mathcal{Y}_\ell$ are linear subspaces of $C^\ell(\overline{\Omega})$. If $\nu \in \big(C^\ell(\overline{\Omega})\big)'$, then there exist $\{\nu_\alpha\}_{|\alpha|\le\ell} \subset \mathfrak{M}(\overline{\Omega})$, such that

$$\nu(u) \;=\; \langle u, \nu\rangle \;=\; \sum_{|\alpha|\le\ell} \int_{\overline{\Omega}} D^\alpha u\, d\nu_\alpha \quad \text{for every} \quad u \in C^\ell(\overline{\Omega}).$$

From the Hahn–Banach theorem, it suffices to show that $\mathcal{X}^\perp \subset \mathcal{Y}_\ell^\perp$, i.e.,

$$\textit{if} \quad \nu \in \big(C^\ell(\overline{\Omega})\big)' \quad \textit{and} \quad \underset{\textit{for every } u\,\in\,\mathcal{X}}{\langle u, \nu\rangle = 0} \quad \textit{then} \quad \underset{\textit{for every } u\,\in\,\mathcal{Y}_\ell}{\langle u, \nu\rangle = 0.}$$

Let $\nu \in \big(C^\ell(\overline{\Omega})\big)'$ be such that $\langle u, \nu\rangle = 0$, for every $u \in \mathcal{X}$. In particular, if $\boldsymbol{x} \in \partial\Omega'$, then the function $u(\boldsymbol{y}) = e_1(\boldsymbol{y}-\boldsymbol{x}) = \tau_{-\boldsymbol{x}} e_1(\boldsymbol{y})$ belongs to $\mathcal{X}$ and

$$0 \;=\; \langle u, \nu\rangle \;=\; \langle \tau_{-\boldsymbol{x}} e_1, \nu\rangle \;=\; \nu(\tau_{-\boldsymbol{x}} e_1) \;=\; \nu(\tau_{-\boldsymbol{x}} \check{e}_1) \;=\; \big(e_1 * \nu\big)(\boldsymbol{x}).$$

Thus the convolution $\vartheta = e_1 * \nu$ vanishes on $\partial\Omega'$. Note that ϑ defines a distribution in $\mathbb{R}^n$, as a convolution of two distributions, one of which (namely ν) is of compact support (i.e., $\operatorname{supp}\nu \subset \overline{\Omega}$). Meanwhile, ϑ is real analytic, and in fact, harmonic function in $\mathbb{R}^n \smallsetminus \overline{\Omega}$. Also $-\Delta\vartheta = \nu$ in the sense of distributions in $\mathbb{R}^n$.

Let U be the unbounded connected component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$. Since Ω' embraces Ω, there is a and U' be a connected component U' of $\mathbb{R}^n \smallsetminus \overline{\Omega}'$ such that $\overline{U}' \subset U$. Clearly, $\partial U' \subset \partial\Omega'$, and therefore ϑ vanishes on $\partial U'$. If U' is bounded, then ϑ vanishes in U', from the maximum principle. Consequently,

ϑ vanishes in the whole of U, being a real analytic function. If U' is unbounded, then for $n \geq 3$ we have

$$\vartheta(\boldsymbol{x}) \;=\; \langle \tau_{-\boldsymbol{x}} e_1, \nu \rangle \;=\; \sum_{|\alpha| \leq \ell} \int_{\overline{\Omega}} D^{\alpha}_{\boldsymbol{y}} e_1(\boldsymbol{y}-\boldsymbol{x})\, d\nu_\alpha(\boldsymbol{y}),$$

and thus

$$\big|\vartheta(\boldsymbol{x})\big| \;\leq\; \Big(\sum_{|\alpha|\leq \ell} \|\nu_\alpha\| \Big) \cdot \Big(\sup_{x \in \overline{\Omega}} |D^\alpha e_1(\boldsymbol{y}-\boldsymbol{x})| \Big). \tag{3.2}$$

It is not hard to show that for $\boldsymbol{x}$ large (and $\boldsymbol{y}$ in $\overline{\Omega}$), we have $D^\alpha e_1(\boldsymbol{y}-\boldsymbol{x}) \;=\; \mathcal{O}\big(|\boldsymbol{x}|^{2-n-|\alpha|}\big)$, which combined with (3.2) provides that $\vartheta(\boldsymbol{x}) \;=\; \mathcal{O}\big(|\boldsymbol{x}|^{2-n}\big)$. Therefore $\lim_{\boldsymbol{x}\to\infty} \vartheta(\boldsymbol{x}) = 0$. Since ϑ vanishes also on $\partial U'$ and is arbitrarily small on $S_R = \big\{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| = R\big\}$, for R sufficiently large, then by the maximum principle, ϑ vanishes in the whole of U'. Thus ϑ vanishes in the whole of U, since ϑ vanishes in U', a nonempty open subset of U.

If U is a bounded component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$, then according to Definition 4, there is an open component U' of $\mathbb{R}^n \smallsetminus \overline{\Omega}'$ such that $\overline{U}' \subset U$. If particular, $\partial U' \subset \partial\Omega'$ and thus ϑ vanishes in $\partial U'$. Therefore, ϑ vanishes in the whole of U', and, since ϑ is harmonic in U, it has to vanish in the whole of U.

Consequently, ϑ vanishes in $\mathbb{R}^n \smallsetminus \overline{\Omega}$, and thus $\operatorname{supp}\vartheta \subset \overline{\Omega}$.

Lemma 1 implies that there exists a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $-\Delta\psi_k \to \nu$ in the weak* sense of $C^\ell(\overline{\Omega})$, as $k \to \infty$, i.e., for every $u \in C^\ell(\overline{\Omega})$

$$-\lim_{k\to\infty} \int_\Omega u(\boldsymbol{x})\, \Delta\psi_k(x)\, d\boldsymbol{x} \;=\; \nu(u).$$

Let $u \in \mathcal{Y}_\ell$. Then

$$\nu(u) \;=\; -\lim_{k\to\infty} \int_\Omega u\, \Delta\psi_k\, d\boldsymbol{x} \;=\; -\lim_{k\to\infty} \int_\Omega \Delta u\, \psi_k\, d\boldsymbol{x} \;=\; 0,$$

which concludes the proof in the case $n \geq 3$.

If $n = 2$, we have also assumed that the function $1_{\overline{\Omega}}$ belongs to $\mathcal{X}$. Therefore

$$0 \;=\; \langle 1_{\overline{\Omega}}, \nu \rangle \;=\; \sum_{|\alpha|\leq\ell} \int_{\overline{\Omega}} D^\alpha 1_{\overline{\Omega}}\, d\nu_\alpha \;=\; \int_{\overline{\Omega}} 1_{\overline{\Omega}}\, d\nu_0 \;=\; \nu_0(\overline{\Omega}).$$

Let V be the unbounded component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$ and $\boldsymbol{x} \in V$, then

$$\vartheta(\boldsymbol{x}) \;=\; -\frac{1}{2\pi} \langle \log|\cdot - \boldsymbol{x}|, \nu\rangle \;=\; -\frac{1}{2\pi}\big(\langle \log|\cdot - \boldsymbol{x}|, \nu\rangle - \langle \log|\boldsymbol{x}|, \nu\rangle\big) - \frac{1}{2\pi}\langle \log|\boldsymbol{x}|, \nu\rangle.$$

Clearly, $\langle \log|\boldsymbol{x}|, \nu\rangle = \log|\boldsymbol{x}| \langle 1_{\overline{\Omega}}, \nu\rangle = 0$ and, for $\boldsymbol{x}$ large and $\boldsymbol{y} \in \overline{\Omega}$ we have

$$\big|\log|\boldsymbol{y}-\boldsymbol{x}| - \log|\boldsymbol{x}|\big| \;\leq\; \log\left(1 + \frac{|\boldsymbol{y}|}{|\boldsymbol{x}|}\right) \;=\; \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right), \tag{3.3}$$

whereas, for $|\alpha| \geq 1$, we have

$$D^\alpha_{\boldsymbol{y}}\big(\log|\boldsymbol{y}-\boldsymbol{x}| - \log|\boldsymbol{x}|\big) \;=\; \mathcal{O}\big(|\boldsymbol{x}|^{-|\alpha|}\big). \tag{3.4}$$

Therefore, $\lim_{\boldsymbol{x}\to\infty} \vartheta(\boldsymbol{x}) = 0$, which implies that ϑ vanishes in V. $\square$

An interesting consequence of this proof is the following corollary:

Corollary 1. *Let Ω be an open bounded domain in $\mathbb{R}^n$ satisfying the Segment Condition. If a measure $\mu \in \mathfrak{M}(\overline{\Omega})$ annihilates the space $\mathcal{Y}_0 = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \,:\, \Delta u = 0 \text{ in } \Omega\}$, then there exists a function $\vartheta \in W_0^{1,q}(\Omega)$, with $1 < q < n/(n-1)$, satisfying the equation $-\Delta\vartheta = \mu$, in the sense of distributions. In particular, μ is the weak* limit of a sequence of the form $\{-\Delta\psi_k\}_{k\in\mathbb{N}}$, where $\{\psi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$.* □

Remark 3.1. *If $\mathcal{L} = -\sum_{k,\ell=1}^n a_{k,\ell}\partial_k\partial_\ell$ is an elliptic operator, then a fundamental solution of $\mathcal{L}$ is given by* (1.4) *with $|\boldsymbol{x}|$ replaced by $\left(\sum_{k,\ell=1}^n A_{k,\ell}x_k x_\ell\right)^{1/2}$, where $A_{k,\ell}$ is the cofactor of $a_{k,\ell}$ in the matrix $(a_{k,\ell})$. Theorem* 1 *also holds for the operator $\mathcal{L}$.*

3.2. **Necessity of the constant functions when $n = 2$.**

3.2.1. *A counterexample.* A plausible question to ask, in the case $n = 2$, is whether the constant functions can be approximated by fundamental solutions of the Laplacian with singularities on a given pseudo–boundary. Unfortunately, the answer can be negative as happens in the case in which Ω' is the unit disk D_1 and

$$\varphi(\mathbf{0}, \boldsymbol{y}) \;=\; -\frac{1}{2\pi}\log|\boldsymbol{y}| \;=\; 0 \qquad \text{for every} \quad \boldsymbol{y} \in \partial D_1.$$

Thus every $v \in \mathcal{X}$, where $\mathcal{X}$ is the set of all linear combinations of fundamental solutions with singularities lying on the unit circle, vanishes at the origin, and therefore $\overline{\mathcal{X}} \subsetneq \mathcal{Y}_\ell$. On the other hand, in the case of D_ϱ, the disk of radius $\varrho \neq 1$, the constant function can be approximated by fundamental solutions. In fact the sequence of MFS solutions

$$u_N(\boldsymbol{x}) \;=\; \frac{1}{N\log\varrho}\sum_{j=1}^{N}\log\left|\boldsymbol{x} - \varrho\, \mathrm{e}^{2\pi \mathrm{i} j/N}\right|, \qquad N \in \mathbb{N},$$

converges to $u \equiv 1$, uniformly in the compact subsets of D_ϱ. (See [SK04b].)

3.2.2. *Rarity of pseudo–boundaries for which $\overline{\mathcal{X}} \subsetneq \mathcal{Y}_\ell$.* Pseudo–boundaries $\partial\Omega'$ with the property that linear combinations of fundamental solutions with singularities lying on them are not dense in the space of harmonic functions are very rare. From the proof of Theorem 1, we conclude that, if $\partial\Omega'$ has this property, then the function $\vartheta = e_1 * \mu$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, tends to infinity, as $|\boldsymbol{x}|$ tends to infinity, and vanishes on $\partial\Omega'$. This implies that $\partial\Omega'$ is subset of the level set of a harmonic function and thus it is a subset of a real analytic boundary.

It is noteworthy that, if the unbounded connected component U of $\mathbb{R}^n \setminus \overline{\Omega}$ contains a *bounded* connected component U' of $\mathbb{R}^n \setminus \overline{\Omega}'$, then the function ϑ in the proof of Theorem 1 vanishes on $\partial U'$, and thus in $\overline{U}'$, and consequently in $\mathbb{R}^n \setminus \overline{\Omega}$. Therefore, constant functions are not necessary in such case.

3.2.3. *A case where the constant functions are not necessary.* The constant functions are approximated by linear combinations of fundamental solutions if the pseudo–boundary is included in a unit disk. We have the following result:

Proposition 1. *Let Ω, Ω' be open and bounded domains in $\mathbb{R}^2$ with Ω satisfying the Segment Condition and Ω' embracing Ω. If $\overline{\Omega}'$ is a subset of a unit disk, then the constant function $f = 1_{\overline{\Omega}}$ can be approximated, in the sense of the supremum norm, by linear combinations of the form $\sum_{j=1}^N c_j \log|\boldsymbol{x} - \boldsymbol{y}_j|$, where $\{\boldsymbol{y}_j\}_{j=1}^N \subset \partial\Omega'$.*

Proof. Without loss of generality we may assume that $\overline{\Omega}'$ is a subset of the unit disk D_1, which is centered at the origin. Let $\mathcal{X}$ be the space of linear combinations of the form $\sum_{j=1}^{N} c_j \log|\boldsymbol{x}-\boldsymbol{y}_j|$, where $\{\boldsymbol{y}_j\}_{j=1}^{N} \subset \partial\Omega'$. Clearly, the elements of $\mathcal{X}$, when restricted to $\overline{\Omega}$, are also elements of $C(\overline{\Omega})$. Assume that $1_{\overline{\Omega}} \notin \overline{\mathcal{X}}$. Then, due to the Hahn–Banach theorem, there is a measure $\mu \in \mathfrak{M}(\overline{\Omega}) = \big(C(\overline{\Omega})\big)'$, satisfying

$$\int_{\overline{\Omega}} 1_{\overline{\Omega}}\, d\mu(\boldsymbol{y}) \;=\; \mu(\overline{\Omega}) \;=\; 1$$

and annihilating $\mathcal{X}$. This, as in the proof of Theorem 1, implies that the function

$$u(\boldsymbol{x}) \;=\; \int_{\overline{\Omega}} \log|\boldsymbol{x}-\boldsymbol{y}|\, d\mu(\boldsymbol{y}),$$

which is harmonic in $\mathbb{R}^2 \smallsetminus \overline{\Omega}$, vanishes on $\partial\Omega'$. Also, for $|\boldsymbol{x}|$ large we have

$$u(\boldsymbol{x}) \;=\; \int_{\overline{\Omega}} \log|\boldsymbol{x}|\, d\mu(\boldsymbol{y}) + \int_{\overline{\Omega}} \log\frac{|\boldsymbol{x}-\boldsymbol{y}|}{|\boldsymbol{x}|}\, d\mu(\boldsymbol{y}) \;=\; \log|\boldsymbol{x}| + \mathcal{O}\bigg(\frac{1}{|\boldsymbol{x}|}\bigg),$$

which implies that $\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = +\infty$. Due to the maximum principle we have that $u(\boldsymbol{x}) > 0$, for every $\boldsymbol{x} \in \mathbb{R}^2 \smallsetminus \overline{\Omega}'$. If we let (identifying $\mathbb{R}^2$ with $\mathbb{C}$)

$$\tilde{u}(\boldsymbol{x}) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_{\overline{\Omega}} \log|\boldsymbol{x}-\mathrm{e}^{\mathrm{i}\vartheta}\boldsymbol{y}|\, d\mu(\boldsymbol{y})\, d\vartheta,$$

then $\tilde{u}$ is harmonic in $\mathbb{R}^2 \smallsetminus \widetilde{\Omega}$, where $\widetilde{\Omega} = \cup_{\vartheta\in[0,2\pi]} \mathrm{e}^{i\vartheta}\Omega$. In particular, $\widetilde{\Omega}$ is an open disk of radius $\varrho < 1$ centered at the origin, where $1-\varrho = \mathrm{dist}\big(\overline{\Omega}', \partial D_1\big)$. Also, $\tilde{u}$ is radial, i.e., $\tilde{u}(\boldsymbol{x}) = v(|\boldsymbol{x}|)$. In fact, $\tilde{u}$ has to be of the form

$$\tilde{u}(\boldsymbol{x}) \;=\; A\log|\boldsymbol{x}| + B, \tag{3.5}$$

where A, B real constants with $A > 0$. Clearly, $\tilde{u}(\boldsymbol{x}) > 0$, for every $|\boldsymbol{x}| = 1$, since the same holds for u. Thus $B > 0$. If we let $R = |\boldsymbol{x}| > |\boldsymbol{y}| = r$, then we have $\big($[GR00, p. 585]$\big)$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\boldsymbol{x}-\mathrm{e}^{\mathrm{i}\vartheta}\boldsymbol{y}|\, d\vartheta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log\big(R^2-2rR\cos\vartheta+r^2\big)\, d\vartheta \;=\; \log R \;=\; \log|\boldsymbol{x}|.$$

Using Fubini's theorem we obtain

$$\tilde{u}(\boldsymbol{x}) \;=\; \frac{1}{2\pi}\int_{\overline{\Omega}}\int_0^{2\pi} \log|\boldsymbol{x}-\mathrm{e}^{\mathrm{i}\vartheta}\boldsymbol{y}|\, d\vartheta\, d\mu(\boldsymbol{y}) \;=\; \mu(\overline{\Omega})\,\log|\boldsymbol{x}| \;=\; \log|\boldsymbol{x}|,$$

which contradicts (3.5), and in particular the fact that $B > 0$. □.

