

Application of approximation and interpolation methods to the examination of entire functions of n complex variables

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Abstract. The object of this paper is to give a characterization of the order and type and of the order and type systems of the entire function $f: C^n \rightarrow C$ by means of the Čebyšev best approximation to f on compact sets $E \subset C^n$ by polynomials.

The methods of proof are based on the properties of the Leja-Siciak extremal function $\Phi(z, E)$.

I. Introduction. Let E be a bounded closed set in the space C^n of n complex variables $z = (z_1, \dots, z_n)$.

We put

$$\|f\|_E = \sup\{|f(z)|: z \in E\}$$

for a function f defined and bounded on E .

Let \mathcal{P}_ν denote the set of polynomials in z of degree $\leq \nu$. Write

$$\mathcal{E}_\nu(f, E) = \inf\{\|f - p\|_E: p \in \mathcal{P}_\nu\}.$$

In the case of one complex variable the following theorems are known (see [2], [9] and [11]):

THEOREM 1. *A function f , defined and bounded on a closed set E with a positive transfinite diameter d , can be continued to an entire function \tilde{f} of order ρ ($0 < \rho < \infty$) and of type σ ($0 < \sigma < \infty$), if and only if*

$$(1) \quad \limsup_{\nu \rightarrow \infty} \nu^{1/\rho} (\mathcal{E}_\nu(f, E))^{1/\nu} = d(e\sigma\rho)^{1/\rho}.$$

THEOREM 2. *If $d > 0$, then the order ρ of \tilde{f} is given by*

$$\rho = \limsup_{\nu \rightarrow \infty} \frac{\nu \ln \nu}{-\ln \mathcal{E}_\nu(f, E)}.$$

The object of this paper is to extend these results to the case $n \geq 2$.

Let B be a complex Banach space with a norm $\|\cdot\|$. Let $f: C^n \rightarrow B$ be an entire function. Write

$$S_f(r) = \sup\{\|f(z)\|: \|z\| = r\},$$

$$\varrho = \limsup_{r \rightarrow \infty} \frac{\ln \ln S_f(r)}{\ln r},$$

$$\sigma = \limsup_{r \rightarrow \infty} \frac{\ln S_f(r)}{r^\sigma}, \quad \text{when } 0 < \varrho < \infty.$$

In connection with Theorem 1 the following question arises: does there exist a number $\bar{d} = \bar{d}(E)$ independent of the entire function f (of order ϱ and type σ) such that

$$\bar{d}(f, E) = \limsup_{\nu \rightarrow \infty} \frac{\nu^{1/\varrho} \sqrt[\nu]{\mathcal{E}_\nu(f, E)}}{(e\sigma\varrho)^{1/\varrho}} = \bar{d}(E).$$

The answer is negative. Indeed, if we take $E = E_1 \times E_2$, where E_j ($j = 1, 2$) is a bounded closed set in the complex z_j -plane with a positive transfinite diameter $\bar{d}_j = \bar{d}(E_j)$ ($\bar{d}_1 \neq \bar{d}_2$) and if

$$f_1(z_1, z_2) = g(z_1), \quad f_2(z_1, z_2) = g(z_2),$$

where g is an entire function of the order ϱ ($0 < \varrho < \infty$) and type σ ($0 < \sigma < \infty$), we have

$$\bar{d}(f_1, E) = \bar{d}_1, \quad \bar{d}(f_2, E) = \bar{d}_2.$$

Therefore $\bar{d}(f, E)$ depends on f .

In the case $n > 1$, we replace the assumption $\bar{d}(E) > 0$ by the assumption of local boundedness in C^n of the extremal function $\Phi(z, E)$. For such a set E Theorem 2 is true, but in Theorem 1 type σ cannot be determined by (1). It appears that the order ϱ of \tilde{f} is given by

$$\varrho = \inf\{\mu > 0: \psi(\mu) < \infty\},$$

where $\psi(\mu) = \limsup_{\nu \rightarrow \infty} \nu^{1/\mu} \sqrt[\nu]{\mathcal{E}_\nu(f, E)}$.

Moreover, if $0 < \varrho < \infty$, then the type of \tilde{f} is:

- (a) minimal, when $\psi(\varrho) = 0$;
- (b) normal, when $0 < \psi(\varrho) < \infty$;
- (c) maximal, when $\psi(\varrho) = \infty$.

Furthermore, the function \tilde{f} is given by

$$\tilde{f}(z) = \lim_{\nu \rightarrow \infty} L_\nu(z), \quad z \in C^n,$$

where L_ν is the ν -th Lagrange interpolation polynomial with nodes at extremal points of E (def. see [10]).

If $n > 1$, the type σ of \tilde{f} cannot be characterized by means of the measure of the Čebyšev best approximation to f on E by polynomials of degree $\leq \nu$ with respect to all variables. So we have to consider the measures $\mathcal{E}_k^*(f, E)$, $k = (k_1, \dots, k_n)$, of the Čebyšev best approximation to f on $E = E_1 \times \dots \times E_n$ by polynomials of the degree $\leq k_j$ with respect to the j -th variable, $j = 1, \dots, n$, where E_j is a bounded closed set with a positive transfinite diameter $d_j = d(E_j)$ in the complex z_j -plane.

In the case of $n \geq 1$ the following theorem will be proved:

THEOREM. *Two systems $\varrho = (\varrho_1, \dots, \varrho_n)$, $\sigma = (\sigma_1, \dots, \sigma_n)$, $\varrho_j, \sigma_j > 0$, $j = 1, \dots, n$ are order and type systems of an entire function f , respectively, if and only if*

$$\limsup_{\min(k_j) \rightarrow \infty} \sqrt[|k|]{\frac{\mathcal{E}_k^*(f, E)}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/e}} = 1,$$

where $|k| = k_1 + \dots + k_n$, $d^k = d_1^{k_1} \dots d_n^{k_n}$ and

$$\left(\frac{k}{e\sigma\varrho}\right)^{k/e} = \left(\frac{k_1}{e\sigma_1\varrho_1}\right)^{k_1/e_1} \dots \left(\frac{k_n}{e\sigma_n\varrho_n}\right)^{k_n/e_n}.$$

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2. Extremal function. As in [10], let us denote by $A_\nu(E)$ the set of all polynomials p of degree $\leq \nu$ such that

$$\|p\|_E \leq 1.$$

We define the extremal function [10]

$$\Phi(z) = \Phi(z, E) = \lim_{\nu \rightarrow \infty} \left\{ \sup \{ |p(z)|^{1/\nu} : p \in A_\nu(E) \} \right\}, \quad z \in C^n.$$

It follows from the definition that $\Phi(z) \geq 1$ for $z \in O^n$ and $\Phi(z) = 1$ for $z \in E$. The following property is known [10].

