APPLICATION OF AN INTEGRAL FORMULA TO CR-SUBMANIFOLDS OF COMPLEX HYPERBOLIC SPACE

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The purpose of this paper is to study n-dimensional compact CR-submanifolds of complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$, and especially to characterize geodesic hypersphere in $\mathrm{CH}^{(n+1)/2}$ by an integral formula.

1. Introduction

Let $\bar{M}$ be a complex space form of constant holomorphic sectional curvature c and let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in $\bar{M}$. Then M has an almost contact metric structure (F,U,u,g) (see Section 2) induced from the canonical complex structure of $\bar{M}$. Hence on an n-dimensional CR-submanifold of $(n-1)$ CR-dimension, we can consider two structures, namely, almost contact structure F and a submanifold structure represented by second fundamental form A. In this point of view, many differential geometers have classified M under the conditions concerning those structures (cf. [3, 5, 8, 9, 10, 11, 12, 14, 15, 16]). In particular, Montiel and Romero [12] have classified real hypersurfaces M of complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ which satisfy the commutativity condition

(C)

$$FA = AF \tag{1.1}$$

by using the S^1-fibration $\pi : H_1^{n+2} \to \mathrm{CH}^{(n+1)/2}$ of the anti-de Sitter space H_1^{n+2} over $\mathrm{CH}^{(n+1)/2}$, and obtained Theorem 4.1 stated in Section 2. We notice that among the model spaces in Theorem 4.1, the geodesic hypersphere is only compact.

In this paper, we will investigate n-dimensional compact CR-submanifold of $(n-1)$ CR-dimension in complex hyperbolic space and provide a characterization of the geodesic hypersphere, which is equivalent to condition (C), by using the following integral formula established by Yano [17, 18]:

$$\int_M \operatorname{div}\{(\nabla_X X)-(\operatorname{div}X)X\} * 1 = \int_M \left\{ \operatorname{Ric}(X,X)+\frac{1}{2}\|\mathcal{L}_X g\|^2 - \|\nabla X\|^2 - (\operatorname{div}X)^2 \right\} * 1 = 0, \tag{1.2}$$

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where X is an arbitrary vector field tangent to M. Our results of the paper are complex hyperbolic versions of those in [6, 15].

2. Preliminaries

Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension isometrically immersed in a complex space form $\bar{M}^{(n+p)/2}(c)$. Denoting by $(J,\bar{g})$ the Kähler structure of $\bar{M}^{(n+p)/2}(c)$, it follows by definition (cf. [5, 6, 8, 9, 13, 16]) that the maximal J-invariant subspace

$$\mathcal{D}_x := T_xM \cap JT_xM \tag{2.1}$$

of the tangent space T_xM of M at each point x in M has constant dimension $(n-1)$. So there exists a unit vector field U_1 tangent to M such that

$$\mathcal{D}_x^{\perp} = \operatorname{Span}\{U_1\}, \quad \forall x \in M, \tag{2.2}$$

where $\mathcal{D}_x^{\perp}$ denotes the subspace of T_xM complementary orthogonal to $\mathcal{D}_x$. Moreover, the vector field ξ_1 defined by

$$\xi_1 := JU_1 \tag{2.3}$$

is normal to M and satisfies

$$JTM \subset TM \oplus \operatorname{Span}\{\xi_1\}. \tag{2.4}$$

Hence we have, for any tangent vector field X and for a local orthonormal basis $\{\xi_1, \xi_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M, the following decomposition in tangential and normal components:

$$JX = FX + u^1(X)\xi_1, \tag{2.5}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1,\dots,p. \tag{2.6}$$