Remark 3.2. *Proposition* 1 *is valid even in the case of approximation in the sense of the norm of the space* $C^{\ell}(\overline{\Omega})$, *where* ℓ *is any nonnegative integer.*

3.2.4. *A rescaling argument.* The functions

$$\varphi_R(\boldsymbol{x}) \;=\; -\frac{1}{2\pi}\log\bigg(\frac{|\boldsymbol{x}|}{R}\bigg), \tag{3.6}$$

are also fundamental solutions of the Laplacian for any $R > 0$. Proposition 2 provides that if the pseudo–boundary is a subset of a disk of radius R, then the harmonic functions in any domain Ω embraced by Ω', with boundary satisfying the Segment Condition, can be approximated by linear combinations of the form

$$v(\boldsymbol{x}) \;=\; \sum_{j=1}^{N} c_j\, \varphi_R(\boldsymbol{x}-\boldsymbol{y}_j),$$

where $\{\boldsymbol{y}_j\}_{j=1}^{N} \subset \partial\Omega'$. This observation allows us to avoid using constant functions in the MFS approximation by rescaling the fundamental solution.

3.2.5. *Introduction of a second pseudo–boundary.* Constant functions can be avoided by introducing a second pseudo–boundary. Let us assume that Ω' embraces Ω and Ω'' embraces Ω'. If our function space $\mathcal{X}$ contains the linear combinations of fundamental solutions with singularities on $\partial\Omega' \cup \partial\Omega''$, then in the proof of Theorem 1, the function ϑ vanishes on $\partial\Omega'$ and $\partial\Omega''$, and thus in $\Omega'' \smallsetminus \overline{\Omega}'$, due to the maximum principle. Consequently, ϑ vanishes in the whole of the unbounded connected component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$.

In fact, the constant functions can be avoided by adding to the pseudo–boundary just one additional point $\boldsymbol{y}_0 \in \mathbb{R}^2 \smallsetminus \overline{\Omega}'$. In such case, if $\nu \in \mathcal{X}^\perp$, then $\vartheta = e_1 * \nu$ will be vanishing on $\partial\Omega'$ and $\boldsymbol{y}_0$. If the constant functions do not belong to $\overline{\mathcal{X}}$, then $\nu\big(1_{\overline{\Omega}}\big) \neq 0$, where $1_{\overline{\Omega}}$ is the function $u \equiv 1$, and

$$\lim_{|\boldsymbol{x}|\to\infty} \vartheta(\boldsymbol{x}) = \begin{cases} -\infty & \text{if } \nu\big(1_{\overline{\Omega}}\big) < 0, \\ +\infty & \text{if } \nu\big(1_{\overline{\Omega}}\big) > 0. \end{cases}$$

If $\nu\big(1_{\overline{\Omega}}\big) > 0$, then the maximum principle implies $\vartheta(\boldsymbol{x}) > 0$, for every $\boldsymbol{x}$ in the unbounded component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$, which leads to a contradiction.

3.3. $W^{k,p}-$**density results.** Analogous density results can be obtained with respect to the norms of the spaces $W^{k,p}(\Omega)$, where k is a nonnegative integer and $p \in (1,\infty)$. In fact, the following version of Theorem 1 holds:

THEOREM $1'$ *Let Ω, Ω' be open domains in $\mathbb{R}^n$ with Ω bounded and satisfying the Segment Condition and Ω' embracing Ω. Let $k \in \mathbb{N}$ and $p \in (1,\infty)$. Then the space $\mathcal{X}$ of finite linear combinations of the form $\sum_{j=1}^N c_j\, e_1(\boldsymbol{x}-\boldsymbol{y}_j)$, where e_1 is given by* (1.4) *is dense in*

$$\mathcal{Y} = \big\{u \in C^2(\Omega) \,:\, \Delta u = 0 \text{ in } \Omega\big\} \cap W^{k,p}(\Omega),$$

with respect to the norm of $W^{k,p}(\Omega)$ if $n \geq 3$. If $n = 2$ then the linear sum $\mathcal{X} \oplus \{c \,|\, c \in \mathbb{R}\}$ is dense in $\mathcal{Y}_\ell$ also with respect to the same norm.

SKETCH OF PROOF. The proof of Theorem $1'$ is almost identical to the proof of Theorem 1, and in fact slightly simpler. Here we use the fact that the elements of the dual of $W^{k,p}(\Omega)$ can be expressed, in the sense of distributions, as

$$\nu = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha v_\alpha,$$

where $\{v_\alpha\} \subset L^q(\Omega)$ and $1/p + 1/q = 1$. As in the proof of Theorem 1, if $\nu \in \big(W^{k,p}(\Omega)\big)'$, then $\vartheta = e_1 * \nu$ vanishes in $\mathbb{R}^n \smallsetminus \overline{\Omega}$. A modification of Lemma 1, which is simpler to prove, provides that, if $\vartheta = e_1 * \nu$ vanishes on $\partial\Omega'$ and eventually in $\mathbb{R}^n \smallsetminus \overline{\Omega}$, then there is a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$, such that $\{-\Delta\psi_k\}_{k\in\mathbb{N}}$ converges to ν in the weak* sense of $\big(W^{k,p}(\Omega)\big)'$. □

4. BIHARMONIC AND $m-$HARMONIC PROBLEMS

A function u is called *biharmonic* if $\Delta^2 u = 0$. More generally, u is called *$m-$harmonic* (or *polyharmonic*) if $\Delta^m u = 0$. In this section, we shall investigate density questions in the case of approximations of solutions of biharmonic and $m-$harmonic operators, i.e., $\mathcal{L} = (-\Delta)^m$, by linear combinations of their fundamental solutions.

4.1. **Biharmonic problems.** The function

$$e_2(\boldsymbol{x}) \;=\; \begin{cases} \dfrac{1}{8\pi}|\boldsymbol{x}|^2(\log|\boldsymbol{x}|-1) & \text{if} \quad n=2, \\[2ex] \dfrac{1}{4\,\omega_3}\log|\boldsymbol{x}| & \text{if} \quad n=4, \\[2ex] \dfrac{|\boldsymbol{x}|^{4-n}}{2(4-n)(2-n)\,\omega_{n-1}} & \text{if} \quad n\neq 2,4, \end{cases} \tag{4.1}$$

is a fundamental solution of the biharmonic operator $\mathcal{L}=\Delta^2$. In fact, if e_1 is the fundamental solution of $-\Delta$ given by (1.4), then

$$-\Delta e_2 \;=\; e_1,$$

in the sense of distributions. (See [ACL83].)

4.1.1. *A non–density result.* In general, a biharmonic function $u\in C^4(\Omega)\cap C(\overline{\Omega})$, where Ω is an open bounded domain, cannot be approximated by linear combinations of the fundamental solutions e_2 in (4.1) with singularities located on a given pseudo–boundary $\partial\Omega'$ embracing Ω. Let for example Ω be $B(0,\varrho)$, the three–dimensional ball of radius ϱ centered at the origin and $\partial\Omega'$ be $S(0,R)$ the sphere of radius R, where $\varrho<R$, also centered at the origin. We shall show that the constant function in $\overline{B}(0,\varrho)$, which is biharmonic, cannot be approximated by linear combination of the form

$$u_N(\boldsymbol{x}) \;=\; \sum_{j=1}^N c_j\,|\boldsymbol{x}-\boldsymbol{y}_j| \quad \text{where} \quad \{\boldsymbol{y}_j\}_{j=1}^N\subset S(0,R).$$

If not, then for every $\varepsilon>0$, there exist $\{\boldsymbol{y}_j\}_{j=1}^N\subset S(0,R)$, such that

$$1-\varepsilon \;<\; \sum_{j=1}^N c_j\,|\boldsymbol{x}-\boldsymbol{y}_j| \;<\; 1+\varepsilon,$$

for every $\boldsymbol{x}\in\overline{B}(0,\varrho)$. If U is a unitary matrix in $\mathbb{R}^{3\times3}$, i.e., $U^TU=I$, then clearly,

$$1-\varepsilon \;<\; \sum_{j=1}^N c_j\,|U^T\boldsymbol{x}-\boldsymbol{y}_j| \;=\; \sum_{j=1}^N c_j\,|\boldsymbol{x}-U\boldsymbol{y}_j| \;<\; 1+\varepsilon, \tag{4.2}$$

for every $\boldsymbol{x}\in\overline{B}(0,\varrho)$ as well. The set $SO(3)$ of unitary matrices in $\mathbb{R}^{3\times3}$ is a compact group, with respect to the matrix multiplication. Therefore, $SO(3)$ possesses a Haar measure; an invariant, with respect to multiplication, positive measure μ, which we normalize to be a probability measure. (See [Fol99].) Integrating (4.2) over $SO(3)$ we obtain

$$1-\varepsilon \;<\; \sum_{j=1}^N c_j\int_{SO(3)}|\boldsymbol{x}-U\boldsymbol{y}_j|\,d\mu(U) \;<\; 1+\varepsilon \quad \text{for every} \quad \boldsymbol{x}\in\overline{B}(0,\varrho). \tag{4.3}$$

Moreover, the integral $\int_{SO(3)}|\boldsymbol{x}-U\boldsymbol{y}_j|\,d\mu(U)$ does not depend on $\boldsymbol{y}_j$, due to the invariance of μ and the fact that, for every $\boldsymbol{y}\in S(0,R)$, there exists a $V\in SO(3)$, such that $V\boldsymbol{y}_j=\boldsymbol{y}$. Therefore, (4.3) yields that

$$1-\varepsilon \;<\; s_N\int_{SO(3)}|\boldsymbol{x}-U\boldsymbol{y}|\,d\mu(U) \;<\; 1+\varepsilon \quad \text{for every } \boldsymbol{x}\in\overline{B}(0,\varrho) \text{ and } \boldsymbol{y}\in S(0,R), \tag{4.4}$$

where $s_N = \sum_{j=1}^{N} c_j$. Setting $f(\boldsymbol{x}) = \int_{SO(3)} |\boldsymbol{x}-U\boldsymbol{y}|\, d\mu(U)$, then by using Fubini's theorem we obtain

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{4\pi R^2}\int_{S(0,R)}\left(\int_{SO(3)}|\boldsymbol{x}-U\boldsymbol{y}|\,d\mu(U)\right)d\boldsymbol{y} \;=\; \frac{1}{4\pi R^2}\int_{SO(3)}\left(\int_{S(0,R)}|\boldsymbol{x}-U\boldsymbol{y}|\,d\boldsymbol{y}\right)d\mu(U) \\ &= \frac{1}{4\pi R^2}\int_{S(0,R)}|\boldsymbol{x}-\boldsymbol{y}|\,d\boldsymbol{y}. \end{aligned}$$

Clearly, f is a function of $r = |\boldsymbol{x}|$. In fact, if $\boldsymbol{x} = r(1,0,0)$ and $\boldsymbol{y} = (s, y_1, y_2) \in S(0,R)$, then $|\boldsymbol{x}-\boldsymbol{y}| = (R^2 - 2rs + r^2)^{1/2}$ and integration over the sphere $S(0,R)$ reduces to

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{4\pi R^2}\int_{-R}^{R} 2\pi R\,(R^2-2rs+r^2)^{1/2}ds \;=\; \frac{1}{2R}\cdot\frac{2}{3}\cdot\frac{1}{2r}\cdot\left(R^2+2rs+r^2\right)^{3/2}\Big|_{s=-R}^{s=R} \\ &= \frac{1}{6rR}\left((R^2+2rR+r^2)^{3/2}-(R^2-2rR+r^2)^{3/2}\right) \;=\; \frac{1}{6rR}\left((R+r)^3-(R-r)^3\right) \\ &= \frac{1}{6rR}\left(6rR^2+2r^3\right) \;=\; R+\frac{r^2}{3R} \;=\; R+\frac{|\boldsymbol{x}|^2}{3R}, \end{aligned}$$

which contradicts (4.4).

Remark 4.1. *A biharmonic function $u \in C^4(\Omega) \cap C(\overline{\Omega})$, where Ω is an open bounded domain, can be approximated by linear combinations of the fundamental solutions e_2 in* (4.1) *with singularities located, not on a given pseudo–boundary $\partial\Omega'$ embracing Ω, but in an open neighborhood of $\partial\Omega'$ (see Theorem* 5*).*

4.1.2. *The standard implementation of the MFS.* In the MFS for biharmonic problems, the approximate solution is a linear combination of two types of fundamental solutions, the fundamental solutions of the Laplacian as well as the fundamental solutions of the biharmonic operator, i.e.,

$$u_N(\boldsymbol{x};\boldsymbol{c},\boldsymbol{d}) \;=\; \sum_{j=1}^{N}\left\{c_j\, e_1(\boldsymbol{x},\boldsymbol{y}_j) + d_j\, e_2(\boldsymbol{x},\boldsymbol{y}_j)\right\}, \qquad \boldsymbol{x}\in\overline{\Omega},$$

where the singularities $\boldsymbol{y}_j$, $j = 1,\ldots,N$, lie outside of $\overline{\Omega}$. (See [Bog85, FK98, FKS05, KF87, Kup65, MC74].)

We have the following density result:

Theorem 2. *Let Ω, Ω' be open bounded domains in $\mathbb{R}^n$, $n \geq 3$, with boundary of Ω satisfying the Segment Condition and Ω' embracing Ω, and let ℓ be a nonnegative integer. Then the space $\mathcal{X}$ of all finite linear combinations of the form*

$$v(\boldsymbol{x}) \;=\; \sum_{j=1}^{N}\left\{c_j\, e_1(\boldsymbol{x},\boldsymbol{y}_j) + d_j\, e_2(\boldsymbol{x},\boldsymbol{y}_j)\right\}, \tag{4.5}$$

restricted in $\overline{\Omega}$, where $\{\boldsymbol{y}_j\}_{j=1}^{N} \subset \partial\Omega'$, is dense in

$$\mathcal{Y}_\ell \;=\; \left\{u \in C^4(\Omega) \,:\, \Delta^2 u = 0 \text{ in } \Omega\right\} \cap C^\ell(\overline{\Omega}) \tag{4.6}$$

with respect to the norm of $C^\ell(\overline{\Omega})$.

If $n = 2$ then the same density result holds for the linear sum $\mathcal{X} \oplus \left\{c_1 + c_2|\boldsymbol{x}|^2 \,:\, c_1,\, c_2 \in \mathbb{R}\right\}$.

Proof. Let $\nu \in \left(C^\ell(\overline{\Omega})\right)'$ annihilating all the elements of $\mathcal{X}$, and $\vartheta_j = e_j * \nu$, $j = 1,2$. As in the proof of Theorem 1, we obtain

$$\vartheta_1\big|_{\partial\Omega'} \;=\; \vartheta_2\big|_{\partial\Omega'} \;=\; 0. \tag{4.7}$$

Also,

$$-\Delta\,\vartheta_2 \;=\; -\Delta(e_2{*}\nu) \;=\; (-\Delta e_2){*}\nu \;=\; e_1{*}\nu \;=\; \vartheta_1,$$

in the sense of distributions, since $-\Delta\, e_2 = e_1$, also in the sense of distributions. As in the proof of Theorem 1, we obtain $\vartheta_1\big|_{\mathbb{R}^n\smallsetminus\overline{\Omega}} = 0$. In particular, since ϑ_1, ϑ_2 are real analytic functions in $\mathbb{R}^n\smallsetminus\overline{\Omega}$, we deduce that $\Delta\vartheta_2 = 0$ in $\mathbb{R}^n\smallsetminus\overline{\Omega}$.