PROPERTY 1. *If $E = E_1 \times \dots \times E_n$, where E_j is a compact set in the complex z_j -plane, $j = 1, \dots, n$, then*

$$\Phi(z, E) = \max \{ \Phi(z_1, E_1), \dots, \Phi(z_n, E_n) \}.$$

PROPERTY 2. *If $E = \{z \in C^n : \|z\| = r\}$, then $\Phi(z, E) = \max(1, \|z\|/r)$.*

PROPERTY 3. *If E, F ($E \subset F$) are compact sets in C^n , then*

$$\Phi(z, F) \leq \Phi(z, E), \quad z \in C^n.$$

PROPERTY 4. *If $p: C^n \rightarrow B$ (B is a complex Banach space with a norm $\|\cdot\|$) is a polynomial of degree $\leq \nu$, then*

$$(2.1) \quad \|p(z)\| \leq \|p\|_E \Phi^\nu(z), \quad z \in C^n.$$

Property 4 follows from the definition of the extremal function and from the Hahn-Banach theorem [1].

THEOREM 2.1. *If the extremal function $\Phi(z, E)$ is locally bounded in C^n and if $R = \sup\{\|z\|: z \in E\}$, then there exists an $S \geq 1$ such that*

$$(2.2) \quad \max\left(1, \frac{\|z\|}{R}\right) \leq \Phi(z, E) \leq S \max\left(1, \frac{\|z\|}{R}\right), \quad z \in C^n.$$

Proof. The first inequality follows from Property 3.

Let $p_r \in A_r(E)$. Put

$$S = \sup\{\Phi(z, E): z \in B(R)\}.$$

By Property 4 of the extremal function it follows that

$$|p_r(z)|^{1/r} \leq S\Phi(z, B(R)), \quad z \in C^n.$$

Hence and from the definition of Φ we get

$$\Phi(z, E) \leq S\Phi(z, B(R)).$$

3. Order and type of an entire function. Let E be a bounded closed set in C^n such that $\Phi(z, E)$ is locally bounded.

Let

$$E_r = \{z \in C^n: \Phi(z, E) = r\}, \quad r > 1.$$

Write

$$M_E(r, f) = \sup\{\|f(z)\|: z \in E_r\}, \quad r > 1,$$

$$S_f(r) = \sup\{\|f(z)\|: \|z\| = r\}, \quad r > 0$$

for an entire function $f: C^n \rightarrow B$, where B is a complex Banach space. If $E = B(1) = \{z \in C^n: \|z\| = 1\}$, then

$$M_E(r, f) = S_f(r).$$

The order and type of an entire function f will be determined in the same way as in the case of $n = 1$.

DEFINITION 3.1. We call $\rho = \rho(f)$ the *order* of f if

$$(3.1) \quad \rho(f) = \limsup_{r \rightarrow \infty} \frac{\ln \ln S_f(r)}{\ln r}.$$

We have $0 \leq \rho(f) \leq \infty$. If $0 < \rho(f) < \infty$, we say that f is of the *type* $\sigma = \sigma(f)$ if

$$(3.2) \quad \sigma(f) = \limsup_{r \rightarrow \infty} \frac{\ln S_f(r)}{r^\rho}.$$

An entire function f is said to be of

- a) *minimal type* when $\sigma = 0$,
- b) *normal type* when $0 < \sigma < \infty$,
- c) *maximal type* when $\sigma = \infty$.

If in (3.1), (3.2) we replace $S_f(r)$ by $M_E(r, f)$, then we get ρ_E, σ_E , respectively. We call ρ_E and σ_E the E -order and E -type, respectively. It appears that ρ_E is independent of E . Indeed, it follows from Theorem 2.1 that

$$(3.3) \quad S_f\left(r \frac{R}{s}\right) \leq M_E(r, f) \leq S_f(rR).$$

Hence $\rho_E = \rho$ and also we get

$$(3.4) \quad (R/s)^{\rho} \sigma \leq \sigma_E \leq R^{\rho} \sigma.$$

Therefore an entire function f is of minimal (normal, maximal) type if and only if $\sigma_E = 0$ ($0 < \sigma_E < \infty$, $\sigma_E = \infty$).

LEMMA 3.1. *Let $\{p_\nu\}$ be a sequence of polynomials of degree $\leq \nu$, respectively.*

If there exist $\lambda \geq 0$, $\alpha > 0$, $\nu_0 \in N$, and $j \in N$ such that

$$(3.5) \quad \|p_{\nu+j}\|_E \leq \lambda \nu^{-\alpha} \quad \text{for } \nu > \nu_0,$$

then $f(z) = \sum_{\nu=0}^{\infty} p_\nu(z)$, $z \in C^n$, is an entire function, and there exist $A \geq 0$, $B > 0$, $\beta > 0$ such that

$$\|f(z)\| \leq \Phi^\beta(z, E) (A + B \exp(\alpha \Phi^{1/\alpha}(z, E))), \quad z \in C^n.$$

Proof. By (3.5) and Property 4 we have

$$(3.6) \quad \|f(z)\| \leq \Phi^{\nu_1}(z, E) \sum_{\nu < \nu_1} \|p_\nu\|_E + \lambda \Phi^j(z, E) \sum_{\nu \geq \nu_0} \left(\frac{\alpha \Phi^{1/\alpha}(z, E)}{\alpha \nu} \right)^{\alpha \nu}, \quad z \in C^n,$$

where $\nu_1 = \nu_0 + j$.

Denote by m_ν the entire part of $\alpha \nu$. The sequence $\{m_\nu\}$ is an increasing sequence of natural numbers. It follows from the definition that each number of this sequence is repeated no more than k_0 ($k_0 \geq 1/\alpha$) times.

If $\alpha \geq 1$, then from (3.6) we obtain

$$\begin{aligned} \|f(z)\| &\leq \Phi^{\nu_1}(z) \sum_{\nu < \nu_1} \|p_\nu\|_E + \lambda \Phi^j(z) \sum_{\nu = \nu_1}^{\infty} \frac{(\alpha \Phi^{1/\alpha}(z))^{\alpha \nu + 1}}{m_\nu^{\alpha \nu}} \\ &\leq \Phi^{\nu_1}(z) \sum_{\nu < \nu_1} \|p_\nu\|_E + \alpha \lambda (\Phi(z))^{j+1/\alpha} \sum_{\nu=0}^{\infty} \frac{(\alpha \Phi^{1/\alpha}(z))^\nu}{\nu!} \\ &\leq \Phi^\beta(z) (A + B \exp(\alpha \Phi^{1/\alpha}(z))), \quad z \in C^n, \end{aligned}$$

where $\beta = \max(\nu_1, j + 1/\alpha)$, $A = \sum_{\nu < \nu_1} \|p_\nu\|_E$, $B = \alpha \lambda k_0$.