Since the structure $(J,\bar{g})$ is Hermitian and $J^2 = -I$, we can easily see from (2.5) and (2.6) that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively, and that

$$g(FU_\alpha, X) = -u^1(X)\bar{g}(\xi_1, P\xi_\alpha), \tag{2.7}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \bar{g}(P\xi_\alpha, P\xi_\beta), \tag{2.8}$$

where $T_xM^{\perp}$ denotes the normal space of M at x and g the metric on M induced from $\bar{g}$. Furthermore, we also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha}, \tag{2.9}$$

and consequently,

$$g(U_1, X) = u^1(X), \qquad U_\alpha = 0, \quad \alpha = 2,\dots,p. \tag{2.10}$$

Next, applying J to (2.5) and using (2.6) and (2.10), we have

$$F^2X = -X + u^1(X)U_1, \qquad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \tag{2.11}$$

from which, taking account of the skew-symmetry of P and (2.7),

$$u^1(FX) = 0, \qquad FU_1 = 0, \qquad P\xi_1 = 0. \tag{2.12}$$

Thus (2.6) may be written in the form

$$J\xi_1 = -U_1, \qquad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2,\dots,p. \tag{2.13}$$

These equations tell us that (F,g,U_1,u^1) defines an almost contact metric structure on M (cf. [5, 6, 8, 9, 16]), and consequently, $n = 2m+1$ for some integer m.

We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connection on $\bar{M}^{(n+p)/2}(c)$ and M, respectively. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.14}$$

$$\bar{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \quad \alpha = 1,\dots,p, \tag{2.15}$$

for any vector fields X, Y tangent to M. Here $\nabla^\perp$ denotes the normal connection induced from $\bar{\nabla}$ in the normal bundle $TM^\perp$ of M, and h and A_α the second fundamental form and the shape operator corresponding to ξ_α, respectively. It is clear that h and A_α are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha. \tag{2.16}$$

We put

$$\nabla^\perp_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta. \tag{2.17}$$

Then $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Now, using (2.14), (2.15), and (2.17), and taking account of the Kähler condition $\bar{\nabla}J = 0$, we differentiate (2.5) and (2.6) covariantly and compare the tangential and normal parts. Then we can easily find that

$$(\nabla_X F)Y = u^1(Y)A_1X - g(A_1Y,X)U_1, \tag{2.18}$$

$$(\nabla_X u^1)(Y) = g(FA_1X,Y), \tag{2.19}$$

$$\nabla_X U_1 = FA_1X, \tag{2.20}$$

$$g(A_\alpha U_1,X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)\bar{g}(P\xi_\beta,\xi_\alpha), \quad \alpha = 2,\dots,p, \tag{2.21}$$

for any X, Y tangent to M.

In the rest of this paper, *we suppose that the distinguished normal vector field* ξ_1 *is parallel with respect to the normal connection* $\nabla^{\perp}$. Hence (2.17) gives

$$s_{1\alpha} = 0, \quad \alpha = 2,\dots,p, \tag{2.22}$$

which, together with (2.21), yields

$$A_\alpha U_1 = 0, \quad \alpha = 2,\dots,p. \tag{2.23}$$

On the other hand, the ambient manifold $\bar{M}^{(n+p)/2}(c)$ is of constant holomorphic sectional curvature c and consequently, its Riemannian curvature tensor $\bar{R}$ satisfies

$$\bar{R}_{\bar{X}\bar{Y}}\bar{Z} = \frac{c}{4}\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(J\bar{Y},\bar{Z})J\bar{X} - \bar{g}(J\bar{X},\bar{Z})J\bar{Y} - 2\bar{g}(J\bar{X},\bar{Y})J\bar{Z}\} \tag{2.24}$$

for any $\bar{X}$, $\bar{Y}$, $\bar{Z}$ tangent to $\bar{M}^{(n+p)/2}(c)$ (cf. [1, 2, 4, 19]). So, the equations of Gauss and Codazzi imply that

$$\begin{aligned} R_{XY}Z = {} & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\} \\ & + \sum_\alpha \{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\}, \end{aligned} \tag{2.25}$$