If $n > 4$, we have

$$e_2(x) \;=\; \frac{1}{2(4-n)(2-n)\,\omega_{n-1}}|x|^{4-n} \quad\text{and}\quad D^\alpha e_2(x) \;=\; \mathcal{O}\big(|x|^{4-n-|\alpha|}\big),$$

for every multi–index α. Therefore, for $|x|$ large,

$$\vartheta_2(x) \;=\; \big(e_2{*}\nu\big)(x) \;=\; \langle \tau_x\check{e}_2,\nu\rangle \;=\; \langle \tau_x e_2,\nu\rangle \;=\; \mathcal{O}\big(|x|^{4-n}\big).$$

Thus $\lim_{|x|\to\infty}\vartheta_2(x) = 0$ and combining this with (4.7) we conclude that $\vartheta_2\big|_{\mathbb{R}^n\smallsetminus\overline{\Omega}} = 0$.

If $n = 4$, then $e_2(x) = \dfrac{\log|x|}{4\,\omega_3}$ and $D^\alpha e_2(x) = \mathcal{O}\big(|x|^{-|\alpha|}\big)$, for every multi–index α, and since the function $u\equiv 1$ belongs to $\overline{\mathcal{X}}$, the closure of $\mathcal{X}$ with respect to the norm of $C^\ell(\overline{\Omega})$ (which is obtained by Theorem 1), repeating what was done in (3.3)–(3.4) we obtain, for $|x|$ large,

$$\vartheta_2(x) \;=\; \mathcal{O}\bigg(\frac{1}{|x|}\bigg),$$

which also implies that $\lim_{|x|\to\infty}\vartheta_2(x) = 0$ and consequently that $\vartheta_2\big|_{\mathbb{R}^4\smallsetminus\overline{\Omega}} = 0$.

If $n = 3$, then $e_2(x) = -\dfrac{1}{8\,\pi}|x|$, and for $|x|$ large and $y\in\overline{\Omega}$ we have

$$|x-y| - |x| + \frac{1}{|x|}\,x\cdot y \;=\; \frac{|y|^2\big(|x|^2 + |x|\cdot|x-y| + 2x\cdot y\big) - 4\big(x\cdot y\big)^2}{|x|\big(|x-y| + |x|\big)^2} \;=\; \mathcal{R}(x,y), \tag{4.8}$$

and, clearly, for $|x|$ large,

$$D^\alpha_y\mathcal{R}(x,y) \;=\; \mathcal{O}\bigg(\frac{1}{|x|}\bigg), \tag{4.9}$$

for all multi–indices α. Also, since $1, y_1, y_2, y_3 \in \overline{\mathcal{X}}$, are harmonic, we obtain

$$\langle 1_{\overline{\Omega}},\nu\rangle \;=\; \langle y_1,\nu\rangle \;=\; \langle y_2,\nu\rangle \;=\; \langle y_3,\nu\rangle \;=\; 0. \tag{4.10}$$

Therefore, for $|x|$ large, combining (4.8)–(4.10), we obtain

$$\langle |x-\cdot|,\nu\rangle \;=\; \langle |x|\cdot 1_{\overline{\Omega}},\nu\rangle - \frac{1}{|x|}\sum_{j=1}^{3}\langle y_j,\nu\rangle + \langle\mathcal{R}(x,\cdot),\nu\rangle \;=\; \mathcal{O}\bigg(\frac{1}{|x|}\bigg).$$

Therefore $\vartheta_2(x) = \langle\tau_x e_2,\nu\rangle = -\dfrac{1}{8\,\pi}\langle|x-\cdot|,\nu\rangle \to 0$ as $|x|\to\infty$ and consequently $\vartheta_2\big|_{\mathbb{R}^3\smallsetminus\overline{\Omega}} = 0$.

If $n = 2$, we have that $1, y_1, y_2, y_1y_2, y_1^2 - y_2^2 \in \overline{\mathcal{X}}$, are harmonic, and we also assumed that $|y|^2 = y_1^2 + y_2^2 \in \mathcal{X}$, thus $y_1^2, y_2^2 \in \overline{\mathcal{X}}$. Therefore for every $x\in\mathbb{R}^n$, we have

$$|x-y|^2 \;=\; |x|^2\cdot 1 - 2\,x_1\cdot y_1 - 2\,x_2\cdot y_2 + y_1^2 + y_2^2 \in \overline{\mathcal{X}}. \tag{4.11}$$

Also, for $|x|$ large, the Taylor expansion around x yields

$$\begin{aligned}|x-y|^2\log|x-y|^2 =& |x|^2\log|x|^2 - 2\big(\log|x|^2+1\big)(x\cdot y) + \big(\log|x|^2+1\big)|y|^2\\ &+ \frac{6}{|x|^2}(x\cdot y)^2 + \mathcal{R}(x,y),\end{aligned} \tag{4.12}$$

with

$$D_{\boldsymbol{y}}^{\alpha}\mathcal{R}(\boldsymbol{x},\boldsymbol{y}) \;=\; \mathcal{O}\big(|\boldsymbol{x}|^{-1-|\alpha|}\big), \tag{4.13}$$

for all multi–indices α. Combining (4.11)–(4.13), we obtain

$$\vartheta_2(\boldsymbol{x}) \;=\; \langle \tau_{\boldsymbol{x}} e_2, \nu\rangle \;=\; \frac{1}{8\pi}\langle |\boldsymbol{x}-\cdot|^2 \log|\boldsymbol{x}-\cdot|, \nu\rangle - \frac{1}{8\pi}\langle |\boldsymbol{x}-\cdot|^2, \nu\rangle \;=\; \mathcal{O}\Big(\frac{1}{|\boldsymbol{x}|}\Big).$$

Thus $\lim_{|\boldsymbol{x}|\to\infty}\vartheta_2(\boldsymbol{x}) = 0$ and consequently $\vartheta_2\big|_{\mathbb{R}^2\smallsetminus\overline{\Omega}} = 0$.

Lemma 1 implies that $\vartheta_2 \in W_0^{3-\ell,q}(\Omega)$, and that there exists a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$, such that $\Delta^2\psi_k \to \nu$ in the weak* sense of the space $C^\ell(\overline{\Omega})$. If $u \in \mathcal{Y}_\ell$, then

$$\langle u, \nu\rangle \;=\; \lim_{k\to\infty}\int_{\overline{\Omega}} u(\boldsymbol{x})\,\Delta^2\psi_k(\boldsymbol{x})\,d\boldsymbol{x} \;=\; \lim_{k\to\infty}\int_{\overline{\Omega}} \Delta^2 u(\boldsymbol{x})\,\psi_k(\boldsymbol{x})\,d\boldsymbol{x} \;=\; 0,$$

which concludes the proof. □

4.2. $m-$**harmonic problems.** It can be readily shown that the function

$$e_m(\boldsymbol{x}) \;=\; \begin{cases} \dfrac{(-1)^m|\boldsymbol{x}|^{2m-2}\big(\log|\boldsymbol{x}| - \gamma_{m-1}\big)}{2^{2m-1}\pi\big((m-1)!\big)^2}, & \text{if} \quad n = 2, \\[2ex] \dfrac{(-1)^{m-1}|\boldsymbol{x}|^{2m-3}}{4\pi(2m-3)!}, & \text{if} \quad n = 3, \end{cases}$$

where $\gamma_0 = 0$ and $\gamma_m = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{m}$, for $m \geq 1$, is a fundamental solution of the $m-$harmonic operator $\mathcal{L} = (-\Delta)^m$. In fact, these fundamental solutions satisfy the equations

$$(-\Delta)^j e_k \;=\; e_{k-j}, \quad \text{for} \quad k > j, \tag{4.14}$$

in the sense of distributions. For $n > 3$, the functions

$$e_m(\boldsymbol{x}) \;=\; |\boldsymbol{x}|^{2m-n}\big(a_{m,n}\log|\boldsymbol{x}| + b_{m,n}\big),$$

are fundamental solutions of $(-\Delta)^m$, for suitable coefficients $a_{m,n}$, $b_{m,n}$. In particular, there are unique $a_{m,n}$ and $b_{m,n}$, so that (1.2) and (4.14) are satisfied. In fact, $a_{m,n} = 0$ if $2m < n$ or if n is odd, whereas $b_{m,n} = 0$ if $m \geq 0$ and n is even $\big($[ACL83]$\big)$.

In the MFS the solutions of $(-\Delta)^m u = 0$ are approximated by linear combinations of the form

$$v(\boldsymbol{x}) \;=\; \sum_{k=1}^{m}\sum_{j=1}^{N} c_j^k\, e_k(\boldsymbol{x}-\boldsymbol{y}_j), \tag{4.15}$$

where the $\boldsymbol{y}_j$'s lie on prescribed pseudo–boundary.

The following theorem is a predictable generalization of Theorems 1 and 2.

Theorem 3. *Let Ω, Ω' be open bounded domains in $\mathbb{R}^n$, $n \geq 3$, with the boundary of Ω satisfying the Segment Condition and Ω' embracing Ω, and let ℓ be a nonnegative integer. Then the space $\mathcal{X}$ of all finite linear combinations of the form* (4.15)*, when restricted in $\overline{\Omega}$, where $\{\boldsymbol{y}_j\}_{j=1}^N \subset \partial\Omega'$, is dense in*

$$\mathcal{Y}_\ell \;=\; \big\{u \in C^{2m}(\Omega) \,:\, (-\Delta)^m u = 0 \text{ in } \Omega\big\} \cap C^\ell(\overline{\Omega}), \tag{4.16}$$

with respect to the norm of $C^\ell(\overline{\Omega})$.

If $n = 2$ then the same density result holds for the sum

$$\mathcal{X} \oplus \big\{c_1 + c_2|\boldsymbol{x}|^2 + \cdots + c_m|\boldsymbol{x}|^{2m-2} \,:\, c_1, c_2, \ldots, c_m \in \mathbb{R}\big\}.$$ □

Corollary 2. *Let Ω be an open bounded domain in $\mathbb{R}^n$ satisfying the Segment Condition. If a functional $\nu \in \big(C^\ell(\overline{\Omega})\big)'$ annihilates the space*

$$\mathcal{Y}_\ell \;=\; \big\{u\in C^{2m}(\Omega)\,:\,(-\Delta)^m u=0 \text{ in } \Omega\big\}\cap C^\ell(\overline{\Omega}),$$

then there exists a $\vartheta\in W_0^{2m-\ell-1,q}(\Omega)$, with $1<q<n/(n-1)$, satisfying $(-\Delta)^m\vartheta=\nu$, in the sense of distributions. In particular, ν is the weak* limit of a sequence of the form $\{(-\Delta)^m\psi_k\}_{k\in\mathbb{N}}$, where $\{\psi_k\}_{k\in\mathbb{N}}\subset C_0^\infty(\Omega)$. □

Remark 4.2. *It can be readily seen that, in the two space dimensions, Proposition 1 allows us to avoid the use of the functions 1, $|\boldsymbol{x}|^2,\ldots,|\boldsymbol{x}|^{2m-2}$ in the MFS approximation provided that the fundamental solution of the Laplacian is suitably rescaled. In particular, if $\overline{\Omega}'\subset D_R$, where D_R is a disk of radius R, then linear combinations of the form*

$$u_N(\boldsymbol{x}) \;=\; \sum_{j=1}^N c_j^1\,\varphi_R(\boldsymbol{x}-\boldsymbol{y}_j)+\sum_{k=2}^m\sum_{j=1}^N c_j^k\,e_k(\boldsymbol{x}-\boldsymbol{y}_j),$$

where φ_R is given by (3.6) and $\{\boldsymbol{y}_j\}\subset\partial\Omega'$, are dense in

$$\mathcal{Y}_\ell \;=\; \big\{u\in C^{2m}(\Omega)\,:\,(-\Delta)^m u=0 \text{ in } \Omega\big\}\cap C^\ell(\overline{\Omega}),$$

with respect to the norm of $C^\ell(\overline{\Omega})$.

4.3. **Using Almansi's representation.** Almansi [Alm96] showed that every biharmonic function w in $B_r(\boldsymbol{x}_0)$, the ball of radius r and center $\boldsymbol{x}_0$, can be represented as

$$w(\boldsymbol{x}) \;=\; u_1(\boldsymbol{x})+|\boldsymbol{x}-\boldsymbol{x}_0|^2u_2(\boldsymbol{x}), \tag{4.17}$$

where u_1 and u_2 are harmonic functions in the same ball. In [Alm98], Almansi showed that every $m-$harmonic function w in B_r can be represented as

$$w(\boldsymbol{x}) \;=\; u_1(\boldsymbol{x})+|\boldsymbol{x}-\boldsymbol{x}_0|^2u_2(\boldsymbol{x})+\cdots+|\boldsymbol{x}-\boldsymbol{x}_0|^{2m-2}u_m(\boldsymbol{x}), \tag{4.18}$$

where $u_1,\ldots,u_m$ are harmonic in B_r. Nicolescu [Nic36] proved that Almansi's representation holds even when $B_r(\boldsymbol{x}_0)$ is replaced by a star–shaped domain with center $\boldsymbol{x}_0$. Karageorghis and Fairweather [KF88] used the fundamental solutions of Laplace's operator and Almansi's representation (4.17) in order to approximate the solutions of the biharmonic equation.

The following theorem justifies the approach of [KF88].

Theorem 4. *Let Ω, Ω' be open bounded domains in $\mathbb{R}^n$, $n\ge 3$, with the boundary of Ω satisfying the Segment Condition and Ω' embracing Ω. We further assume that Ω is a star–shaped domain with center $\boldsymbol{x}_0$. Then, for every nonnegative integer ℓ, the space $\mathcal{X}$ of all finite linear combinations of the form*

$$v(\boldsymbol{x}) \;=\; \sum_{k=0}^{m-1}\sum_{j=1}^N c_j^k\,|\boldsymbol{x}-\boldsymbol{x}_0|^{2k}e_1(\boldsymbol{x}-\boldsymbol{y}_j), \tag{4.19}$$

restricted in $\overline{\Omega}$, $\{\boldsymbol{y}_j\}_{j=1}^N\subset\partial\Omega'$, where e_1 is given by (1.4), is dense in

$$\mathcal{Y}_\ell \;=\; \big\{u\in C^{2m}(\Omega)\,:\,\Delta^m u=0 \text{ in } \Omega\big\}\cap C^\ell(\overline{\Omega})$$

with respect to the norm of $C^\ell(\overline{\Omega})$. If $n=2$, then the same density result holds for the sum

$$\mathcal{X}\oplus\big\{c_1+c_2|\boldsymbol{x}|^2+\cdots+c_m|\boldsymbol{x}|^{2m-2}\,:\,c_1,c_2,\ldots,c_m\in\mathbb{R}\big\}.$$

Proof. Let $u \in \mathcal{Y}_\ell$ and v^ε be a linear combination of the form (4.15) such that $|v^\varepsilon - u|_\ell < \varepsilon$ (such an $m-$harmonic function v^ε exists as a consequence of Theorem 3). Let also Ω'' be a star–shaped open domain with center $\boldsymbol{x}_0$ such that

$$\overline{\Omega} \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega'.$$

One way to define such a domain Ω'' is as follows. Let $2d = \operatorname{dist}(\partial\Omega, \mathbb{R}^n \setminus \Omega') > 0$ and

$$\Omega'' = \big\{\boldsymbol{x} \in \mathbb{R}^n : \exists\, \boldsymbol{y} \in \partial\Omega \text{ such that } \boldsymbol{x} - \boldsymbol{x}_0 = \alpha(\boldsymbol{y} - \boldsymbol{x}_0) \text{ for some } \alpha \in (0, 1+d)\big\}.$$