If $0 < \alpha < 1$, we consider two cases:

$$1^\circ \alpha \Phi^{1/\alpha}(z) \geq 1,$$

$$2^\circ 0 < \alpha \Phi^{1/\alpha}(z) < 1,$$

and reason as in the case of $\alpha \geq 1$.

CONCLUSION. *The function f is an entire function of the order $\rho \leq 1/\alpha$.*

LEMMA 3.2. *If there exist $K \geq 0$, $\mu > 0$, $j \in N$, $\lambda \geq 0$ and ν_0 such that*

$$\|p_{\nu+j}\|_E \leq \lambda \left(\frac{ek\mu}{\nu} \right)^{1/\mu} \quad \text{for } \nu > \nu_0,$$

then the function

$$f(z) = \sum_{\nu=1}^{\infty} p_{\nu}(z), \quad z \in C^n$$

is an entire function, and for all $\varepsilon > 0$ there exists an $r_0 = r_0(\varepsilon)$ such that

$$\ln M_E(r, f) \leq (K + \varepsilon)r^\mu \quad \text{for } r > r_0.$$

This Lemma may be proved in the same way as in the case $n = 1$ (see [11], Lemma 3.3).

Put

$$\rho_E^* = \limsup_{\nu \rightarrow \infty} \frac{\nu \ln \nu}{-\ln \mathcal{E}_\nu(f, E)}.$$

We shall prove that $\rho_E^* = \rho$. Let $\rho' > \rho_E^*$. It follows from the definition of limsup that

$$\frac{\nu \ln \nu}{-\ln \mathcal{E}_\nu(f, E)} < \rho' \quad \text{and} \quad \mathcal{E}_\nu(f, E) < 1$$

for a sufficiently large ν , say $\nu > \nu_0$. Hence, by Lemma 3.1 the function f is an entire function of order $\rho \leq \rho_E^*$.

It remains to prove that $\rho \geq \rho_E^*$. Without loss of generality we can assume that $E \subset \{z: |z_j| \leq 1, j = 1, \dots, n\}$.

Let $L_k(z)$, $k = (k_1, \dots, k_n)$ be the Lagrange interpolation polynomial for f with nodes $\eta^{(1)} \times \dots \times \eta^{(n)}$ of degree $\leq k_j$ with respect to the j -th variable, where $\eta^{(j)} = \{\eta_{j0}, \dots, \eta_{jk_j}\}$ is a system of $k_j + 1$ extremal points of the circle $|z_j| \leq 1$.

It can be proved that

$$f(z) - L_k(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=r} \dots \int_{|\zeta_n|=r} \frac{w_k(z)}{w_k(\zeta)} \frac{f(\zeta)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d(\zeta), \quad z \in E,$$

where $w_k(z) = \prod_{j=1}^n (z_j - \eta_{j0}) \dots (z_j - \eta_{jk_j})$, $d\zeta = d\zeta_1 \dots d\zeta_n$.

Hence

$$\|f(z) - L_k(z)\|_E \leq 2^{3nv} \frac{\tilde{M}_f(r)}{r^\nu}, \quad k_1 + \dots + k_n \leq \nu, \quad r > 2,$$

where

$$\tilde{M}_f(r) = \sup\{\|f(z)\|: |z_j| \leq r, j = 1, \dots, n\}.$$

Therefore

$$(3.7) \quad \mathcal{E}_\nu(f, E) \leq 2^{3nv} \frac{\tilde{M}_f(r)}{r^\nu}, \quad \nu = 1, 2, \dots, r > 2.$$

If we take $\varrho_1 < \varrho_E^*$, then for any sequence $\{\nu_j\}_{j \in \mathbb{N}}$

$$\varrho_1 < \frac{\nu_j \ln \nu_j}{-\ln \mathcal{E}_{\nu_j}(f, E)} \quad \text{for } j = 1, 2, \dots$$

From this and by (3.7) we get

$$(3.8) \quad \ln \tilde{M}_f(r) > \nu_j \left(\ln \frac{r}{8^n} - \frac{1}{\varrho_1} \ln \nu_j \right), \quad j = 1, 2, \dots, r > 2.$$

Let $r_j = 8^n (e\nu_j)^{1/\varrho_1}$. Then

$$\ln \tilde{M}_f(r_j) > \frac{\nu_j}{\varrho_1} = A r_j^{\varrho_1}, \quad j = 1, 2, \dots,$$

where

$$A = \frac{8^{-n\varrho_1}}{e\varrho_1}.$$

Hence

$$(3.9) \quad \limsup_{r \rightarrow \infty} \frac{\ln \ln \tilde{M}_f(r)}{\ln r} \geq \varrho_E^*$$

It is clear that

$$B(r) \subset \{z \in C^n: |z_j| \leq r, j = 1, \dots, n\} \subset B(r\sqrt{n}),$$

where $B(r) = \{z \in C^n: \|z\| \leq r\}$. Therefore

$$(3.10) \quad S_f(r) \leq \tilde{M}_f(r) \leq S_f(r\sqrt{n}).$$

Now, it is clear that (3.10) implies

$$(3.11) \quad \limsup_{r \rightarrow \infty} \frac{\ln \ln \tilde{M}_f(r)}{\ln r} = \varrho.$$

From (3.11) and (3.9) it follows that $\varrho \geq \varrho_E^*$.