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = \frac{c}{4}\{g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1\}, \tag{2.26}$$

for any X, Y, Z tangent to M with the aid of (2.22), where R denotes the Riemannian curvature tensor of M. Moreover, (2.11) and (2.25) yield

$$\operatorname{Ric}(X,Y) = \frac{c}{4}\{(n+2)g(X,Y) - 3u^1(X)u^1(Y)\} + \sum_\alpha \{(\operatorname{tr}A_\alpha)g(A_\alpha X,Y) - g(A_\alpha^2 X,Y)\}, \tag{2.27}$$

$$\rho = \frac{c}{4}(n+3)(n-1) + n^2\|\mu\|^2 - \sum_\alpha \operatorname{tr}{A_\alpha}^2, \tag{2.28}$$

where Ric and ρ denote the Ricci tensor and the scalar curvature, respectively, and

$$\mu = \frac{1}{n}\sum_\alpha (\operatorname{tr}A_\alpha)\xi_\alpha \tag{2.29}$$

is the mean curvature vector (cf. [1, 2, 4, 19]).

3. Codimension reduction of CR-submanifolds of $\mathrm{CH}^{(n+p)/2}$

Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ with constant holomorphic sectional curvature $c = -4$.

Applying the integral formula (1.2) to the vector field U_1, we have

$$\int_M \left\{ \operatorname{Ric}(U_1,U_1) + \frac{1}{2}\|\mathcal{L}_{U_1}g\|^2 - \|\nabla U_1\|^2 - (\operatorname{div}U_1)^2 \right\} * 1 = 0. \tag{3.1}$$

Now we take an orthonormal basis $\{U_1, e_a, e_{a^*}\}_{a=1,\dots,(n-1)/2}$ of tangent vectors to M such that

$$e_{a^*} := Fe_a, \quad a = 1,\dots,\frac{n-1}{2}. \tag{3.2}$$

Then it follows from (2.11) and (2.20) that

$$\operatorname{div} U_1 = \operatorname{tr}(FA_1) = \sum_{a=1}^{(n-1)/2} \{g(FA_1e_a, e_a) + g(FA_1e_{a^*}, e_{a^*})\} = 0. \tag{3.3}$$

Also, using (2.20), we have

$$\|\nabla U_1\|^2 = g(FA_1U_1, FA_1U_1) + \sum_{a=1}^{(n-1)/2} \{g(FA_1e_a, FA_1e_a) + g(FA_1e_{a^*}, FA_1e_{a^*})\}, \tag{3.4}$$

from which, together with (2.11) and (2.12), we can easily obtain

$$\|\nabla U_1\|^2 = \operatorname{tr} A_1^2 - \|A_1U_1\|^2. \tag{3.5}$$

Furthermore, (2.20) yields

$$(\mathcal{L}_{U_1}g)(X,Y) = g(\nabla_X U_1, Y) + g(\nabla_Y U_1, X) = g((FA_1 - A_1F)X, Y), \tag{3.6}$$

and consequently,

$$\|\mathcal{L}_{U_1}g\|^2 = \|FA_1 - A_1F\|^2. \tag{3.7}$$

On the other hand, (2.27) and (2.28) with $c = -4$ yield

$$\operatorname{Ric}(U_1, U_1) = -(n-1) + u^1(A_1U_1)(\operatorname{tr} A_1) - \|A_1U_1\|^2, \tag{3.8}$$

$$\operatorname{tr}(A_1^2) = -\rho - (n+3)(n-1) + n^2\|\mu\|^2 - \sum_{\alpha=2}^{p} \operatorname{tr} A_\alpha{}^2. \tag{3.9}$$

Substituting (3.3), (3.5), (3.7), (3.8), and (3.9) into (3.1), we have

$$\begin{aligned}\int_M \bigg\{ \frac{1}{2}\|FA_1 - A_1F\|^2 + \operatorname{Ric}(U_1, U_1) + \rho - n^2\|\mu\|^2 \\ + \|A_1U_1\|^2 + (n+3)(n-1) + \sum_{\alpha=2}^{p} \operatorname{tr} A_\alpha{}^2 \bigg\} * 1 = 0,\end{aligned} \tag{3.10}$$

or equivalently,

$$\int_M \left\{ \frac{1}{2}\|FA_1 - A_1F\|^2 + u^1(A_1U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1) \right\} * 1 = 0. \tag{3.11}$$

Thus we have the following lemma.