The $m-$harmonic function v^ε can be expressed as

$$v^\varepsilon(\boldsymbol{x}) = u_1^\varepsilon(\boldsymbol{x}) + |\boldsymbol{x}-\boldsymbol{x}_0|^2 u_2^\varepsilon(\boldsymbol{x}) + \cdots + |\boldsymbol{x}-\boldsymbol{x}_0|^{2m-2} u_m^\varepsilon(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega'',$$

where $u_1^\varepsilon, \ldots, u_m^\varepsilon$ are harmonic in Ω''. According to Theorem 1, each of the u_k^ε's, when restricted in $\overline{\Omega}$, can be approximated, with respect to the $|\cdot|_\ell-$norm, by linear combinations of the form $u_k^{\varepsilon,\delta}(\boldsymbol{x}) = \sum_{j=1}^{N_{\varepsilon,\delta}} c_{k,j}\, e_1(\boldsymbol{x}-\boldsymbol{y}_j)$, where $\{\boldsymbol{y}_j\}_{j=1}^{N_{\varepsilon,\delta}} \subset \partial\Omega'$. In particular, these $u_k^{\varepsilon,\delta}$'s can be chosen so that $|v^\varepsilon - v^{\varepsilon,\delta}|_\ell < \delta$, where

$$v^{\varepsilon,\delta}(\boldsymbol{x}) = u_1^{\varepsilon,\delta}(\boldsymbol{x}) + |\boldsymbol{x}-\boldsymbol{x}_0|^2 u_2^{\varepsilon,\delta}(\boldsymbol{x}) + \cdots + |\boldsymbol{x}-\boldsymbol{x}_0|^{2m-2} u_m^{\varepsilon,\delta}(\boldsymbol{x}). \qquad \square$$

4.4. **Approximation by fundamental solutions with singularities in a neighborhood of the pseudo–boundary.** It is noteworthy that we can approximate the solutions of the $m-$harmonic equation $\mathcal{L}\, u = (-\Delta)^m u = 0$ in $\Omega \subset \mathbb{R}^n$, by linear combinations of fundamental solutions e_m of the operator $\mathcal{L} = (-\Delta)^m$, with singularities lying in an open neighborhood of the pseudo–boundary $\partial\Omega'$ of arbitrarily small thickness. This approximation does not include the fundamental solutions e_k, $k = 1, \ldots, m-1$. In particular, if $n = 2$, then the constant functions, and perhaps other polynomial functions, are not required in the approximation. Indeed, if $\delta > 0$, the set

$$\Omega'_\delta = \big\{\boldsymbol{x} \in \mathbb{R}^n : \operatorname{dist}(\boldsymbol{x}, \partial\Omega') < \delta\big\} \tag{4.20}$$

is an open neighborhood $\partial\Omega'$ of thickness $2\delta > 0$. Let u be an arbitrary $m-$harmonic function in Ω which is continuous in $\overline{\Omega}$, let ℓ be a nonnegative integer and u_ε an $m-$harmonic function in Ω'', with $\overline{\Omega}' \subset \Omega''$, for which $\max_{\boldsymbol{x}\in\overline{\Omega}} \big|u_\varepsilon(\boldsymbol{x}) - u(\boldsymbol{x})\big|_\ell < \varepsilon$. Such a u_ε exists due to Theorem 3. If we construct a function $\psi_\delta \in C_0^\infty(\mathbb{R}^n)$, such that

$$\psi_\delta\big|_{\mathbb{R}^n \setminus (\Omega' \cup \Omega'_\delta)} = 0 \quad \text{and} \quad \psi_\delta\big|_{\Omega' \setminus \Omega'_\delta} = 1,$$

and set

$$v_\varepsilon(\boldsymbol{x}) = e_m * (\psi_\delta\, u_\varepsilon)(\boldsymbol{x}) = \int_{\mathbb{R}^n} e_m(\boldsymbol{x}-\boldsymbol{y})\, \psi_\delta(\boldsymbol{y})\, u_\varepsilon(\boldsymbol{y})\, d\boldsymbol{y},$$

then we have $\big(-\Delta\big)^m v_\varepsilon(\boldsymbol{x}) = u_\varepsilon(\boldsymbol{x})$ for all $\boldsymbol{x} \in \overline{\Omega}$. On the other hand,

$$\begin{aligned} \big(-\Delta\big)^m v_\varepsilon(\boldsymbol{x}) &= \big(-\Delta\big)^m \int_{\mathbb{R}^2} e_m(\boldsymbol{y})\, \psi_\delta(\boldsymbol{x}-\boldsymbol{y})\, u_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^m} e_m(\boldsymbol{y})\, \big(-\Delta_{\boldsymbol{x}}\big)^m \big(\psi_\delta(\boldsymbol{x}-\boldsymbol{y})\, u_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\big)\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^m} e_m(\boldsymbol{x}-\boldsymbol{y})\, \big(-\Delta\big)^m \big(\psi_\delta(\boldsymbol{y})\, u_\varepsilon(\boldsymbol{y})\big)\, d\boldsymbol{y}. \end{aligned}$$

Therefore,

$$u_\varepsilon(\boldsymbol{x}) = \int_{\Omega'_\delta} e_m(\boldsymbol{x}-\boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} \quad \text{for every} \quad \boldsymbol{x} \in \overline{\Omega}, \tag{4.21}$$

where $f(\boldsymbol{y}) = \big(-\Delta\big)^m \big(\psi_\delta(\boldsymbol{y})\, u_\varepsilon(\boldsymbol{y})\big) \in C\big(\overline{\Omega}'_\delta\big)$. The left hand side of (4.21) can be approximated by linear combinations of fundamental solutions e_m with singularities in Ω'_δ. We thus have the result:

Theorem 5. *Let Ω, Ω' be as in Theorem 3, ℓ be a nonnegative integer, $\mathcal{Y}_\ell$ as in (4.16), $\delta > 0$ and Ω'_δ as in (4.20). Then the space of linear combinations of fundamental solutions with singularities in Ω'_δ is dense, with respect to the norm of the space $C^\ell(\overline{\Omega})$, in $\mathcal{Y}_\ell$.* □

5. Approximations by fundamental solutions of the modified Helmholtz operator

5.1. The modified Helmholtz equation.

The elliptic partial differential operator $\mathcal{L} = \Delta - \kappa^2$, for $\kappa > 0$, corresponds to the *modified Helmholtz equation*:

$$\mathcal{L}u \;=\; (\Delta - \kappa^2)\, u \;=\; 0, \tag{5.1}$$

and has as fundamental solution the function

$$e_1\big(\boldsymbol{x}, \kappa^2\big) \;=\; \begin{cases} -\dfrac{K_0(\kappa|\boldsymbol{x}|)}{2\pi} & \text{if} \quad n = 2, \\[2ex] -\dfrac{\mathrm{e}^{-\kappa|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} & \text{if} \quad n = 3, \end{cases} \tag{5.2}$$

where $K_0(r)$ is the modified Bessel function of the second kind. In fact, the function $e_1\big(\boldsymbol{x}, \kappa^2\big)$ given by (5.2) is a fundamental solution for $\mathcal{L} = \Delta - \kappa^2$, even in the case $\kappa \in \mathbb{C}$ for $n = 3$ and $\kappa \in \mathbb{C} \smallsetminus \{0\}$ for $n = 2$. The function K_0 is real analytic in $(0, \infty)$, it blows up at $r = 0$ and

$$\frac{d^j}{dr^j} K_0(r) \;\simeq\; \sqrt{\frac{\pi}{2\,r}}\, \mathrm{e}^{-r}, \tag{5.3}$$

for r large and for every nonnegative integer j. (See [AS92, p. 374–378].)

We have the following density result:

Theorem 6. *Let Ω, Ω' be open domains in $\mathbb{R}^n$, $n = 2, 3$, with Ω bounded and satisfying the Segment Condition and Ω' embracing Ω, and let ℓ be a nonnegative integer. Then the space $\mathcal{X}$ of finite linear combinations of the form $\sum_{j=1}^N c_j\, e\big(\boldsymbol{x} - \boldsymbol{y}_j, \kappa^2\big)$, where κ positive, $e_1\big(\cdot, \kappa^2\big)$ is given by (5.2) and $\{\boldsymbol{y}_j\}_{j=1}^N$ lie on $\partial\Omega'$, is dense in*

$$\mathcal{Y}_\ell \;=\; \big\{ u \in C^2(\Omega) \,:\, (\Delta - \kappa^2)\, u = 0 \text{ in } \Omega \big\} \cap C^\ell(\overline{\Omega})$$

with respect to the norm of $C^\ell(\overline{\Omega})$.

Proof. We only need to show that, for every $\nu \in \mathcal{Y}_\ell^\perp \subset \big(C^\ell(\overline{\Omega})\big)'$, the convolution $\vartheta = e_1\big(\cdot, \kappa^2\big) * \nu$, which satisfies $(\Delta - \kappa^2)\, \vartheta = 0$ in $\mathbb{R}^n \smallsetminus \overline{\Omega}$, vanishes in $\mathbb{R}^n \smallsetminus \overline{\Omega}$, and the rest of the proof is a tedious repetition of the proof of Theorem 1. In order to prove that $\vartheta\big|_{\mathbb{R}^n \smallsetminus \overline{\Omega}} = 0$, we need the following facts:

(i) $\lim_{|\boldsymbol{x}| \to \infty} D^\alpha e_1\big(\boldsymbol{x}, \kappa^2\big) = 0$, for every multi–index α, which implies that $\lim_{|\boldsymbol{x}| \to \infty} \vartheta(\boldsymbol{x}) = 0$.

(ii) the maximum principle for the modified Helmholtz equation:

Let V be an open bounded subset of $\mathbb{R}^n$. If $u \in C^2(V)$ satisfies (5.1) in V, then

$$\sup_{\boldsymbol{x} \in V} |u(\boldsymbol{x})| \;=\; \sup_{\boldsymbol{x} \in \partial V} |u(\boldsymbol{x})|.$$

Fact (i) is a consequence of the definition of $e_1\big(\cdot, \kappa^2\big)$ and (5.3). For a proof of fact (ii) see [GT83]. □

5.2. **Approximation of solutions of Poly–Helmholtz equations.** The higher order elliptic equation

$$(\Delta-\kappa_1^2)\cdots(\Delta-\kappa_m^2)\,u \;=\; 0,$$

which is also known as *Poly–Helmholtz equation*, arises from $m-$porosity media as well as from $m-$layered aquifer systems. See [CAO94] and references therein.

5.2.1. *Construction of fundamental solutions.* Following the technique of Trèves [Trè66] (see also [CAO94]), fundamental solutions of operators of the form $\mathcal{L}=\prod_{j=1}^{m}\big(\Delta-\kappa_j^2\big)$, where $\kappa_1,\ldots,\kappa_m$ are distinct non-negative reals, can be constructed as linear combinations of fundamental solutions of the operators $\mathcal{L}_j=\Delta-\kappa_j^2$, $j=1,\ldots,m$. Let $\varphi_m(\boldsymbol{x})=\sum_{j=1}^{m}\beta_j\,e_1\big(\boldsymbol{x},\kappa_j^2\big)$, where $e_1(\cdot,\kappa^2)$ is given by (5.2) and $\kappa_1^2,\ldots,\kappa_m^2$ distinct. Then

$$\begin{aligned}(\Delta-\kappa_1^2)\,\varphi_m \;&=\; \sum_{j=1}^{m}\beta_j\,(\Delta-\kappa_1^2)\,e_1\big(\cdot,\kappa_j^2\big) \;=\; \sum_{j=1}^{m}c_j\,\Big(\delta+\big(\kappa_j^2-\kappa_1^2\big)\,e_1\big(\cdot,\kappa_j^2\big)\Big)\\ &=\; \Big(\sum_{j=1}^{m}\beta_j\Big)\,\delta+\sum_{j=2}^{m}\beta_j\,\big(\kappa_j^2-\kappa_1^2\big)\,e_1\big(\cdot,\kappa_j^2\big).\end{aligned}$$

We obtain the first equation for the β_j's by setting the coefficient of δ to be equal to zero, i.e., $\sum_{j=1}^{m}\beta_j=0$. After applying the first ℓ factors of the operator $\mathcal{L}$ we get

$$\prod_{j=1}^{\ell}(\Delta-\kappa_j^2)\,\varphi_m \;=\; \left(\sum_{j=\ell}^{m}\beta_j\,\left(\prod_{i=1}^{\ell-1}(\kappa_j^2-\kappa_i^2)\right)\right)\,\delta+\sum_{j=\ell+1}^{m}\beta_j\left(\prod_{i=1}^{\ell}(\kappa_j^2-\kappa_i^2)\right)e_1\big(\cdot,\kappa_j^2\big),$$

and the $\ell^{\text{th}}-$equation is

$$\sum_{j=\ell}^{m}\beta_j\,\left(\prod_{i=1}^{\ell-1}(\kappa_j^2-\kappa_i^2)\right) \;=\; 0.$$

Finally, applying all factors of the operator $\mathcal{L}$ we have

$$\mathcal{L}\,\varphi_m \;=\; \beta_m(\kappa_m^2-\kappa_1^2)\cdots(\kappa_m^2-\kappa_{m-1}^2)\,\delta,$$

and the last equation is $\beta_m(\kappa_m^2-\kappa_1^2)\cdots(\kappa_m^2-\kappa_1^2)=1$. We have thus obtained an upper diagonal system, with unknowns $\beta_1,\ldots,\beta_m$, the solution of which is

$$\beta_j \;=\; \left(\prod_{\substack{\ell=1\\ \ell\neq j}}^{m}(\kappa_j^2-\kappa_\ell^2)\right)^{-1},\quad j=1,\ldots,m.$$

One can easily show that

$$\big(\Delta-\kappa_m^2\big)\,\varphi_m \;=\; \varphi_{m-1}, \tag{5.4}$$

in the sense of distributions, where φ_{m-1} is the fundamental solution of $\big(\Delta-\kappa_1^2\big)\cdots\big(\Delta-\kappa_{m-1}^2\big)$, constructed analogously.

If $m=2$, then $e^{\mathcal{L}}(\boldsymbol{x})=\dfrac{e_1(\boldsymbol{x},\lambda^2)-e_1(\boldsymbol{x},\kappa^2)}{\lambda^2-\kappa^2}$, is a fundamental solution of $\mathcal{L}=(\Delta-\kappa^2)(\Delta-\lambda^2)$, provided $\kappa^2\neq\lambda^2$. Letting $\lambda\to\kappa$, we obtain a fundamental solution of $\mathcal{L}=(\Delta-\kappa^2)^2$, namely,

$$e_2\big(\boldsymbol{x},\kappa^2\big) \;=\; \lim_{\lambda\to\kappa}\frac{e_1\big(\boldsymbol{x},\lambda^2\big)-e_1\big(\boldsymbol{x},\kappa^2\big)}{\lambda^2-\kappa^2} \;=\; \begin{cases}\dfrac{|\boldsymbol{x}|\,K_0'(\kappa|\boldsymbol{x}|)}{4\kappa\pi} & \text{if}\quad n=2,\\[2ex] \dfrac{\mathrm{e}^{-\kappa|\boldsymbol{x}|}}{8\kappa\pi} & \text{if}\quad n=3.\end{cases} \tag{5.5}$$

In particular, when $\lambda = 0$ and $\kappa \neq 0$, the function

$$\varphi^{\mathcal{L}}(\boldsymbol{x}) \;=\; \begin{cases} \dfrac{K_0(\kappa|\boldsymbol{x}|) + \log|\boldsymbol{x}|}{2\pi\kappa^2} & \text{if} \quad n = 2, \\[2ex] \dfrac{\mathrm{e}^{-\kappa|\boldsymbol{x}|} - 1}{4\pi\kappa^2|\boldsymbol{x}|} & \text{if} \quad n = 3, \end{cases}$$

is a fundamental solution of the operator $\mathcal{L} = \Delta^2 - \kappa^2\Delta$.