So finally we have

$$(3.12) \quad \varrho = \varrho_E = \varrho_E^*.$$

If we put

$$\tilde{\sigma} = \limsup_{r \rightarrow \infty} \frac{\ln \tilde{M}(r)}{r^2},$$

then from (3.10) we obtain

$$(3.13) \quad \sigma \leq \tilde{\sigma} \leq \sqrt{n}\sigma.$$

Let us take a function $f: E \rightarrow B$ and write

$$\Gamma(f, E) = \{\mu > 0: \psi(\mu) < \infty\},$$

where $\psi(\mu) = \limsup_{\nu \rightarrow \infty} \nu^{1/\mu} \sqrt[\nu]{\mathcal{E}_\nu(f, E)}$. Put

$$\tilde{\varrho}_E = \begin{cases} \inf \Gamma(f, E), & \text{when } \Gamma(f, E) \neq \emptyset, \\ \infty, & \text{when } \Gamma(f, E) = \emptyset \text{ and } \limsup_{\nu \rightarrow \infty} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} = 0. \end{cases}$$

If $0 < \tilde{\varrho}_E < \infty$, then there exists exactly one number σ_E^* such that

$$\limsup_{\nu \rightarrow \infty} \nu^{1/\tilde{\varrho}_E} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} = (\sigma_E^* \tilde{\varrho}_E)^{1/\tilde{\varrho}_E}.$$

Now, we shall define the ν -th extremal system $\eta^{(\nu)}$ of a compact set $E \subset C^n$ and the ν -th Lagrange interpolation polynomial for $f: E \rightarrow B$ in the same way as in [10].

Let k_{1l}, \dots, k_{nl} , $l = 1, 2, \dots, \nu_*$, denote the sequence of all solutions in non-negative integers of the inequality $k_1 + \dots + k_n \leq \nu$.

Let $p^{(\nu)} = \{p_1, p_2, \dots, p_{\nu_*}\}$ be a system of ν_* points

$$p_i = (z_{1i}, \dots, z_{ni}), \quad i = 1, \dots, \nu_*$$

such that the determinant

$$(3.14) \quad V(p^{(\nu)}) = \det[z_{1i}^{k_{1l}}, \dots, z_{ni}^{k_{nl}}], \quad i, l = 1, \dots, \nu_*$$

is different from zero.

We shall consider a new determinant (3.14), say $V_i(z, p^{(\nu)})$, which corresponds to the system of points

$$\{p_1, \dots, p_{i-1}, z, p_{i+1}, \dots, p_{\nu_*}\},$$

z being an arbitrary point of C^n . Let

$$(3.15) \quad L^{(i)}(z, p^{(\nu)}) = \frac{V_i(z, p^{(\nu)})}{V(p^{(\nu)})}, \quad i = 1, \dots, \nu_*.$$

DEFINITION. If f is a function defined on a set E and if $p^{(\nu)} \subset E$, then

$$L_\nu(z) = \sum_{i=1}^{\nu_*} f(p_i) L^{(i)}(z, p^{(\nu)}), \quad z \in C^n,$$

is called the ν -th Lagrange interpolation polynomial for f with nodes $p^{(\nu)}$.

If E is a compact set in C^n , then there exists a system $\eta^{(\nu)} = \{\eta_{\nu 0}, \dots, \eta_{\nu \nu_*}\} \subset E$, $\nu = 1, 2, \dots$, such that

$$(3.16) \quad V(\eta^{(\nu)}) = \sup\{V(p^{(\nu)}): p^{(\nu)} \subset E\}.$$

DEFINITION. We call $\eta^{(\nu)} \subset E$ satisfying (3.16) the ν -th *extremal system* of E .

Let $E \subset C^n$ be a compact set such that $V(\eta^{(\nu)}) \neq 0$ for $\nu = 1, 2, \dots$

LEMMA 3.3. If $f: E \rightarrow B$ is defined and bounded on E , then

$$(3.17) \quad \mathcal{E}_\nu(f, E) \leq \|f - L_\nu\|_E \leq (1 + \nu_*) \mathcal{E}_\nu(f, E) \quad \text{for } \nu = 1, 2, \dots,$$

where L_ν is the ν -th Lagrange interpolation polynomial for f with nodes a extremal points $\eta^{(\nu)}$ of E .

Proof. Let p_ν be a polynomial of degree $\leq \nu$ and let P_ν be the ν -th Lagrange interpolation polynomial for the function $g(z) = f(z) - p_\nu(z)$.

Since

$$L_\nu(\eta_{\nu j}) = p_\nu(\eta_{\nu j}) + P_\nu(\eta_{\nu j}) \quad \text{for } j = 0, \dots, \nu_*,$$

we have

$$L_\nu(z) = p_\nu(z) + P_\nu(z) \quad \text{for } z \in C^n.$$

Therefore

$$\begin{aligned} \mathcal{E}_\nu(f, E) &\leq \|f - L_\nu\|_E \leq \|f - p_\nu\|_E + \|P_\nu\|_E \\ &\leq \|f - p_\nu\|_E \left(1 + \sum_{j=1}^{\nu_*} \|L^{(j)}\|_E\right) \leq \|f - p_\nu\|_E (1 + \nu_*). \end{aligned}$$

This implies the assertion of Lemma 3.3.

THEOREM 3.1. If $\Phi(z, E)$ is locally bounded in C^n , then $f: E \rightarrow B$ is the restriction to E of an entire function \tilde{f} of the order ϱ if and only if

$$\tilde{\varrho}_E = \varrho.$$

Moreover, if $0 < \varrho < \infty$, then \tilde{f} is of

- a) minimal type when $\psi(\varrho) = 0$,
- b) normal type when $0 < \psi(\varrho) < \infty$,
- c) maximal type when $\psi(\varrho) = \infty$.

Proof. (Divided into 3 parts).

1° We shall prove that if f is an entire function of order $\varrho < \infty$, then $\Gamma(f, E) \neq \emptyset$ and $\varrho = \tilde{\varrho}_E$.

Taking (if necessary) $g(z) = f(az)$ we can assume that $E \subset \{z: \|z\| \leq 1\}$. It follows immediately from (3.7) and (3.11) that for every $\varepsilon_1 > \varepsilon$ there exists an $r_0 > 2$ such that

$$(3.18) \quad \mathcal{E}_\nu(f, E) \leq 8^{n\nu} \frac{e^{r_1^\varrho}}{r^\nu} \quad \text{for } r > r_0.$$

If we take a sufficiently large ν_0 , we have

$$\left(\frac{\nu}{\varrho_1}\right)^{1/\varrho_1} > r_0 \quad \text{for } \nu > \nu_0.$$

Putting $r = (\nu/\varrho_1)^{1/\varrho_1}$ into (3.18) gives

$$\mathcal{E}_\nu(f, E) \leq 8^{n\nu} \left(\frac{e\varrho_1}{\nu}\right)^{\nu/\varrho_1}.$$

From this it follows that $I(f, E) \neq \emptyset$ and $\tilde{\varrho}_E \leq \varrho$.