Lemma 3.1. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\left|\left|A_1U_1\right|\right|^2+(n+3)(n-1)\geq 0 \tag{3.12}$$

holds on M, then

$$A_1F=FA_1 \tag{3.13}$$

and $A_\alpha=0$ for $\alpha=2,\dots,p$.

Corollary 3.2. *Let M be a compact orientable real hypersurface of* $\mathrm{CH}^{(n+1)/2}$ *over which the inequality*

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\left|\left|A_1U_1\right|\right|^2+(n+3)(n-1)\geq 0 \tag{3.14}$$

holds. Then M satisfies the commutativity condition (C).

Combining Lemma 3.1 and the codimension reduction theorem proved in [7, Theorem 3.2, page 126], we have the following theorem.

Theorem 3.3. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\left|\left|A_1U_1\right|\right|^2+(n+3)(n-1)\geq 0 \tag{3.15}$$

holds on M, then there exists a totally geodesic complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ *immersed in* $\mathrm{CH}^{(n+p)/2}$ *such that* $M\subset\mathrm{CH}^{(n+1)/2}$. *Moreover M satisfies the commutativity condition (C) as a real hypersurface of* $\mathrm{CH}^{(n+1)/2}$.

Proof. Let

$$N_0(x):=\{\eta\in T_xM^\perp \mid A_\eta=0\} \tag{3.16}$$

and let $H_0(x)$ be the maximal holomorphic subspace of $N_0(x)$, that is,

$$H_0(x)=N_0(x)\cap JN_0(x). \tag{3.17}$$

Then, by means of Lemma 3.1,

$$H_0(x)=N_0(x)=\mathrm{Span}\,\{\xi_2,\dots,\xi_p\}. \tag{3.18}$$

Hence, the orthogonal complement $H_1(x)$ of $H_0(x)$ in $TM^\perp$ is $\mathrm{Span}\{\xi_1\}$ and so, $H_1(x)$ is invariant under the parallel translation with respect to the normal connection and $\dim H_1(x)=1$ at any point $x\in M$. Thus, applying the codimension reduction theorem in [4] proved by Kawamoto, we verify that there exists a totally geodesic complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ immersed in $\mathrm{CH}^{(n+p)/2}$ such that $M\subset\mathrm{CH}^{(n+1)/2}$. Therefore, M can be regarded as a real hypersurface of $\mathrm{CH}^{(n+1)/2}$ which is totally geodesic in $\mathrm{CH}^{(n+p)/2}$. Tentatively, we denote $\mathrm{CH}^{(n+1)/2}$ by M', and by i_1 we denote the immersion of M into M', and by i_2 the totally geodesic immersion of M' into $\mathrm{CH}^{(n+p)/2}$. Then it is clear from (2.14) that

$$\nabla'_{i_1X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi', \tag{3.19}$$

where ∇' is the induced connection on M' from that of $\mathrm{CH}^{(n+p)/2}$, h' the second fundamental form of M in M', and A' the corresponding shape operator to a unit normal vector field ξ' to M in M'. Since $i = i_2 \circ i_1$ and M' is totally geodesic in $\mathrm{CH}^{(n+p)/2}$, we can easily see that (2.15) and (3.19) imply that

$$\xi_1 = i_2\xi', \qquad A_1 = A'. \tag{3.20}$$

Since M' is a holomorphic submanifold of $\mathrm{CH}^{(n+p)/2}$, for any X in TM,

$$J i_2 X = i_2 J' X \tag{3.21}$$

is valid, where J' is the induced Kähler structure on M'. Thus it follows from (2.5) that