Fundamental solutions of operators of the form

$$\mathcal{L} \;=\; (\Delta - \kappa_1^2)^{\nu_1} \cdots (\Delta - \kappa_m^2)^{\nu_m} \tag{5.6}$$

can be obtained in a similar fashion. It can be shown that the function (see [Trè66])

$$\varphi^{\mathcal{L}}(\boldsymbol{x}) \;=\; \sum_{j=1}^{m} \frac{1}{(\nu_j - 1)!} \, \frac{\partial^{\nu_j - 1}\big(\beta_j \, e_1(\boldsymbol{x}, \lambda)\big)}{\partial \lambda^{\nu_j - 1}}\bigg|_{\lambda = \kappa_j^2}, \tag{5.7}$$

where

$$\beta_j \;=\; \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m} \frac{1}{(\kappa_j^2 - \kappa_\ell^2)^{\nu_j}}, \quad j = 1, \ldots, m,$$

is a fundamental solution of the operator in (5.6). In particular, the function

$$e_m\big(\boldsymbol{x}, \kappa^2\big) \;=\; \frac{\partial^{m-1} e_1(\boldsymbol{x}, \lambda)}{\partial \lambda^{m-1}}\bigg|_{\lambda = \kappa^2} \tag{5.8}$$

is a fundamental solution of $(\Delta - \kappa^2)^m$ and the following equation

$$(\Delta - \kappa^2)^j \, e_\ell\big(\cdot, \kappa^2\big) \;=\; e_{\ell - j}\big(\cdot, \kappa^2\big), \tag{5.9}$$

is satisfied in the sense of distributions. Note that $\varphi^{\mathcal{L}}$ in (5.7) is a linear combination of the functions

$$e_\ell\big(\cdot; \kappa_j^2\big), \quad j = 1, \ldots, m, \;\; \ell = 1, \ldots, \nu_j.$$

5.2.2. *A density result.* Following the lines of the proof of Theorem 2, we obtain the following result:

Theorem 7. *Let Ω, Ω' be open domains in $\mathbb{R}^n$, $n = 2, 3$, with Ω bounded and satisfying the Segment Condition and Ω' embracing Ω, and let $\ell \in \mathbb{N}$. Further, assume that $0 \leq \kappa_1 < \cdots < \kappa_m$. If $n = 3$ or $k_1 > 0$, then the space $\mathcal{X}$ of finite linear combinations of the form*

$$u^N(\boldsymbol{x}) \;=\; \sum_{i=1}^{N} \sum_{j=1}^{m} c_{ij} \, e_1\big(\boldsymbol{x} - \boldsymbol{y}_i, \kappa_j^2\big),$$

where $e_1\big(\cdot, \kappa^2\big)$ is given by (5.2) and $\boldsymbol{y}_i$, $i = 1, \ldots, N$, lying on $\partial\Omega'$, is dense in

$$\mathcal{Y}_\ell \;=\; \big\{u \in C^{2m}(\Omega) \,:\, \mathcal{L}\, u = 0 \text{ in } \Omega\big\} \cap C^\ell(\overline{\Omega}),$$

with respect to the norm of $C^\ell(\overline{\Omega})$, where $\mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2)$. If $n = 2$ and $k_1 = 0$, the same result holds, provided that the constant functions are included in $\mathcal{X}$.

Proof. Assume first that $\kappa_1 > 0$ or $n = 3$. If $\nu \in \big(C^\ell(\overline{\Omega})\big)'$ annihilates $\mathcal{X}$, then

$$\nu\big(\tau_{\boldsymbol{y}} e_1(\cdot;\kappa_j^2)\big) \;=\; 0,$$

for every $\boldsymbol{y} \in \partial\Omega'$ and $j = 1, \ldots, m$. As in the proof of Theorem 2, let

$$\vartheta_j(\boldsymbol{x}) \;=\; \nu\big(\tau_{\boldsymbol{x}}\varphi_j\big) \;=\; \big(\varphi_j * \nu\big)(\boldsymbol{x}), \quad j = 1, \ldots, m,$$

where φ_j is the fundamental solution of $\mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_j^2)$ constructed so that (5.4) holds. We need to show that $\vartheta_m|_{\mathbb{R}^n \smallsetminus \overline{\Omega}} = 0$, and the rest of the proof is as in Theorem 1. Clearly, (5.4) implies that

$$\big(\Delta - \kappa_j^2\big)\, \vartheta_j \;=\; \vartheta_{j-1}, \quad \text{for } j = 2, \ldots, m, \quad \text{and } \big(\Delta - \kappa_1^2\big)\, \vartheta_1 \;=\; \nu,$$

in the sense of distributions. Also,

$$\vartheta_1\big|_{\partial\Omega'} \;=\; \cdots \;=\; \vartheta_m\big|_{\partial\Omega'} \;=\; 0,$$

since

$$\vartheta_j(\boldsymbol{x}) \;=\; \nu\big(\tau_{\boldsymbol{x}}\varphi_j\big) \;=\; \nu\Big(\sum_{\ell=1}^{j} \beta_\ell\, \tau_{\boldsymbol{x}} e_1(\cdot;\kappa_\ell^2)\Big) \;=\; \sum_{\ell=1}^{j} \beta_\ell\, \nu\Big(\tau_{\boldsymbol{x}} e_1(\cdot;\kappa_\ell^2)\Big) \;=\; 0.$$

As in the proof of Theorem 6, we obtain $\vartheta_1|_{\mathbb{R}^n \smallsetminus \overline{\Omega}} = 0$. Therefore, $(\Delta - \kappa_2^2)\, \vartheta_2 = 0$ in $\mathbb{R}^n \smallsetminus \overline{\Omega}$. Since $\vartheta_2|_{\partial\Omega'} = 0$, then, using once again Theorem 6, we obtain $\vartheta_2|_{\mathbb{R}^n \smallsetminus \overline{\Omega}} = 0$, and, inductively, that $\vartheta_m|_{\mathbb{R}^n \smallsetminus \overline{\Omega}} = 0$.

In $n = 2$ and $\kappa_1 = 0$, then, as in the proof of Theorem 1, $\vartheta_1|_{\mathbb{R}^2 \smallsetminus \overline{\Omega}} = 0$, provided that the constant functions are included in $\mathcal{X}$ and the rest of the proof is identical. $\square$.

Remark 5.1.

(i) *The function $e_m(\cdot;\kappa^2)$ given by (5.8) is a fundamental solution of $\mathcal{L} = (\Delta - \kappa^2)^m$, where $\kappa > 0$. With Ω, Ω' and $\boldsymbol{y}_i$ as in Theorem 7, one can show, following the lines of the proof of Theorem 7, that linear combinations of the form*

$$u^N(\boldsymbol{x}) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{m} c_{ij}\, e_j\big(\boldsymbol{x} - \boldsymbol{y}_i, \kappa^2\big),$$

are dense in $\mathcal{Y}_\ell = \big\{u \in C^{2m}(\Omega) : \mathcal{L}\, u = 0 \text{ in } \Omega\big\} \cap C^\ell(\overline{\Omega})$ with respect to the norm of $C^\ell(\overline{\Omega})$.

(ii) *More generally, the fundamental solution of the operator*

$$\mathcal{L} \;=\; (\Delta - \kappa_1^2)^{\nu_1} \cdots (\Delta - \kappa_m^2)^{\nu_m},$$

given by (5.7), is a linear combination of the functions $e_\ell(\cdot;\kappa_j^2)$, $j = 1, \ldots, m$, $\ell = 1, \ldots, \nu_j$. A similar density result, with approximations of the form

$$u^N(\boldsymbol{x}) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{m}\sum_{\ell=1}^{\nu_j} c_{ij\ell}\, e_\ell\big(\boldsymbol{x} - \boldsymbol{y}_i, \kappa_j^2\big),$$

holds for the solutions of $\mathcal{L}u = 0$.

(iii) *In particular, if for every $j = 1, \ldots, m$ the linear space $\mathcal{X}^{j,\nu_j}$ is dense in*

$$\mathcal{Y}_\ell^{j,\nu_j} \;=\; \big\{u \in C^{2\nu_j}(\Omega) \,:\, (\Delta - \kappa_j^2)^{\nu_j}\, u = 0 \text{ in } \Omega\big\} \cap C^\ell(\overline{\Omega}),$$

then $\mathcal{X}^{1,\nu_1} \oplus \cdots \oplus \mathcal{X}^{m,\nu_m}$ is dense in $\mathcal{Y}_\ell$ given by

$$\mathcal{Y}_\ell \;=\; \Big\{u \in C^{2\nu_1 + \cdots + 2\nu_m}(\Omega) \,:\, \prod_{j=1}^{m} (\Delta - \kappa_j^2)^{\nu_j}\, u = 0 \text{ in } \Omega\Big\} \cap C^\ell(\overline{\Omega}),$$

whenever $k_1^2, \dots, k_m^2$ *are distinct complex numbers.*

6. Applications of the MFS to elliptic systems

6.1. **The MFS for systems of PDEs.**

6.1.1. *Fundamental solutions of linear systems.* Fundamental solutions are also defined for systems of partial differential equations and the MFS has been applied for the solution of boundary value problems in which the corresponding equations constitute an elliptic system. Let $\mathcal{L}\boldsymbol{u} = \mathbf{0}$ be a $d \times d$ linear homogeneous system with

$$\mathcal{L}\boldsymbol{u} \;=\; \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1d} \\ \vdots & & \vdots \\ \mathcal{L}_{d1} & \cdots & \mathcal{L}_{dd} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \;=\; \begin{pmatrix} \sum_{j=1}^d \mathcal{L}_{1j} u_j \\ \vdots \\ \sum_{j=1}^d \mathcal{L}_{dj} u_j \end{pmatrix}, \tag{6.1}$$

where $\boldsymbol{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$, $\mathcal{L} = (\mathcal{L}_{ij})_{i,j=1}^d$ and $\mathcal{L}_{ij} = \sum_{|\alpha| \le m} a_\alpha^{ij} D^\alpha$ are scalar partial differential operators in $\mathbb{R}^n$ with constant coefficients. Alternatively, $\mathcal{L} = \sum_{|\alpha| \le m} A_\alpha D^\alpha$, where $A_\alpha = (a_\alpha^{ij})_{i,j=1}^d$ are constant matrices. A fundamental solution of $\mathcal{L}$ is a matrix $E = (e_{ij})_{i,j=1}^d$, where e_{ij} are real–valued functions, smooth in $\mathbb{R}^n \smallsetminus \{0\}$, satisfying $\mathcal{L}E = \delta\, I$, in the sense of distributions, where δ is the Dirac measure with unit mass at the origin and I is the identity matrix in $\mathbb{R}^{d\times d}$. This means that

$$\sum_{j=1}^d \mathcal{L}_{ij}\, e_{jk} \;=\; \begin{cases} \delta & \text{if} \quad i = k, \\ 0 & \text{if} \quad i \neq k, \end{cases} \tag{6.2}$$

or equivalently,

$$\int_{\mathbb{R}^n} E(\boldsymbol{x}-\boldsymbol{y})\, \mathcal{L}^\star \boldsymbol{\psi}(\boldsymbol{y})\, d\boldsymbol{y} \;=\; \boldsymbol{\psi}(\boldsymbol{x}),$$

for every $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{\psi} = (\psi_1, \dots, \psi_d)$ with $\psi_i \in C_0^\infty(\mathbb{R}^n)$, where $\mathcal{L}^\star$ is the adjoint operator of $\mathcal{L}$. If $f_1, \dots, f_d$ are measurable functions and

$$u_i(\boldsymbol{x}) \;=\; \sum_{j=1}^d \int_{\mathbb{R}^n} e_{ij}(\boldsymbol{x}-\boldsymbol{y})\, f_j(\boldsymbol{y})\, d\boldsymbol{y}, \quad i = 1, \dots, d, \tag{6.3}$$

then $\mathcal{L}\boldsymbol{u} = \boldsymbol{f}$, where $\boldsymbol{f} = (f_1, \dots, f_d)$ and $\boldsymbol{u} = (u_1, \dots, u_d)$, provided that the integrals on the right–hand side of (6.3) are meaningful. Formulae (6.3) can be written in vector form as

$$\boldsymbol{u}(\boldsymbol{x}) \;=\; \int_{\mathbb{R}^n} E(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{f}(\boldsymbol{y})\, d\boldsymbol{y}, \tag{6.3$'$}$$

or even simpler as $\boldsymbol{u} = E * \boldsymbol{f}$.

6.1.2. *The MFS formulation.* In the MFS for second order elliptic systems, the components of the approximate solution $\boldsymbol{u}^N = (u_1^N, \dots, u_d^N)$ is a linear combination of the form

$$u_i^N(\boldsymbol{x}) \;=\; \sum_{k=1}^d \sum_{j=1}^N c_j^k\, e_{ik}(\boldsymbol{x}-\boldsymbol{y}_j), \quad i = 1, \dots, d, \tag{6.4}$$

with $\{\boldsymbol{y}_j\}_{j=1}^N$, the singularities, lying on a given pseudo–boundary (see Kupradze and Aleksidze [KA63]). Alternatively, (6.4) can be written as

$$\boldsymbol{u}^N(\boldsymbol{x}) \;=\; \sum_{j=1}^N E(\boldsymbol{x}-\boldsymbol{y}_j)\, \boldsymbol{c}_j, \tag{6.4$'$}$$

where $\boldsymbol{c}_j = (c_j^1, \ldots, c_j^d)$, or equivalently

$$\boldsymbol{u}^N(\boldsymbol{x}) \;=\; \sum_{k=1}^{d}\sum_{j=1}^{N} c_j^k\, \boldsymbol{e}_k(\boldsymbol{x}-\boldsymbol{y}_j), \tag{6.4$''$}$$

with $\boldsymbol{e}_k$, $k = 1, \ldots, d$, the columns of the matrix E.

6.1.3. *Spaces of vector–valued functions and their duals.* In the case of systems, the density results we seek are with respect to the norms of the spaces $C^\ell(\overline{\Omega}\,;\mathbb{R}^d)$ which contain vector–valued functions $\boldsymbol{u} = (u_1, \ldots, u_d) : \overline{\Omega} \to \mathbb{R}^d$, for which $u_i \in C^\ell(\overline{\Omega})$, $i = 1, \ldots, d$. The norm of $C^\ell(\overline{\Omega}\,;\mathbb{R}^d)$ is defined as

$$|\boldsymbol{u}|_\ell \;=\; \max_{i=1,\ldots,d} |u_i|_\ell.$$

Analogously, $C^\ell(\Omega\,;\mathbb{R}^d)$ is the space of vector–valued functions $\boldsymbol{u} = (u_1, \ldots, u_d) : \Omega \to \mathbb{R}^d$, for which $u_i \in C^\ell(\Omega)$, $i = 1, \ldots, d$. If $\boldsymbol{\nu}$ is an element of the dual of $C^\ell(\overline{\Omega}\,;\mathbb{R}^d)$, then $\boldsymbol{\nu}$ can be represented as

$$\boldsymbol{\nu}(\boldsymbol{u}) \;=\; \sum_{i=1}^{d} \nu_i(u_i), \quad \boldsymbol{u} \in C^\ell(\overline{\Omega}\,;\mathbb{R}^d),$$

where $\nu_1, \ldots, \nu_d \in \big(C^\ell(\overline{\Omega})\big)'$.

6.1.4. *Distributions of vector–values functions and their convolutions.* The set of test functions on Ω with values in $\mathbb{R}^d$ is denoted by $\mathscr{D}(\Omega\,;\mathbb{R}^d)$ and its dual by $\mathscr{D}'(\Omega\,;\mathbb{R}^d)$. If $\boldsymbol{T} \in \mathscr{D}'(\Omega\,;\mathbb{R}^d)$, then there exist $T_1, \ldots, T_d \in \mathscr{D}'(\Omega)$ such that

$$\boldsymbol{T}(\boldsymbol{\psi}) \;=\; T_1(\psi_1) + \cdots + T_d(\psi_d), \quad \text{where} \quad \boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathscr{D}(\Omega\,;\mathbb{R}^d).$$

If $\boldsymbol{\nu}$ is a distribution with compact support, and $E = E(\boldsymbol{x})$ is a fundamental solution of the operator $\mathcal{L}$, then the convolution $\boldsymbol{\vartheta} = E * \boldsymbol{\nu}$ defines a distribution in $\mathscr{D}'(\mathbb{R}^n\,;\mathbb{R}^d)$ according to

$$\boldsymbol{\vartheta}(\boldsymbol{\psi}) \;=\; (\boldsymbol{e}_1 * \boldsymbol{\nu})(\psi_1) + \cdots + (\boldsymbol{e}_d * \boldsymbol{\nu})(\psi_d) \;=\; \sum_{i=1}^{d}\sum_{j=1}^{n} \big(e_{ji} * \nu_j\big)(\psi_i),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the columns of E and $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathscr{D}(\mathbb{R}^n\,;\mathbb{R}^d)$. Also, $\mathcal{L}\boldsymbol{\vartheta} = \boldsymbol{\nu}$ in the sense of distributions, i.e.,

$$\big(\mathcal{L}\boldsymbol{\vartheta}\big)(\boldsymbol{\psi}) \;=\; \boldsymbol{\vartheta}(\mathcal{L}^\star \boldsymbol{\psi}) \;=\; \boldsymbol{\nu}(\boldsymbol{\psi}) \quad \text{for every} \quad \boldsymbol{\psi} \in \mathscr{D}(\mathbb{R}^n\,;\mathbb{R}^d).$$

6.1.5. *Ellipticity in the sense of Agmon–Douglis–Nirenberg.* A partial differential operator of the form $\mathcal{L} = \sum_{|\alpha| \le m} A_\alpha D^\alpha$, where A_α are constant $d \times d$ matrices, is said to be elliptic in the sense of Agmon–Douglis–Nirenberg [ADN64] if their *principal symbol*

$$\sigma(\mathcal{L})(\boldsymbol{\xi}) \;=\; \sum_{|\alpha|=m} \boldsymbol{\xi}^\alpha A_\alpha$$

is a nonsingular matrix for every $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\boldsymbol{\xi}^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. In the rest of this section, we refer to elliptic operators (resp., systems) in the sense of Agmon–Douglis–Nirenberg simply as elliptic operators (resp., systems). The solutions of homogeneous elliptic systems with constant coefficients (more generally, with analytic coefficients) are real analytic (vector valued) functions. (See [Tar95, Example 4.3.4].)