Suppose that $\tilde{\varrho}_E < \varrho$ and take ϱ_2 such that $\tilde{\varrho}_E < \varrho_2 < \varrho$. For a sufficiently large ν we obtain

$$\nu^{1/\varrho} \sqrt[\nu]{\|f - L_\nu\|_E} < \psi(\varrho_2) + \varepsilon.$$

From Lemma 3.2 and after developing the function f into a series

$$f(z) = L_1(z) + \sum_{\nu=1}^{\infty} (L_{\nu+1}(z) - L_\nu(z)), \quad z \in O^n,$$

it would follow that $\varrho \leq \varrho_2$ and this contradicts the definition of ϱ_2 .

2° We shall prove that if R and S are defined as in Theorem 1 and $0 < \varrho < \infty$, then

$$(3.19) \quad \left(\frac{R}{S}\right)^{\varrho} \sigma \leq \sigma_E \leq \sigma_E^* \leq 8^{n\varrho} \sqrt{n} \sigma.$$

Let $\sigma < \infty$. Then $\tilde{\sigma} < \infty$. Let us fix $K > \tilde{\sigma}$ and $R > 2$ such that

$$\tilde{M}_f(R) < e^{Kr^{\varrho}} \quad \text{for } r > R.$$

Taking sufficiently large ν_0 we have

$$\left(\frac{\nu}{K\varrho}\right)^{1/\varrho} > R \quad \text{for } \nu > \nu_0.$$

Putting $r = (\nu/K\varrho)$ in (3.7) gives

$$\mathcal{E}_\nu(f, E) \leq 8^{n\nu} \left(\frac{eK\varrho}{\nu}\right)^{\nu/\varrho} \quad \text{for } \nu > \nu_0.$$

Hence and from (3.13) we conclude that

$$\limsup_{\nu \rightarrow \infty} \nu^{1/\varrho} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} \leq 8^n (e\tilde{\sigma}\varrho)^{1/\varrho} \leq 8^n (e\sigma\varrho)^{1/\varrho}.$$

Suppose that $\sigma_E^* < \sigma_E$ and take σ_1 such that $\sigma_E^* < \sigma_1 < \sigma_E$. For a sufficiently large ν we would obtain

$$\|f - L_\nu\|_E \leq \left(\frac{e\sigma_1\varrho}{\nu}\right)^{\nu/\varrho}.$$

Hence and from Lemma 3.2 it would follow that $\sigma_E < \sigma_1$ and this contradicts our assumption. Therefore

$$(e\sigma_E \varrho)^{1/e} \leq (e\sigma_E^* \varrho)^{1/e} \leq 8^n n^{1/2e} (e\sigma \varrho)^{1/e}.$$

From this and (3.4) we get

$$\left(\frac{R}{S}\right)^e \sigma \leq \sigma_E \leq \sigma_E^* \leq 8^{ne} \sqrt[n]{n\sigma}.$$

If $\sigma = \infty$, then the above Theorem results from Lemma 3.2. This completes the proof of the sufficient condition.

3° If $\Gamma(f, E) \neq \emptyset$, then for every $\mu \in \Gamma(f, E)$

$$\limsup_{\nu \rightarrow \infty} \nu^{1/\mu} \sqrt[\nu]{\mathcal{E}_\nu(f, E)} = \psi(\mu) < \infty.$$

Hence for every $\varepsilon > 0$ there exists a $\nu_1 = \nu_1(\varepsilon, \mu)$ such that

$$\|f - L_\nu\|_E \leq \left(\frac{(\psi(\mu) + \varepsilon)^\mu}{\nu}\right)^{\nu/\mu} \quad \text{for } \nu > \nu_1.$$

By Lemma 3.2 the function

$$f(z) = L_1(z) + \sum_{\nu=1}^{\infty} (L_{\nu+1}(z) - L_\nu(z)) \quad \text{for } z \in O^n$$

is an entire function of a finite order ϱ and $\tilde{f}(z) = f(z)$ for $z \in E$.

Now, applying parts 1° and 2° of this proof, we can prove the necessary condition.

4. Order and type systems. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ be two systems of n real or complex numbers.

We shall use the following notation:

$$\alpha\beta = (\alpha_1\beta_1, \dots, \alpha_n\beta_n),$$

$$\frac{\alpha}{\beta} = \left(\frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_n}{\beta_n}\right), \quad \text{when } \beta_j \neq 0, j = 1, \dots, n,$$

$$\alpha^\beta = \alpha_1^{\beta_1} \cdot \dots \cdot \alpha_n^{\beta_n},$$

$$|\alpha| = |\alpha_1| + \dots + |\alpha_n|,$$

$$\alpha < \beta \Leftrightarrow \alpha_j < \beta_j \quad \text{for } j = 1, \dots, n,$$

$$\alpha \leq \beta \Leftrightarrow \alpha_j \leq \beta_j \quad \text{for } j = 1, \dots, n,$$

$$\alpha^+ = (|\alpha_1|, \dots, |\alpha_n|).$$

Let B be a complex Banach space and let $f: C^n \rightarrow B$ be an entire function.

The order and type surfaces of f will be determined in the same way as in [7].

Let

$$M_f^*(r) = \sup \{ \|f(z)\| : z^+ \leq r \},$$

where $r = (r_1, \dots, r_n) > (0, \dots, 0)$.

Let $P_f \subset R^n$ be a set of points $\mu \in R^n$ such that for every $\mu \in P_f$ there exists an $r^{(0)} = (r_1^{(0)}, \dots, r_n^{(0)})$ such that

$$(4.1) \quad \ln M_f^*(r) \leq r_1^{\mu_1} + \dots + r_n^{\mu_n} \quad \text{for } r > r^{(0)}.$$

It follows from the definition that the set P_f satisfies the following condition:

$$(W) \quad \begin{cases} 1^\circ & \text{If } \mu \in P_f, \text{ then } \{ \mu' \in R^n : \mu' \geq \mu \} \subset P_f. \\ 2^\circ & \text{If } \mu \in P_f, \text{ then } \{ \mu' \in R^n : \mu' < \mu \} \subset (R^n \setminus P_f). \end{cases}$$

DEFINITION 1. The boundary ∂P_f of the set P_f is called the *order surface of the entire function f* . A point $\rho \in \partial P_f$ is called the *order system of f* .

Let us take $\rho \in \partial P_f$ and denote by $T_f(\rho)$ the set of all $\gamma \in R^n$ such that

$$(4.2) \quad \ln M_f^*(r) \leq \gamma_1 r_1^{\rho_1} + \dots + \gamma_n r_n^{\rho_n} \quad \text{for } r > r^{(1)}, r \in R^n.$$

One can easily check that the set $T_f(\rho)$ satisfies condition (W).