$$\begin{aligned} JiX = Ji_2 \circ i_1 X = i_2 J' i_1 X &= i_2\left(i_1 F'X + u'(X)\xi'\right) \\ = iF'X + u'(X) i_2 \xi' &= iF'X + u'(X)\xi_1 \end{aligned} \tag{3.22}$$

for any vector field X tangent to M. Comparing this equation with (2.5), we have $F = F'$ and $u^1 = u'$, which, together with Lemma 3.1, implies that

$$A'F' = F'A'. \tag{3.23}$$

□

4. An integral formula on the model space $M^h_{2p+1,2q+1}(r)$

We first explain the model hypersurfaces of complex hyperbolic space due to Montiel and Romero for later use (for the details, see [12]).

Consider the complex $(n+3)/2$-space $C_1^{(n+3)/2}$ endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{m} dz_j d\bar{z}_j, \qquad \left(m+1 := \frac{n+3}{2}\right), \tag{4.1}$$

where $\bar{z}_k$ denotes the complex conjugate of z_k.

On $C_1^{(n+3)/2}$, we define

$$F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k \bar{w}_k. \tag{4.2}$$

Put

$$H_1^{n+2} = \left\{ z = (z_0, z_1, \dots, z_m) \in C_1^{(n+3)/2} : \langle z, z\rangle = -1 \right\}, \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $C_1^{(n+3)/2}$ induced from g_0. Then it is known that H_1^{n+2}, together with the induced metric, is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space. Moreover, H_1^{n+2} is a principal S^1-bundle over $\mathrm{CH}^{(n+1)/2}$ with projection $\pi : H_1^{n+2} \to \mathrm{CH}^{(n+1)/2}$ which is a Riemannian submersion with fundamental tensor J and time-like totally geodesic fibers.

Given p, q integers with $2p+2q = n-1$ and $r \in R$ with $0 < r < 1$, we denote by $M_{2p+1,2q+1}(r)$ the Lorentz hypersurface of H_1^{n+2} defined by the equations

$$-\left|z_0\right|^2+\sum_{k=1}^{m}\left|z_k\right|^2=-1, \qquad r\left(-\left|z_0\right|^2+\sum_{k=1}^{p}\left|z_k\right|^2\right)=-\sum_{k=p+1}^{m}\left|z_k\right|^2, \tag{4.4}$$

where $z = (z_0, z_1, \dots, z_m) \in C_1^{(n+3)/2}$. In fact, $M_{2p+1,2q+1}(r)$ is isometric to the product

$$H_1^{2p+1}\left(\frac{1}{r-1}\right)\times S^{2q+1}\left(\frac{r}{1-r}\right), \tag{4.5}$$

where $1/(r-1)$ and $r/(1-r)$ denote the squares of the radii and each factor is embedded in H_1^{n+2} in a totally umbilical way. Since $M_{2p+1,2q+1}(r)$ is S^1-invariant, $M^h_{2p+1,2q+1}(r) := \pi(M_{2p+1,2q+1}(r))$ is a real hypersurface of $\mathrm{CH}^{(n+1)/2}$ which is complete and satisfies the condition (C).

As already mentioned in Section 1, Montiel and Romero [12] have classified real hypersurfaces M of $\mathrm{CH}^{(n+1)/2}$ which satisfy the condition (C) and obtained the following classification theorem.