Remark 6.1. *The MFS for systems was introduced in* [KA63]. *Since then, several formulations of the method, in numerous applications, have been used. See* [BK01, FKM03, PS82].

6.2. **A linear model in the theory of elastostatics.**

6.2.1. *The Cauchy–Navier equations.* The governing equations of equilibrium for a homogeneous, isotropic, linear-elastic solid $\Omega \subset \mathbb{R}^3$, in the absence of body forces, are the Cauchy–Navier equations

$$(\lambda+\mu)\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k \partial x_i}\,u_k + \mu\,\frac{\partial^2}{\partial x_k^2}u_i \;=\; 0, \quad i=1,2,3, \quad \text{in } \Omega, \tag{6.5}$$

where u_1, u_2 and u_3 denote the displacements and λ, μ are real constants known as the *Lamé parameters*. The Cauchy–Navier system can be alternatively written as $\Delta^* \boldsymbol{u} = 0$, where

$$\Delta^* \;=\; \mu\,\Delta + (\lambda+\mu)\ \mathrm{grad}\,\mathrm{div}, \tag{6.6}$$

and $\boldsymbol{u} \;=\; (u_1,u_2,u_3)$. Clearly, Δ^* is a self–adjoint operator, i.e., $\big(\Delta^*\big)^\star \;=\; \Delta^*$. System (6.5) is elliptic if $\mu > 0$ and $\lambda+\mu \geq 0$. In order to show this, we need to prove that the principal symbol of the operator Δ^*:

$$\sigma(\Delta^*)(\boldsymbol{\xi}) \;=\; \mu|\boldsymbol{\xi}^2|\,I + (\lambda+\mu)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi}^T,$$

where I is the identity matrix in $\mathbb{R}^{3\times3}$ and $\boldsymbol{\xi}\cdot\boldsymbol{\xi}^T \;=\; (\xi_i\xi_j)_{i,j=1}^3$, is a nonsingular matrix for every $\boldsymbol{\xi} \neq \mathbf{0}$. This follows from the fact that the matrix $I+\varepsilon\,\boldsymbol{\xi}\cdot\boldsymbol{\xi}^T$ is nonsingular for $\varepsilon \geq 0$ since

$$\big(I+\varepsilon\,\boldsymbol{\xi}\cdot\boldsymbol{\xi}^T\big)^{-1} \;=\; I - \frac{\varepsilon}{1+\varepsilon|\boldsymbol{\xi}^2|}\,\boldsymbol{\xi}\cdot\boldsymbol{\xi}^T.$$

Dirichlet boundary conditions

$$u_i \;=\; f_i, \quad i=1,2,3, \quad \text{on } \partial\Omega,$$

guarantee uniqueness for the solutions of (6.5). This is a consequence of Betti's second formula [Kup65]

$$\int_\Omega \boldsymbol{u}\cdot\Delta^*\boldsymbol{u}\,d\boldsymbol{x} \;=\; \int_{\partial\Omega}\boldsymbol{u}\cdot T\,\boldsymbol{\nu}\,ds - \int_\Omega \mathcal{B}(\boldsymbol{u},\boldsymbol{u})\,d\boldsymbol{x}, \tag{6.7}$$

where $\boldsymbol{\nu}$ is the unit exterior normal,

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{u}) \;=\; 2\mu\bigg(2(u_{12}^2+u_{23}^2+u_{31}^2)+\Big(\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}\Big)^2\bigg)+\lambda\,(\mathrm{div}\,\boldsymbol{u})^2,$$

with $u_{k\ell} = \dfrac{1}{2}\left(\dfrac{\partial u_k}{\partial x_\ell}+\dfrac{\partial u_\ell}{\partial x_k}\right)$ and $T \;=\; (\tau_{k\ell})_{k,\ell=1}^3$ is the stress tensor with

$$\tau_{k\ell} \;=\; \begin{cases} \lambda\,\mathrm{div}\,\boldsymbol{u} + 2\mu\,\dfrac{\partial u_k}{\partial x_k} & \text{if} \quad k=\ell, \\ 2\mu\,u_{k\ell} & \text{if} \quad k\neq\ell. \end{cases}$$

Betti's second formula guarantees uniqueness for the exterior problem, provided suitable conditions at infinity are enforced:

Lemma 2. *The exterior boundary value problem for Cauchy–Navier equations,*

$$\begin{cases} \Delta^*\boldsymbol{u} \;=\; 0 & \text{in} \quad \mathbb{R}^3\smallsetminus\overline{\Omega}, \\ \quad \boldsymbol{u} \;=\; \boldsymbol{f} & \text{on} \quad \partial\Omega, \end{cases} \tag{6.8}$$

where Ω in an open bounded domain in $\mathbb{R}^3$ and $\boldsymbol{f} \;=\; (f_1,f_2,f_3)$, enjoys uniqueness provided the solution satisfies the following conditions at infinity:

(*i*) $\lim_{|\boldsymbol{x}|\to\infty}\boldsymbol{u}(\boldsymbol{x}) = 0$ *and*

(*ii*) $\varrho\dfrac{\partial\boldsymbol{u}}{\partial\varrho} = \mathcal{O}(1)$ *when* $\varrho = |\boldsymbol{x}|$ *is large.*

Proof. Replacing in (6.7) the domain Ω by $(\mathbb{R}^3 \smallsetminus \overline{\Omega}) \cap B_R$, where B_R is the ball of radius R centered at the origin, and using conditions (i) and (ii), we obtain, letting $R \to \infty$, that, if $\boldsymbol{u}$ is a solution of (6.8) with zero boundary conditions, then $\int_{\mathbb{R}^3 \smallsetminus \overline{\Omega}} B(\boldsymbol{u}, \boldsymbol{u})\, dx = 0$, which implies that $\boldsymbol{u} \equiv 0$ in $\mathbb{R}^3 \smallsetminus \overline{\Omega}$. □

Remark 6.2. *Betti's second formula guarantees uniqueness, up to additive constants, to the boundary value problem of Cauchy–Navier equations with* natural conditions, *i.e., conditions of Neumann type prescribing the stresses on the boundary, which are of the form*

$$\lambda\,(\operatorname{div} \boldsymbol{u})\, \boldsymbol{\nu} + 2\,\mu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} + \mu\, \boldsymbol{\nu} \times \operatorname{curl} \boldsymbol{u} \;=\; \boldsymbol{p},$$

where $\boldsymbol{\nu}$ is the unit external normal and $\boldsymbol{p}$ is the pressure.

6.2.2. *Approximation by fundamental solutions.* The matrix

$$E(\boldsymbol{x}) \;=\; \big(e_{ij}(\boldsymbol{x})\big)_{i,j=1}^3 \;=\; -\frac{1}{8\pi\mu\,(2\mu+\lambda)} \left(\frac{3\mu+\lambda}{|\boldsymbol{x}|}\, I + \frac{\mu+\lambda}{|\boldsymbol{x}|^3}\, \boldsymbol{x} \cdot \boldsymbol{x}^T \right) \tag{6.9}$$

is a fundamental solution of Δ^*, where $\boldsymbol{x} \cdot \boldsymbol{x}^T \;=\; (x_i x_j)_{i,j=1}^3 \in \mathbb{R}^3$. The expression (6.9) is due to Lord Kelvin (see [Lov44]). For further details and the derivation of (6.9), see [Kyt96].

We have the following density result:

Theorem 8. *Let Ω, Ω' be open domains in $\mathbb{R}^3$, with Ω bounded and satisfying the Segment Condition and Ω' with smooth boundary and embracing Ω. Also, let ℓ be a nonnegative integer. Then the space $\mathcal{X}$ of finite linear combinations of the form $\sum_{j=1}^N E(\boldsymbol{x}-\boldsymbol{y}_j)\, \boldsymbol{c}_j$, where $E(\boldsymbol{x})$ is given by (6.9) and $\{\boldsymbol{y}_j\}_{j=1}^N$ lie on $\partial\Omega'$, is dense in*

$$\mathcal{Y}_\ell \;=\; \big\{ \boldsymbol{u} \in C^2(\Omega\,;\mathbb{R}^3) \text{ satisfying } (6.5) \text{ in } \Omega \big\} \cap C^\ell(\overline{\Omega}\,;\mathbb{R}^3) \tag{6.10}$$

with respect to the norm of $C^\ell(\overline{\Omega}\,;\mathbb{R}^3)$.

Proof. The elements of the dual of $C^\ell(\overline{\Omega}\,;\mathbb{R}^3)$ are represented by linear functionals of the form $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$, where $\nu_1, \nu_2, \nu_3 \in \big(C^\ell(\overline{\Omega})\big)'$, such that

$$\boldsymbol{\nu}(\boldsymbol{u}) \;=\; \nu_1(u_1) + \nu_2(u_2) + \nu_3(u_3),$$

for every $\boldsymbol{u} = (u_1, u_2, u_3) \in C^\ell(\overline{\Omega}\,;\mathbb{R}^3)$. As in the proof of Theorem 1, we need to show that whenever a functional $\boldsymbol{\nu}$ in $\big(C^\ell(\overline{\Omega}\,;\mathbb{R}^3)\big)'$ annihilates $\mathcal{X}$, then $\boldsymbol{\nu}$ annihilates $\mathcal{Y}_\ell$ as well. Assume $\boldsymbol{\nu} \in \big(C^\ell(\overline{\Omega})\big)'$ annihilates $\mathcal{X}$. Then, in particular,

$$\boldsymbol{\nu}\big(\tau_{\boldsymbol{y}} E\, \boldsymbol{c}\big) \;=\; 0, \quad \text{for every} \quad \boldsymbol{y} \in \partial\Omega' \text{ and } \boldsymbol{c} \in \mathbb{R}^3. \tag{6.11}$$

Note that $E\,\boldsymbol{c} = c_1 \boldsymbol{e}_1 + c_2 \boldsymbol{e}_2 + c_2 \boldsymbol{e}_2$, where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the columns of E and $\boldsymbol{c} = (c_1, c_2, c_3)$. Thus (6.11) implies that

$$\boldsymbol{\nu}\big(\tau_{\boldsymbol{y}} \boldsymbol{e}_1\big) \;=\; \boldsymbol{\nu}\big(\tau_{\boldsymbol{y}} \boldsymbol{e}_2\big) \;=\; \boldsymbol{\nu}\big(\tau_{\boldsymbol{y}} \boldsymbol{e}_3\big) \;=\; 0 \quad \text{for every} \quad \boldsymbol{y} \in \partial\Omega'.$$

Equivalently $\big($since $\boldsymbol{e}_i = (e_{1i}, e_{2i}, e_{3i})\big)$ for $i = 1, 2, 3$, we have

$$\vartheta_i(\boldsymbol{y}) \;=\; \nu_1\big(\tau_{\boldsymbol{y}} e_{1i}\big) + \nu_2\big(\tau_{\boldsymbol{y}} e_{2i}\big) + \nu_3\big(\tau_{\boldsymbol{y}} e_{3i}\big) \;=\; 0 \quad \text{for every} \quad \boldsymbol{y} \in \partial\Omega'. \tag{6.12}$$

Clearly, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3)$ is a smooth function and, in fact, $\Delta^* \boldsymbol{\vartheta} = 0$ in $\mathbb{R}^3 \smallsetminus \overline{\Omega}$. It is easy to show that ϑ satisfies conditions (i) and (ii) of Lemma 2. Therefore, Lemma 2 applies and consequently ϑ vanishes in the unbounded component of $\mathbb{R}^3 \smallsetminus \overline{\Omega}'$, and since $\boldsymbol{\vartheta}$ is real analytic in $\mathbb{R}^3 \smallsetminus \overline{\Omega}$, it has to vanish in the whole of $\mathbb{R}^3 \smallsetminus \overline{\Omega}$. Note that, as in the proof of Theorem 1, we need to study separately the unbounded and the bounded connected components of $\mathbb{R}^3 \setminus \overline{\Omega}$. Meanwhile, $\boldsymbol{\vartheta}$ is defined in the whole of $\mathbb{R}^3$ as a distribution ($\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3) = E * \boldsymbol{\nu}$), and satisfies

$$\Delta^* \boldsymbol{\vartheta} \;=\; \Delta^*(E * \boldsymbol{\nu}) \;=\; \boldsymbol{\nu},$$

in the sense of distributions. We now need a lemma analogous to Lemma 1 for Cauchy–Navier system:

Lemma 3. *Let $E = E(\boldsymbol{x})$ be the fundamental solution of Δ^* given by (6.9) and Ω be an open bounded subset of $\mathbb{R}^3$ satisfying the Segment Condition and $\boldsymbol{\nu} \in \big(C^\ell(\overline{\Omega}\,;\mathbb{R}^3)\big)'$. If $\boldsymbol{\vartheta} = E * \boldsymbol{\nu}$ is the convolution of E and $\boldsymbol{\nu}$ and $\operatorname{supp} \boldsymbol{\vartheta} \subset \overline{\Omega}$, then there exists a sequence $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega\,;\mathbb{R}^3)$, such that $\{\Delta^* \boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ converges to $\boldsymbol{\nu}$ in the weak* sense of $\big(C^\ell(\overline{\Omega}\,;\mathbb{R}^3)\big)'$, i.e.,*

$$\boldsymbol{\nu}(\boldsymbol{u}) \;=\; \lim_{k\to\infty} \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \Delta^* \boldsymbol{\psi}_k(\boldsymbol{x})\, d\boldsymbol{x},$$

for every $\boldsymbol{u} \in C^\ell(\overline{\Omega}\,;\mathbb{R}^3)$.

Proof. See Appendix C.

Lemma 1$'$ provides a sequence $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}} \in C_0^\infty(\Omega\,;\mathbb{R}^3)$ for which $\{\Delta^* \boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ converges to $\boldsymbol{\nu}$ in the weak* sense of $C^\ell(\overline{\Omega}\,;\mathbb{R}^3)$. Thus, for $\boldsymbol{u} \in \mathcal{Y}_\ell$, we have

$$\boldsymbol{\nu}(\boldsymbol{u}) \;=\; \lim_{k\to\infty} \int_{\mathbb{R}^n} \Delta^* \boldsymbol{\psi}_k \cdot \boldsymbol{u}\, d\boldsymbol{x} \;=\; \lim_{k\to\infty} \int_{\mathbb{R}^n} \boldsymbol{\psi}_k \cdot \Delta^* \boldsymbol{u}\, d\boldsymbol{x} \;=\; \boldsymbol{0},$$

since Δ^* is self–adjoint. This completes the proof. □

Remark 6.3. *A similar density result is obtainable in the two–dimensional version of the Cauchy–Navier system, where a fundamental solution is given by*

$$E(\boldsymbol{x}) \;=\; -\frac{1}{4\pi\mu(\lambda+2\mu)} \left((\lambda+3\mu)\, \log|\boldsymbol{x}|\, I - \frac{\lambda+\mu}{|\boldsymbol{x}|^2}\, \boldsymbol{x} \cdot \boldsymbol{x}^T \right),$$

provided that the constant functions are included in the space $\mathcal{X}$ of the linear combinations of the columns of E with singularities on the pseudo–boundary.