DEFINITION 4.2. The boundary $\partial T_f(\rho)$ of the set $T_f(\rho)$ is called the *type surface of f corresponding to ρ* . A point $\sigma \in \partial T_f(\rho)$ is called the *type system of f corresponding to ρ* .

Let $E = E^{(1)} \times \dots \times E^{(n)}$, where $E^{(j)}$ ($j = 1, \dots, n$) is a compact set of a positive transfinite diameter $d_j = d(E^{(j)})$, and let $\Phi_j(z_j) = \Phi(z_j, E^{(j)})$ be the extremal function of the compact set $E^{(j)}$ ($j = 1, \dots, n$).

Let us denote

$$\begin{aligned} E_{r_j}^{(j)} &= \{ z_j : d_j \Phi_j(z_j) = r_j \}, \quad r_j > d_j, j = 1, \dots, n; \\ E_r &= E_{r_1}^{(1)} \times \dots \times E_{r_n}^{(n)}; \\ M_f(r) &= \sup \{ \|f(z)\| : z \in E_r \} \quad \text{for } r > d. \end{aligned}$$

It can be proved [11] that there exist $r^{(2)} > d = (d_1, \dots, d_n)$, and numbers $\alpha_j > 0$ such that if $z_j \in E_{r_j}^{(j)}$, then

$$(4.3) \quad r_j - \alpha_j \leq |z_j| \leq r_j + \alpha_j \quad \text{for } r > r^{(2)}, j = 1, \dots, n.$$

Now we shall prove that in the definition of the order and type systems of f , $M_f^*(r)$ may be replaced by $M_f(r)$. In order to do this we assume that \tilde{P}_f and $\tilde{T}_f(\rho)$ are subsets of R^n defined by $M_f(r)$ as P_f and $T_f(\rho)$ by $M_f^*(r)$.

Let $\mu \in P_f$. Applying (4.3), we easily find that

$$(4.4) \quad M_f^*(r-a) \leq M_f(r) \leq M_f^*(r+a) \quad \text{for } r > r^{(2)},$$

where $a = (a_1, \dots, a_n)$. Hence

$$\ln M_f^*(r) \leq (r_1 + a_1)^{\mu_1} + \dots + (r_n + a_n)^{\mu_n} \quad \text{for } r > r(\mu).$$

Let us divide the system (μ_1, \dots, μ_n) into two parts $(\mu_{j_1}, \dots, \mu_{j_s})$ and $(\mu_{j_{s+1}}, \dots, \mu_{j_n})$ such that $\mu_{j_l} \leq 0$ (for $l = 1, \dots, s$) and $\mu_{j_q} > 0$ for $q = s+1, \dots, n$. Taking any $\varepsilon > (0, \dots, 0)$ and sufficiently large r , say $r > r^{(3)}$, we have

$$\ln M_f(r) \leq r_{j_1}^{\mu_{j_1}} + \dots + r_{j_s}^{\mu_{j_s}} + r_{j_{s+1}}^{\mu_{j_{s+1}} + \varepsilon_{j_{s+1}}} + \dots + r_{j_n}^{\mu_{j_n} + \varepsilon_{j_n}}.$$

Hence if $\mu \in \tilde{P}_f$, then $\mu + \varepsilon \in P_f$, and so

$$\tilde{P}_f \subset \text{cl}P_f = \text{closure of } P_f.$$

Applying the right-hand side of inequality (4.3) in the same way, we may prove that $P_f \subset \text{cl}\tilde{P}_f$. Now applying property (W) we obtain

$$(4.5) \quad \partial P_f = \partial \tilde{P}_f.$$

Therefore in the definition of the order system of the function f_1 we can take $M_f^*(r)$ instead of $M_f(r)$.

Let $\varrho \in \partial P_f$ and $\gamma \leq \tilde{T}_f(\varrho)$. By (4.3) for sufficiently large r we have

$$\ln M_f^*(r) \leq \gamma_1(r_1 + a_1)^{\varrho_1} + \dots + \gamma_n(r_n + a_n)^{\varrho_n}.$$

Because for every $\gamma_j \neq 0$ we have

$$\lim_{r_j \rightarrow \infty} \frac{(r_j + a_j)^{\varrho_j}}{r_j^{\varrho_j}} = 1,$$

so for every $\varepsilon > (0, \dots, 0)$ there exists an $r^{(5)}$ such that

$$\ln M_f(r) \leq (\gamma_1 + \varepsilon_1)r_1^{\varrho_1} + \dots + (\gamma_n + \varepsilon_n)r_n^{\varrho_n} \quad \text{for } r > r^{(5)}.$$

Therefore $\tilde{T}_f(\varrho) \subset \text{cl}T_f(\varrho)$ and analogously $T_f(\varrho) \subset \text{cl}\tilde{T}_f(\varrho)$. Thus, the conclusion is analogous as in the case of the adjoint order systems of f .

Write the following two remarks resulting from the definitions of the sets P_f , $T_f(\varrho)$ and from Liouville's Theorem.

1° If $\mu = (\mu_1, \dots, \mu_n) \in P_f$ (or $T_f(\varrho)$) and for any j we have $\mu_j \leq 0$, then f does not depend on the j -th variable.

2° If $\varrho \in \partial P_f$ and for any j we have $\varrho_j \leq 0$, then f does not depend on the j -th variable or $T_f(\varrho) = \emptyset$.

With reference to these remarks and considering the order system ϱ and the type system σ corresponding to ϱ , it suffices to confine ourselves

to the case $\varrho_j > 0$, $\sigma \geq 0$ for $j = 1, \dots, n$. In the opposite case our considerations will reduce to the entire function in C^m , $m < n$.

Suppose that $\partial P_f \neq \emptyset$ and take a straight line $l \subset E^n$ defined by

$$l(\tau) = (\tau, \dots, \tau), \quad \tau \in R.$$

By applying property (W) it can be proved that $\partial P_f \cap l$ is a one-point set.