THEOREM 4.1. *Let M be a complete real hypersurface of $\mathrm{CH}^{(n+1)/2}$ which satisfies the condition (C). Then there exist the following possibilities.*

(1) *M has three constant principal curvatures $\tanh\theta$, $\coth\theta$, $2\coth 2\theta$ with multiplicities $2p$, $2q$, 1, respectively, $2p+2q = n-1$. Moreover, M is congruent to $M^h_{2p+1,2q+1}$ $(\tanh^2\theta)$.*

(2) *M has two constant principal curvatures λ_1, λ_2 with multiplicities $n-1$ and 1, respectively. (i) If $\lambda_1 > 1$, then $\lambda_1 = \coth\theta$, $\lambda_2 = 2\coth 2\theta$ with $\theta > 0$, and M is congruent to a geodesic hypersphere $M^h_{1,n}(\tanh^2\theta)$. (ii) If $\lambda_1 < 1$, then $\lambda_1 = \tanh\theta$, $\lambda_2 = 2\coth 2\theta$ with $\theta > 0$, and M is congruent to $M^h_{n,1}(\tanh^2\theta)$. (iii) If $\lambda_1 = 1$, then $\lambda_2 = 2$ and M is congruent to a horosphere.*

Combining Corollary 3.2 and Theorem 4.1, we have the following theorem.

THEOREM 4.2. *Let M be a compact orientable real hypersurface of $\mathrm{CH}^{(n+1)/2}$ over which the inequality*

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\left\|A_1U_1\right\|^2+(n+3)(n-1)\geq 0 \tag{4.6}$$

holds. Then M is congruent to a geodesic hypersphere $M^h_{1,n}(r)$ in $\mathrm{CH}^{(n+1)/2}$.

Combining Theorems 3.3 and 4.2, we have the following theorem.

Theorem 4.3. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\|A_1U_1\|^2+(n+3)(n-1)\geq 0 \tag{4.7}$$

holds on M, then M is congruent to a geodesic hypersphere $M^h_{1,n}(\tanh^2\theta)$ in $\mathrm{CH}^{(n+1)/2}$.

Remark 4.4. As already shown in (3.10) and (3.11), the equality

$$\begin{aligned}&\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\|A_1U_1\|^2+(n+3)(n-1)\\&\quad=u^1(A_1U_1)(\mathrm{tr}\,A_1)-\mathrm{tr}\,A_1^2-(n-1)\end{aligned} \tag{4.8}$$

holds on M. On the other hand, the geodesic hypersphere $M^h_{1,n}(\tanh^2\theta)$ in Theorem 4.1 has constant principal curvatures $\coth\theta$ and $2\coth 2\theta$ with multiplicities $n-1$ and 1, respectively. Hence we can easily verify the equality

$$u^1(A_1U_1)(\mathrm{tr}\,A_1)-\mathrm{tr}\,A_1^2-(n-1)=0, \tag{4.9}$$

and consequently,

$$\mathrm{Ric}\,(U_1,U_1)+\rho-n^2\|\mu\|^2+\|A_1U_1\|^2+(n+3)(n-1)=0 \tag{4.10}$$

on $M^h_{1,n}(\tanh^2\theta)$.

Remark 4.5. If we put $V:=\nabla_{U_1}U_1-(\mathrm{div}\,U_1)U_1$, then it easily follows from (2.11) that $V=FA_1U_1$. Taking account of (3.3), (3.5), (3.7), and (3.8), we obtain

$$\mathrm{div}\,V=\frac{1}{2}\|FA_1-A_1F\|^2+u^1(A_1U_1)(\mathrm{tr}\,A_1)-\mathrm{tr}\,A_1^2-(n-1). \tag{4.11}$$

Hence if the commutativity condition (C) holds on M, then the vector field V is zero since U_1 is a principal vector of A_1, and consequently,

$$u^1(A_1U_1)(\mathrm{tr}\,A_1)-\mathrm{tr}\,A_1^2-(n-1)=0. \tag{4.12}$$

Thus, on n-dimensional CR-submanifold M of $(n-1)$ CR-dimension in a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ over which the commutativity condition C holds, the function $u^1(A_1U_1)$ cannot be zero at any point of M. A real hypersurface of a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ satisfying the commutativity condition (C) cannot be minimal.

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Journal of Applied Mathematics and Decision Sciences

Special Issue on Decision Support for Intermodal Transport

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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

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Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

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- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
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