6.3. **Equations of the static theory of thermo–elasticity.**

6.3.1. *The three–dimensional model.* The displacements $\boldsymbol{u} = (u_1, u_2, u_3)$ and the temperature ϑ of a thermo–elastic medium are described by the system (see [KGBB76])

$$\Delta^* \boldsymbol{u} \;=\; \gamma\, \mathrm{grad}\, \vartheta, \tag{6.13a}$$

$$\Delta\, \vartheta \;=\; 0, \tag{6.13b}$$

where γ is positive constant. Equations (6.13) constitute a 4×4 elliptic system with unknowns $\boldsymbol{u} = (u_1, u_2, u_3)$ and ϑ. It is readily seen that the corresponding Dirichlet problem for bounded domains, in which the displacements and the temperature are prescribed on the boundary, enjoys uniqueness.

A fundamental solution $G = (g_{ij})_{i,j=1}^4$ of (6.13) is given by (see [Ale91])

$$g_{ij}(\boldsymbol{x}) = -\frac{(1-\delta_{i4})(1-\delta_{j4})}{8\pi\mu(\lambda+2\mu)} \left((\lambda+\mu) \frac{x_i x_j}{|\boldsymbol{x}|^3} + (\lambda+3\mu) \frac{\delta_{ij}}{|\boldsymbol{x}|} \right) - \frac{\gamma\, \delta_{j4}(1-\delta_{i4})}{8\pi(\lambda+2\mu)} \frac{x_i}{|\boldsymbol{x}|} - \frac{\delta_{i4}\delta_{j4}}{4\pi|\boldsymbol{x}|}, \tag{6.14}$$

$i, j = 1, 2, 3, 4$, where δ_{ij} is symbol of Kronecker. The matrix G can be alternatively written in a block form as

$$G(\boldsymbol{x}) = \left(\begin{array}{c|c} E(\boldsymbol{x}) & \eta \, \dfrac{\boldsymbol{x}}{|\boldsymbol{x}|} \\ \hline 0 & -e_1(\boldsymbol{x}) \end{array}\right) \tag{6.15}$$

where $E(\boldsymbol{x})$ is the fundamental solution of the operator Δ^* given by (6.9), $\eta = -\dfrac{\gamma}{4\pi(\lambda+2\mu)}$ and e_1 is the fundamental solution of $-\Delta$ given by (1.4).

6.3.2. *Approximation of the solutions of* (6.13). The temperature ϑ is a harmonic function and it can be thus approximated by linear combinations of the form

$$\vartheta^N(\boldsymbol{x}) = \sum_{k=1}^{N} a_k \, e_1(\boldsymbol{x}-\boldsymbol{y}_k),$$

where $\{\boldsymbol{y}_k\}_{k=1}^N$ lie on a prescribed pseudo–boundary $\partial\Omega'$, $\{a_k\}_{k=1}^N$ are real constants. Equations (6.13a) now become

$$\Delta^* \boldsymbol{u} = \gamma \, \mathrm{grad} \sum_{k=1}^{N} a_k \, e_1(\boldsymbol{x}-\boldsymbol{y}_k) = -\frac{\gamma}{2\pi} \sum_{k=1}^{N} a_k \frac{\boldsymbol{x}-\boldsymbol{y}_k}{|\boldsymbol{x}-\boldsymbol{y}_k|^3}, \tag{6.16}$$

which are inhomogeneous. It is straight–forward that

$$\Delta^* \left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) = -2(\lambda+2\mu) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3},$$

which allows us to obtain a particular solution $\boldsymbol{u}_p$ of (6.16):

$$\boldsymbol{u}_p(\boldsymbol{x}) = \frac{\gamma}{2\pi(\lambda+2\mu)} \sum_{k=1}^{N} a_k \frac{\boldsymbol{x}-\boldsymbol{y}_k}{|\boldsymbol{x}-\boldsymbol{y}_k|}.$$

Clearly, $\Delta^*(\boldsymbol{u}-\boldsymbol{u}_p) = 0$, and by virtue of Theorem 8 the difference $\boldsymbol{v} = \boldsymbol{u}-\boldsymbol{u}_p$ can be approximated by linear combinations of the form

$$\boldsymbol{v}^M(\boldsymbol{x}) = \sum_{j=1}^{M} E(\boldsymbol{x}-\boldsymbol{z}_j) \, \boldsymbol{b}_j,$$

where $E(\boldsymbol{x})$ is given by (6.9), the points $\{\boldsymbol{z}_j\}_{j=1}^M$ also lie on $\partial\Omega'$ and $\{\boldsymbol{b}_j\}_{j=1}^M$ are constant vectors in $\mathbb{R}^3$. Altogether we have the following approximate solution:

$$\boldsymbol{u}^{M,N}(\boldsymbol{x}) = \sum_{j=1}^{M} E(\boldsymbol{x}-\boldsymbol{z}_j) \, \boldsymbol{b}_j + \frac{\gamma}{2\pi(\lambda+2\mu)} \sum_{k=1}^{N} a_k \frac{\boldsymbol{x}-\boldsymbol{y}_k}{|\boldsymbol{x}-\boldsymbol{y}_k|},$$

$$\vartheta^{M,N}(\boldsymbol{x}) = \vartheta^N(\boldsymbol{x}) = \sum_{k=1}^{N} a_k \, e_1(\boldsymbol{x}-\boldsymbol{y}_k).$$

If we set $\{\boldsymbol{x}_\ell\}_{\ell=1}^L = \{\boldsymbol{y}_k\}_{k=1}^N \cup \{\boldsymbol{z}_j\}_{j=1}^M$, the vector $\left(u_1^{M,N}, u_2^{M,N}, u_3^{M,N}, \vartheta^{M,N}\right)$ is a linear combination of the columns of the matrices $G(\boldsymbol{x}-\boldsymbol{x}_m)$, $m = 1, \ldots, L$. We have established the following density result:

Theorem 9. *Let Ω, Ω' be open and bounded domains in $\mathbb{R}^3$, with Ω satisfying the Segment Condition and Ω' with smooth boundary and embracing Ω. Also, let ℓ be a nonnegative integer. Then the space $\mathcal{X}$ of finite linear combinations of the form $\sum_{j=1}^N G(\boldsymbol{x}-\boldsymbol{y}_j) \, \boldsymbol{c}_j$, where $G(\boldsymbol{x})$ is given by* (6.15), *$\{\boldsymbol{y}_j\}_{j=1}^N$ lie on $\partial\Omega'$ and $\boldsymbol{c}_j$ are constant vectors in $\mathbb{R}^4$, is dense in*

$$\mathcal{Y}_\ell = \left\{ (u_1, u_2, u_3, \vartheta) \in C^2(\Omega; \mathbb{R}^4) \ \textit{satisfying} \ (6.13) \ \textit{in} \ \Omega \right\} \cap C^\ell(\overline{\Omega}; \mathbb{R}^4)$$

with respect to the norm of $C^\ell(\overline{\Omega}; \mathbb{R}^4)$. ☐

7. Concluding Remarks

We have extended previous density results for solutions of elliptic partial differential equations and elliptic systems by finite linear combinations of their fundamental solutions, the singularities of which lie on a prescribed pseudo–boundary. In particular, we have proved a lemma which allows us to establish the density of linear combinations of fundamental solutions with respect to the norms of the spaces $C^{\ell}(\overline{\Omega})$. A slight modification of this lemma provides $W^{k,p}-$density results, where k is a nonnegative integer and $p \in (1, \infty)$. In our density results, the domains may possess holes and their boundaries are required to satisfy a rather weak condition, the Segment Condition.

Using our approach, we proved that the finite linear combinations of the fundamental solutions of the Laplacian and $m-$harmonic operators are dense, with respect to any $C^{\ell}-$norm. In the case of the two dimensions, we observed that linear combinations of fundamental solutions of Laplace's equation with singularities on a prescribed pseudo–boundary, are not always dense in the space of harmonic functions. However, if the pseudo–boundary is a subset of a unit disk, then such linear combinations are dense in the space of harmonic functions. This fact allows us to propose an alternative MFS formulation with rescaled fundamental solutions. We also propose alternative MFS formulations for the approximation of the solutions of the $m-$harmonic equation which exploit Almansi's representation. Analogous density results are obtained for the solutions of the modified Helmholtz equation and equations of the form $\mathcal{L}u = \left(\Delta - \kappa_1^2\right)^{\nu_1} \cdots \left(\Delta - \kappa_m^2\right)^{\nu_k} u$. Finally, we extended our density results to elliptic systems and obtained similar results for the Cauchy–Navier system and a system in the linear theory of thermo–elastostatics.

The MFS has been applied, with very satisfactory numerical results, to a variety of boundary value problems in which the corresponding density results have not been established yet. Such problems include the classical formulations for the Helmholtz equation ([Ale91]) and the Navier system in the theory of linear elasticity ([Ale91, FKM03, Kup65]). Also, questions regarding the applicability of the MFS for mixed, external and contact problems remain unanswered.

Of particular interest are error estimates of the MFS approximate solutions which provide how fast the error decreases as the number of singularities increases. So far, such estimates are available only for simple geometries and particular distributions of singularities (see [Bog85, Kat89, Kat90, KO88, Smy, SK04b, TSK, UC03]).

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Appendix

A. Equivalent definitions of the embracing pseudo–boundary.

Proposition 2. *Let Ω, Ω' be open bounded subsets of $\mathbb{R}^n$ satisfying the Segment Condition and assume that Ω' embraces Ω according to Definition $4'$. Then Ω' embraces Ω according to Definition 4.*

Proof. Let $R^n \smallsetminus \overline{\Omega} = V \cap W$, where W is open and $\overline{V} \cap W = V \cap \overline{W} = \varnothing$. It is readily proved that $\operatorname{dist}(\overline{V}, \overline{W}) > 0$, since Ω satisfies the Segment Condition. Also, $\partial\Omega = \partial V \cup \partial W \subset \Omega'$, and thus $\partial V \cap \partial\Omega' = \varnothing$. First we show that $V \cap \left(\mathbb{R}^n \smallsetminus \overline{\Omega}'\right) \neq \varnothing$. Otherwise $V \subset \overline{\Omega}'$. Since Ω' satisfies the Segment Condition, it is not hard to show that Ω' is equal to the interior of its closure, i.e., $\Omega' = \left(\overline{\Omega}'\right)^{\circ}$. Consequently, $V \subset \Omega'$. Also, $V \subset \mathbb{R}^n \smallsetminus \overline{\Omega} \subset \mathbb{R}^n \smallsetminus \Omega$ and thus $\overline{V} \subset \overline{\Omega}' \cap \left(\mathbb{R}^n \smallsetminus \Omega\right) = \overline{\Omega}' \smallsetminus \Omega$. Equivalently

$$\partial V \cup V \subset \left(\partial\Omega' \cup \Omega'\right) \smallsetminus \Omega.$$

But $\partial V \cap \partial\Omega' = \varnothing$, and $V \subset \overline{\Omega}'$. Consequently

$$\overline{V} \subset \Omega' \smallsetminus \Omega \subset \overline{V} \cup \overline{W}.$$

However, $\operatorname{dist}(\overline{V}, \overline{W}) > 0$ and thus $\Omega' \smallsetminus \Omega$ contains a closed connected component, which contradicts the assumptions. Therefore $V \cap \left(\mathbb{R}^n \smallsetminus \overline{\Omega}'\right) \neq \varnothing$. Since $\mathbb{R}^n \smallsetminus \overline{\Omega}' \subset \mathbb{R}^n \smallsetminus \overline{\Omega}$, there exists an open connected component V' of $\mathbb{R}^n \smallsetminus \overline{\Omega}'$ such that $V' \subset V$. Also, $V' \subset \mathbb{R}^n \smallsetminus \overline{\Omega}' \subset \mathbb{R}^n \smallsetminus \Omega'$ and since the last is closed, then $\overline{V}' \subset \mathbb{R}^n \smallsetminus \Omega'$. Finally, $\overline{V}' \subset \mathbb{R}^n \smallsetminus \Omega' \subset \mathbb{R}^n \smallsetminus \overline{\Omega}$ and therefore $\overline{V}'$ has to be a subset of V since it intersects V and it has to be a subset of a connected component of $\mathbb{R}^n \smallsetminus \overline{\Omega}$. □

B. Proof of Lemma 1.

FIRST STEP. *Every signed Borel measure* $\mu \in \mathfrak{M}(\overline{\Omega})$ *defines a distribution in* Ω_1*, where* $\overline{\Omega} \subset \Omega_1$*, of the form* $T_\mu = \sum_{|\beta|\le 1}(-1)^{|\beta|} D^\beta v_\beta$*, with* $\{v_\beta\}_{|\beta|\le 1} \subset L^q(\mathbb{R}^n)$*, for some* $1 < q < n/(n-1)$.

If $\psi \in \mathscr{D}(\Omega_1)$, then its restriction in $\overline{\Omega}$ belongs to $C^\ell(\overline{\Omega})$, and thus μ defines a linear functional $T_\mu(\psi) = \int_{\overline{\Omega}} \psi\, d\mu$ on $\mathscr{D}(\mathbb{R}^n)$, which is clearly continuous. According to the Sobolev Imbedding Theorem:

If V is an open bounded subset of $\mathbb{R}^n$ satisfying the Cone Condition, $m \geq 1$, an integer, $p \in [1, \infty)$ and $mp > n$, then $W^{m,p}(V) \subset C(\overline{V})$ and there is a $\kappa > 0$, independent of u, such that, for every $u \in W^{m,p}(V)$,

$$\max_{x\in\overline{\Omega}} |u(x)| \;\le\; \kappa\, \|u\|_{m,p}.$$

For a proof see [AF03, Theorem 4.12]. In particular, the Sobolev Imbedding Theorem implies that $W^{1,p}(\Omega_1) \subset C(\overline{\Omega}_1)$, for every $p > n$, provided that Ω_1 satisfies the Cone Condition. Thus a measure $\mu \in \mathfrak{M}(\overline{\Omega})$, which is an element of $\left(C(\overline{\Omega}_1)\right)'$, defines a continuous linear functional on $W^{1,p}(\Omega_1)$. Therefore, μ can be represented as

$$\mu(u) \;=\; \sum_{|\beta|\le 1} \int_{\Omega_1} D^\beta u\, v_\beta\, dx,$$

for suitable $\{v_\beta\}_{|\beta|\le 1} \subset L^q(\Omega_1)$, with $1/p + 1/q = 1$. Thus, for every $\psi \in \mathscr{D}(\overline{\Omega}_1)$, we have

$$\int_{\overline{\Omega}} \psi\, d\mu \;=\; \sum_{|\beta|\le 1} \int_{\Omega_1} D^\beta \psi\, v_\beta\, dx \;=\; \sum_{|\beta|\le 1} \int_{\Omega_1} (-1)^{|\beta|} \psi\, D^\beta v_\beta\, dx,$$

which provides the representation $\mu = \sum_{|\beta|\le 1}(-1)^{|\beta|} D^\beta v_\beta \in W^{-1,q}(\Omega_1)$.

SECOND STEP. *If* $\overline{\Omega} \subset \Omega_1$*, then every bounded linear functional* ν *on* $C^\ell(\overline{\Omega})$ *defines a distribution in* Ω_1 *of the form* $T_\nu = \sum_{|\beta|\le \ell+1}(-1)^{|\beta|} D^\beta v_\beta$*, with* $\{v_\beta\}_{|\beta|\le \ell+1} \subset L^q(\mathbb{R}^n)$ *and* $1 < q < n/(n-1)$.

Clearly, if $\psi \in \mathscr{D}(\Omega_1)$, then its restriction in $\overline{\Omega}$ belongs to $C^\ell(\overline{\Omega})$, and thus ν defines a linear functional on $\mathscr{D}(\Omega_1)$. As explained in Section 2.3.1, there exist $\{\nu_\alpha\}_\alpha \subset \mathfrak{M}(\overline{\Omega})$, such that

$$\nu(\psi) \;=\; \sum_{|\alpha|\le \ell} \int_{\overline{\Omega}} D^\alpha \psi\, d\nu_\alpha \;=\; \sum_{|\alpha|\le \ell} T_{\nu_\alpha}(D^\alpha \psi) \;=\; \sum_{|\alpha|\le \ell} (-1)^\alpha D^\alpha T_{\nu_\alpha}(\psi),$$

which is continuous on $\mathscr{D}(\Omega_1)$ and consequently $\nu \in \mathscr{D}'(\Omega_1)$. Using *First Step*, we finally obtain

$$T_\nu \;=\; \sum_{|\alpha|\leq \ell} (-1)^{|\alpha|}\, D^\alpha T_{\nu_\alpha} \;=\; \sum_{|\beta|\leq \ell+1} (-1)^{|\beta|} D^\beta v_\beta,$$

in the sense of distributions, for suitable $\{v_\beta\}_{|\beta|\leq \ell+1} \subset L^q(\Omega_1)$, for some $1 < q < n/(n-1)$. Thus $\nu \in W^{-\ell-1,q}(\Omega_1)$.