DEFINITION 3. We call $p = p(f)$ the *adjoint order* of f if

$$\tilde{p} = (p, \dots, p) \in \partial P_f.$$

We have $0 \leq p \leq \infty$. If $0 < p < \infty$, we say that f is of the *adjoint type* $q = q(f)$ if

$$\tilde{q} = (q, \dots, q) \in \partial T_f(\tilde{p}).$$

Observe that the adjoint order of the entire function f can be determined as the infimum p of the set of numbers p' for which there exists an $r^{(0)} = (r_1^{(0)}, \dots, r_n^{(0)})$ such that

$$\ln M_f(r) \leq r_1^{p'} + \dots + r_n^{p'} \quad \text{for } r > r^{(0)}$$

and, analogously, the adjoint type of f can be determined as the infimum q of the set of numbers q' for which there exists an $r^{(1)} = (r_1^{(1)}, \dots, r_n^{(1)})$ such that

$$\ln M_f(r) \leq q' r_1^p + \dots + q' r_n^p \quad \text{for } r > r^{(1)}.$$

Let us examine the relationship between the order ϱ and the adjoint order p and between the type σ and the adjoint type q . It will follow that

$$(4.6) \quad \varrho = p$$

and, if $0 < \varrho < \infty$, then

$$(4.7) \quad \frac{\sigma}{n} \leq q \leq \sqrt{n}\sigma.$$

Let $\varrho' > \varrho$. From (3.11) it follows that

$$\ln \tilde{M}_f(s) \leq s^{\varrho'} \quad \text{for a sufficiently large } s.$$

Since $\tilde{M}_f(s) = M_f^*(s, \dots, s)$, so

$$(4.8) \quad \ln M_f^*(r) \leq \ln \tilde{M}_f(r_1) + \dots + \ln \tilde{M}_f(r_n).$$

Hence

$$\ln M_f^*(r) \leq r_1^{\varrho'} + \dots + r_n^{\varrho'} \quad \text{for } r_j > s_0, j = 1, \dots, n.$$

Thus, owing to optional $\varrho' > \varrho$, it follows that $\varrho \geq p$.

To obtain the opposite inequality we fix $p' > p$ and $r^{(0)} = (r_1^{(0)}, \dots, r_n^{(0)})$ so that

$$\ln M_f^*(r) \leq r_1^{p'} + \dots + r_n^{p'} \quad \text{for } r > r^{(0)}.$$

In particular, if $s > \max\{r_1^{(0)}, \dots, r_n^{(0)}\}$, then

$$\ln \tilde{M}_f(s) \leq ns^{p'}.$$

Hence, by a standard argument, $p \geq \varrho$ and so (4.6) is proved.

In order to prove (4.7) let us take $\sigma' > \tilde{\sigma}$, where

$$\tilde{\sigma} = \limsup_{s \rightarrow \infty} \frac{\ln \tilde{M}_f(s)}{s^p}.$$

By the definition of $\tilde{\sigma}$ and (4.8) we have

$$\ln M_f^*(r) \leq \sigma' r_1^p + \dots + \sigma' r_n^p \quad \text{for } r > r^{(1)}.$$

Hence, and by (3.13)

$$q \leq \tilde{\sigma} \leq \sqrt[n]{n\sigma}.$$

On the other hand, if $q' > q$, then for a sufficiently large s

$$\ln \tilde{M}_f(s) \leq nq's^q,$$

so $\tilde{\sigma} \leq nq$. But at the same time by (3.13) we have $\sigma \leq \tilde{\sigma}$. Therefore

$$\frac{\sigma}{n} \leq q \leq \sqrt[n]{n\sigma}.$$

5. Best approximation and interpolation in a set $E = E^{(1)} \times \dots \times E^{(n)}$.

Let $\mathcal{P}_k = \mathcal{P}_k(C^n, B)$, $k = (k_1, \dots, k_n)$ be the set of all polynomials $p: C^n \rightarrow B$ of degree $\leq k_j$ with respect to the j -th variable, respectively.

Let E be a compact set in C^n and let $f: E \rightarrow B$ be a function defined and bounded on E .

Write

$$\mathcal{E}_k^*(f, E) = \inf\{\|f - p\|_E: p \in \mathcal{P}_k\}.$$

Let $E = E^{(1)} \times \dots \times E^{(n)}$, where $E^{(j)}$ ($j = 1, \dots, n$) is a compact set in C containing infinitely many different points.

Let $\eta_j^{(k_j)} = (\eta_{j0}, \dots, \eta_{jk_j})$, $j = 1, \dots, n$, be a system of $k_j + 1$ extremal points of $E^{(j)}$ (see [8]).

Let us write

$$L^{(\mu_j)}(z_j) = L^{(\mu_j)}(z_j, E^{(j)}) = \frac{(z_j - \eta_{j0}) \dots}{(\eta_{j\mu_j} - \eta_{j0}) \dots} \Bigg|_{\mu_j} \frac{\dots (z_j - \eta_{jk_j})}{\dots (\eta_{j\mu_j} - \eta_{jk_j})},$$

where $|_{\mu_j}$ means that the factor μ_j is omitted.

The polynomial

$$L_k(z) = \sum_{\mu_1, \dots, \mu_n=0}^{k_1, \dots, k_n} f(\eta_{1\mu_1}, \dots, \eta_{n\mu_n}) L^{(\mu_1)}(z_1) \dots L^{(\mu_n)}(z_n)$$

is the Lagrange interpolation polynomial for f with nodes $\eta_1^{(k_1)} \times \dots \times \eta_n^{(k_n)}$ of degree $\leq k_j$ with respect to the j -th variable.

The inequality

$$(5.1) \quad \mathcal{E}_k^*(f, E) \leq \|f - L_k\|_E \left(1 + \prod_{j=1}^n (k_j + 1)\right) \mathcal{E}_k^*(f, E)$$

can be proved in the same way as (3.17).

Now we shall prove the Lemma, which together with inequality (5.1) and Property 4 of the extremal function (see Section 2) will be of primary importance in successive investigations.

LEMMA 5.1. *Let $k^{(\nu)} = (k_1^{(\nu)}, \dots, k_n^{(\nu)})$, $\nu = 1, 2, \dots$, be an increasing sequence such that $\min\{k_j^{(\nu)} : j = 1, 2, \dots, n\} \rightarrow \infty$, when $\nu \rightarrow \infty$ and $k_j^{(\nu)}$ are natural numbers.*