THIRD STEP. *The convolution* $\vartheta = e{*}\nu$ *belongs to* $W^{m-\ell-1,q}_{\text{loc}}(\mathbb{R}^n)$.

This is a consequence of the following standard L^p-regularity result (for a proof see Theorem 7.9.7 in [Hör83] and the discussion that follows).

Weyl's Lemma. *Let $\mathcal{L}$ be an elliptic operator with constant coefficients of order m. If V is an open subset of $\mathbb{R}^n$, $p\in(1,\infty)$ and $u \in \mathscr{D}'(V)$, then $\mathcal{L}u \in L^p_{\text{loc}}(V)$ implies that $u \in W^{m,p}_{\text{loc}}(V)$.*

In our case $\nu = \mathcal{L}\vartheta \in W^{-\ell-1,q}_{\text{loc}}(\mathbb{R}^n)$, with $\vartheta = e{*}\nu$, and $\nu = \sum_{|\beta|\leq \ell+1}(-1)^{|\beta|}D^\beta v_\beta$, where

$$\{v_\beta\}_{|\beta|\leq \ell+1} \;\subset\; L^q(\Omega_1) \;\subset\; L^q(\mathbb{R}^n),$$

and, in particular, $\vartheta = \sum_{|\beta|\leq \ell+1}(-1)^{|\beta|}D^\beta(e{*}v_\beta)$. Using Weyl's Lemma, we obtain $e{*}v_\beta \in W^{m,q}_{\text{loc}}(\mathbb{R}^n)$, since $\mathcal{L}(e{*}v_\beta) = v_\beta$. Therefore, $\vartheta = e{*}\nu \in W^{m-\ell-1,q}_{\text{loc}}(\mathbb{R}^n)$.

FOURTH STEP. *The convolution* $\vartheta = e{*}\nu$ *belongs to* $W^{m-\ell-1,q}_0(\Omega)$.

We already know that $\operatorname{supp}\vartheta \subset \overline{\Omega}$ and that $\vartheta \in W^{m-\ell-1,q}(\Omega)$. If $m \leq \ell+1$, there is nothing to prove, since $W^{k,q}_0(\Omega) = W^{k,q}(\Omega)$, when $k \leq 0$. On the other hand, if $m-\ell-1 > 0$, then what needs to be proved is a consequence of the following result (for a proof see [AF03, Theorem 5.29]).

Characterization of $W^{k,p}_0(V)$ by Exterior Extension. *Let V be an open subset of $\mathbb{R}^n$ satisfying the Segment Condition and $k \geq 1$. Then a function u belongs to $W^{k,p}_0(V)$ if and only if the zero extension of u belongs to $W^{k,p}(\mathbb{R}^n)$.*

FIFTH STEP. *Construction of a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty_0(\Omega)$, such that $\{\mathcal{L}\psi_k\}_{k\in\mathbb{N}}$ converges to ν in the weak* sense of $C^\ell(\overline{\Omega})$.*

Since Ω satisfies the Segment Condition, then for every $\boldsymbol{x} \in \partial\Omega$, there exist a vector $\boldsymbol{\xi}_{\boldsymbol{x}} \in \mathbb{R}^n \smallsetminus \{0\}$ and an open neighborhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$, such that if $\boldsymbol{y} \in U_{\boldsymbol{x}} \cap \overline{\Omega}$ then $\boldsymbol{y} + t\,\boldsymbol{\xi}_{\boldsymbol{x}} \in \Omega$ for every $t \in (0,1)$. Let $V_{\boldsymbol{x}}$ be an open set in $\mathbb{R}^n$ satisfying

$$\boldsymbol{x} \in \overline{V}_{\boldsymbol{x}} \subset U_{\boldsymbol{x}}. \tag{B.1}$$

Since $\partial\Omega$ is compact, there is a finite collection of such neighborhoods $\{V_j\}_{j=1}^J$, covering $\partial\Omega$. Let $\{U_j\}_{j=1}^J$, be the corresponding $U_{\boldsymbol{x}}$'s in (B.1), i.e., $\overline{V}_j \subset U$. The collection $\{V_j\}_{j=1}^J$ becomes an open cover of $\overline{\Omega}$ with the addition of another open set V_0, such that $\overline{V}_0 \subset \Omega$. Let $\{\psi^j\}_{j=0}^J$ be an infinitely differentiable partition of unity corresponding to the covering $\{V_j\}_{j=0}^J$ of $\overline{\Omega}$. Clearly, $\{\psi_j\vartheta\}_{j=0}^J \subset W^{s,q}_0(\Omega)$, where $s = m-\ell-1$. Moreover, the *Characterization of $W^{k,p}_0(V)$ by Exterior Extension*, provides that $\psi^j\vartheta \in W^{s,q}(\mathbb{R}^n)$. We denote by $\tau_{j,\varepsilon}$ the translation operator by $\varepsilon\boldsymbol{\xi}_j$, where $\varepsilon \in [0,1]$, i.e.,

$$\big(\tau_{j,\varepsilon}\circ w\big)(\boldsymbol{x}) \;=\; \begin{cases} w(\boldsymbol{x}+\varepsilon\boldsymbol{\xi}_j) & \text{if} \quad j = 1,\ldots,J, \\ w(\boldsymbol{x}) & \text{if} \quad j = 0. \end{cases}$$

We also define $\psi^j_\varepsilon = \tau_{j,\varepsilon} \circ \psi^j$ and $\vartheta_{j,\varepsilon} = \tau_{j,\varepsilon} \circ (\psi_j \vartheta)$.

ASSERTION. *For every* $\varepsilon \in (0,1)$,

$$\delta \;=\; \delta(\varepsilon) \;=\; \min_{1\le j\le J} \operatorname{dist}\bigl(\partial\Omega, \tau_{j,\varepsilon}\bigl[\overline{\Omega}\cap V_j\bigr]\bigr) \;>\; 0. \tag{B.2}$$

PROOF OF THE ASSERTION. If $\operatorname{dist}(\partial\Omega, \tau_{j,\varepsilon}[\overline{\Omega}\cap V_j]) = 0$, for some $j = 1,\dots,J$, then there would be sequences $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}} \subset \tau_{j,\varepsilon}[\overline{\Omega}\cap V_j]$ and $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}} \subset \partial\Omega$, such that $|\boldsymbol{x}_k - \boldsymbol{y}_k| \to 0$. The sequence $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}}$ would have a convergent subsequence, with limit $\boldsymbol{y}^\star \in \tau_{j,\varepsilon}[\overline{V_j}\cap\overline{\Omega}]$. The same would be the limit of the corresponding subsequence of $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$. Thus

$$\boldsymbol{y}^\star \in \tau_{j,\varepsilon}\bigl[\overline{V_j}\cap\overline{\Omega}\bigr]\cap\partial\Omega \;\subset\; \tau_{j,\varepsilon}\bigl[U_j\cap\overline{\Omega}\bigr]\cap\partial\Omega \;\subset\; \Omega\cap\partial\Omega \;=\; \varnothing,$$

which leads to a contradiction.

It is clear that

$$\operatorname{supp}\tau_{j,\varepsilon}\circ\bigl(\psi^j\vartheta\bigr) \;=\; \operatorname{supp}\vartheta_{j,\varepsilon} \;\subset\; \overline{\Omega}^{\delta} \;=\; \bigl\{\boldsymbol{x}\in\Omega \,:\, \operatorname{dist}(\boldsymbol{x},\partial\Omega)\ge\delta\bigr\},$$

where δ is given by (B.2). In particular, $\vartheta_{j,\varepsilon} \in W^{s,q}_0(\Omega)$. Also, if we let $\vartheta_\varepsilon = \sum_{j=0}^J \vartheta_{j,\varepsilon}$, then we have $\operatorname{supp}\vartheta_\varepsilon \subset \overline{\Omega}^{\delta}$ and $\vartheta_\varepsilon \in W^{s,q}_0(\Omega)$. Clearly, $\mathcal{L}\,\vartheta_\varepsilon = \sum_{j=0}^J \mathcal{L}\,\vartheta_{j,\varepsilon}$.

We next show that $\lim_{\varepsilon\searrow 0}\sum_{j=0}^J \mathcal{L}\,\vartheta_{j,\varepsilon} = \mathcal{L}\,\vartheta = \nu$, in the weak* sense of $C^\ell(\overline{\Omega})$.

We observe that

$$\mathcal{L}\,\vartheta_{j,\varepsilon} \;=\; \tau_{j,\varepsilon}\circ\bigl(\mathcal{L}(\psi^j\vartheta)\bigr) \;=\; \tau_{j,\varepsilon}\circ\bigl(\mathcal{L}(\psi^j\vartheta)\bigr).$$

Thus,

$$\lim_{\varepsilon\searrow 0}\mathcal{L}\,\vartheta_\varepsilon \;=\; \lim_{\varepsilon\searrow 0}\sum_{j=0}^J \mathcal{L}\,\vartheta_{j,\varepsilon} \;=\; \lim_{\varepsilon\searrow 0}\sum_{j=0}^J \tau_{j,\varepsilon}\circ\bigl(\psi^j\,\nu\bigr) \;=\; \sum_{j=0}^J \psi^j\,\nu \;=\; \nu,$$

in the weak* sense of $\bigl(C^\ell(\overline{\Omega})\bigr)'$.

Let $\zeta \in C^\infty_0(B_1)$, $\zeta \ge 0$ and $\int_{B_1}\zeta = 1$, where $B_\varrho = \bigl\{\boldsymbol{x}\in\mathbb{R}^n \,:\, |\boldsymbol{x}| < \varrho\bigr\}$, and $\zeta_\eta(\boldsymbol{x}) = \eta^{-n}\zeta(\eta^{-1}\boldsymbol{x})$, for $\eta > 0$. Clearly, $\zeta_\eta \in C^\infty_0(B_\eta)$ and $\int_{B_\eta}\zeta_\eta = 1$. Also, $\zeta_\eta * \vartheta_\varepsilon \in C^\infty(\Omega)$, and for every $\eta < \delta$, we have that

$$\operatorname{supp}\zeta_\eta * \vartheta_\varepsilon \;\subset\; \overline{\Omega}^{\delta} + B_\eta \;\subset\; \Omega,$$

and thus $\zeta_\eta * \vartheta_\varepsilon \in C^\infty_0(\Omega)$. The sequence $\{\psi_k\}_{k\in\mathbb{N}}$ we are seeking can be constructed from functions of the form $\zeta_\eta * \vartheta_\varepsilon$, where $\varepsilon = 1/k$ and η is suitably chosen in the interval $(0,\delta)$, where δ is given by (B.2). Indeed, the sequence

$$\mathcal{L}\,\psi_k \;=\; \mathcal{L}(\zeta_\eta * \vartheta_\varepsilon) \;=\; \zeta_\eta * \mathcal{L}\,\vartheta_\varepsilon,$$

converges to ν in the weak* sense of $C^\ell(\overline{\Omega})$, since for every $\varphi \in C^\ell(\overline{\Omega})$

$$\nu(\varphi) \;=\; \lim_{\varepsilon\searrow 0}\mathcal{L}\vartheta_\varepsilon(\varphi) \;=\; \lim_{\varepsilon\searrow 0}\lim_{\eta\searrow 0}\bigl(\zeta_\eta * \mathcal{L}\vartheta_\varepsilon\bigr)(\varphi). \qquad \square$$

C. Sketch of proof of Lemma 3.

All steps of the proof of this lemma are essentially identical to the corresponding steps of the proof of Lemma 1 except for the assertion in the Third Step which is reformulated as:

ASSERTION. *The convolution* $\boldsymbol{\vartheta} = E*\boldsymbol{\nu}$ *belongs to* $W^{2-\ell-1,q}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$, *for some* q *in* $(1,3/2)$.

PROOF OF THE ASSERTION. We have that $\boldsymbol{\vartheta} = (\vartheta_1,\vartheta_2,\vartheta_3)$, with

$$\vartheta_i \;=\; e_{i1}*\nu_1 + e_{i2}*\nu_2 + e_{i3}*\nu_3,$$

where $E=(e_{ij})_{i,j=1}^3$ and $\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)$. Since $\nu_j \in \big(C^\ell(\overline{\Omega})\big)'$, then, as in the Second Step of the proof of Lemma 1, the ν_j, $j=1,2,3$, are also distributions and can be expressed as

$$\nu_j \;=\; \sum_{|\alpha|\le \ell+1} (-1)^{|\alpha|} D^\alpha v^j_\alpha, \quad j=1,2,3,$$

where $v^j_\alpha \in L^q(\Omega)$, for some q in $(1,3/2)$. In particular, ϑ_i can be expressed as

$$\vartheta_i \;=\; \sum_{j=1}^3 e_{ij} * \Bigg(\sum_{|\alpha|\le\ell+1}(-1)^{|\alpha|}D^\alpha v^j_\alpha\Bigg) \;=\; \sum_{\substack{j=1,2,3\\|\alpha|\le\ell+1}} (-1)^{|\alpha|} D^\alpha\big(e_{ij}*v^j_\alpha\big),$$

in the sense of distributions. Clearly, it suffices to show the following:

If $v\in L^q(\Omega)$, *where* $q\in(1,\infty)$, *then* $e_{ij}*v \in W^{2,q}_{\mathrm{loc}}(\Omega)$, *for every* $i,j=1,2,3$.

This is equivalent to showing that

$$e_{ij}*v,\; \frac{\partial}{\partial x_\mu}\big(e_{ij}*v\big),\; \frac{\partial^2}{\partial x_\mu\partial x_\nu}\big(e_{ij}*v\big) \in L^q_{\mathrm{loc}}(\Omega), \quad \text{for every } i,j,\mu,\nu=1,2,3.$$

We shall establish the above only in the case of the second derivatives. For the lower order derivatives the proof is simpler. Tedious calculations provide that the functions

$$p^{\mu\nu}_{ij}(\boldsymbol{x}) \;=\; |\boldsymbol{x}|^3 \frac{\partial^2}{\partial x_\mu \partial x_\nu} e_{ij}(\boldsymbol{x}), \quad i,j,\mu,\nu=1,2,3,$$

satisfy $p^{\mu\nu}_{ij}(\alpha\boldsymbol{x}) = p^{\mu\nu}_{ij}(\boldsymbol{x})$, for every $\alpha>0$, and

$$\int_{S^2} p^{\mu\nu}_{ij}(\boldsymbol{x})\,d\boldsymbol{x} \;=\; 0,$$

where S^2 is the surface of the unit sphere in $\mathbb{R}^3$. The classical result in the theory of singular integral operators by Calderón–Zygmund [CZ52] provides that for every $q\in(1,\infty)$ and $v\in L^q(\mathbb{R}^3)$, the limit

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\smallsetminus B(0,\varepsilon)} v(\boldsymbol{x}-\boldsymbol{y})\frac{p^{\mu\nu}_{ij}(\boldsymbol{y})}{|\boldsymbol{y}|^3}\,d\boldsymbol{y} \;=\; \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\smallsetminus B(0,\varepsilon)} v(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^2}{\partial x_\mu\partial_\nu} e_{ij}(\boldsymbol{y})\,d\boldsymbol{y} \;=\; \big(\mathcal{P}^{\mu\nu}_{ij}\,v\big)(\boldsymbol{x}),$$

exists in $L^q(\mathbb{R}^3)$ and defines a bounded operator, i.e., there exist $c_q>0$, such that

$$\big|\mathcal{P}^{\mu\nu}_{ij} v\big|_{L^q(\mathbb{R}^3)} \;\le\; c_q |v|_{L^q(\mathbb{R}^3)},$$

for every $v\in L^q(\mathbb{R}^3)$. Using integration by parts and standard distribution manipulations, we obtain that

$$\frac{\partial^2}{\partial x_\mu\partial_\nu}\big(e_{ij}*v\big) \;=\; \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\smallsetminus B(0,\varepsilon)} v(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^2}{\partial x_\mu\partial_\nu} e_{ij}(\boldsymbol{y})\,d\boldsymbol{y},$$

in the sense of distributions. Thus $\frac{\partial^2}{\partial x_\mu\partial_\nu}\big(e_{ij}*v\big)\in L^q(\mathbb{R}^3)$. $\square$

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