Let $E = E^{(1)} \times \dots \times E^{(n)}$, where $E^{(j)}$ ($j = 1, \dots, n$) is a compact set with a positive transfinite diameter $d_j = d(E^{(j)})$ in the complex z_j -plane and let $p_k \in \mathcal{P}_k$, $k = (k_1, \dots, k_n)$ be polynomials such that

$$p_k(z) \equiv 0, \quad \text{when } k \notin \{k^{(\nu)}\}.$$

If there exist $K = (K_1, \dots, K_n) \geq 0$, $\mu = (\mu_1, \dots, \mu_n) > 0$, $\nu_0 \in N$ and $\lambda \geq 0$ such that

$$(5.2) \quad \|p_k\|_E \leq \lambda d^{k-\tilde{\gamma}} \left(\frac{eK\mu}{k-\tilde{\gamma}}\right)^{\mu} \quad \text{when } |k| > |k^{(\nu_0)}| := \alpha,$$

where $d = (d_1, \dots, d_n)$, γ is a fixed natural number and $\tilde{\gamma} = (\gamma, \dots, \gamma) \in R^n$, then

$$f(z) = \sum_k p_k(z), \quad z \in C^n,$$

is an entire function, and for all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) > 0$ there exists an $r^{(0)} = (r_1^{(0)}, \dots, r_n^{(0)}) \in R^n$ such that

$$\ln M_f(r) \leq \sum_{j=1}^n (K_j + \varepsilon_j) r_j^{\mu_j} \quad \text{for } r > r^{(0)}.$$

Proof. By Property 4 of the extremal function $\Phi(z, E)$ applied to every variable separately we have

$$(5.3) \quad \|p_k(z)\| \leq \|p_k\|_E \Phi^{k_1}(z_1, E^{(1)}) \dots \Phi^{k_n}(z_n, E^{(n)}) \quad \text{for } z \in C^n.$$

Write $\nu(r) = (2^{\mu_1} e K_1 \mu_1 r_1^{\mu_1}, \dots, 2^{\mu_n} e K_n \mu_n r_n^{\mu_n})$ and take $r^{(1)} = (r_1^{(1)}, \dots, r_n^{(1)}) > (1, \dots, 1)$ in such a way that $\nu(r) > k^{(\nu_0)}$ for $r > r^{(1)}$. Moreover, we assume that $\gamma = 0$ and $\lambda = 1$.

Thus by (5.3) we get

$$(5.4) \quad M_f(r) \leq \sum_{|k| \leq \alpha} \frac{\|p_k\|_E}{d^k} r^k + \sum_{\alpha < |k| < |\nu(r)|} \left(\frac{eK\mu}{k}\right)^{k/\mu} r^k + \sum_{|k| > |\nu(r)|} 2^{\frac{1}{|k|}}$$

$$\leq \beta r^{\tilde{\alpha}} + \sum_{\alpha < |k| < |\nu(r)|} \left(\frac{eK\mu}{k}\right)^{k/\mu} r^k + 2^n \quad \text{for } r > r^{(1)},$$

where β does not depend on r , $\tilde{\alpha} = (\alpha, \dots, \alpha) \in R^n$.

Since the maximum value of the expression

$$\left(\frac{eK_j \mu_j}{k_j}\right)^{k_j/\mu_j} r_j^{k_j}$$

for r_j fixed ($j = 1, \dots, n$) is obtained for $k_j = \mu_j K_j r_j^{\mu_j}$ and is equal to $\exp(K_j r_j^{\mu_j})$, we have

$$M_f(r) \leq \beta r^{\tilde{\alpha}} + \left(\binom{|\nu(r)| + n}{n} - \binom{\hat{\alpha} + n}{n}\right) \exp\left(\sum_{j=1}^n K_j r_j^{\mu_j}\right) + 2^n$$

$$\leq \beta r^{\tilde{\alpha}} + \frac{2^{|\nu(r)|}}{n!} \exp\left(\sum_{j=1}^n K_j r_j^{\mu_j}\right) + 2^n$$

$$\leq \left(\frac{\beta r^{\tilde{\alpha}}}{\exp(\sum K_j r_j^{\mu_j})} + \frac{2^{|\nu(r)|}}{n!} + \frac{2^n}{\exp(\sum K_j r_j^{\mu_j})}\right) \exp\left(\sum_{j=1}^n K_j r_j^{\mu_j}\right)$$

for $r > r^{(1)}$, where $\hat{\delta}$ is the smallest entire number greater than or equal to δ .

Hence, by a standard argument, there exists an $r^{(0)} > r^{(1)}$ such that

$$M_f(r) \leq \exp \sum_{j=1}^n (K_j + \varepsilon_j) r_j^{\mu_j} \quad \text{for } r > r^{(0)}.$$

Since for any $K' > K$ we have $(K'/k)^{k/\mu} > (K/(k - \tilde{\gamma}))^{(k - \tilde{\gamma})/\mu}$ when k is sufficiently large, in the case of $\gamma \neq 0$ or $\lambda \neq 1$ the proof is analogous with the only difference that before the second and the third component of the right-hand side of inequality (5.4) there occur (as factors) positive constants which have no decisive influence on the reasoning.

6. Characterization of the order and type systems by $\mathcal{E}_k^*(f, E)$. Let, as before, $E = E^{(1)} \times \dots \times E^{(n)}$ and $d_j = d(E^{(j)}) > 0$.

Now, we shall present a characterization of the order and type systems of an entire function f by means of the measure $\mathcal{E}_k^*(f, E)$

of the Čebyšev best approximation to f on E by polynomials of degree $\leq k_j$ with respect to the j -th variable.

THEOREM 6.1. *If the transfinite diameter $d_j = d(E^{(j)}) > 0$ ($j = 1, \dots, n$) and $\varrho = (\varrho_1, \dots, \varrho_n) > (0, \dots, 0)$, $\sigma = (\sigma_1, \dots, \sigma_n) > (0, \dots, 0)$ are order and type systems of an entire function f , respectively, then*

$$(6.1) \quad \limsup_{\min(k_j) \rightarrow \infty} \sqrt[|k|]{\frac{\mathcal{E}_k^*(f, E)}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/e}} = 1$$

and

$$(6.2) \quad \limsup_{\min(k_j) \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_E}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/e}} = 1.$$

Proof. Let

$$w_k(z) = \prod_{j=1}^n (z_j - \eta_{j0}) \dots (z_j - \eta_{jk_j}),$$

where $\{\eta_{j0}, \dots, \eta_{jk_j}\}$ is a system of $k_j + 1$ extremal points of the compact set $E^{(j)}$ ($j = 1, \dots, n$).

If r_j is sufficiently large, say $r_j > r_j^{(0)}$, then

$$E_{r_j}^{(j)} = \{z_j: d_j \Phi(z_j, E^{(j)}) = r\}$$

is a union of a finite number of mutually disjoint analytic Jordan curves in the complex z_j -plane; therefore

$$(6.3) \quad f(z) - L_k(z) = \frac{1}{(2\pi i)^n} \int_{E_{r_1}^{(1)}} \dots \int_{E_{r_n}^{(n)}} \frac{w_k(z)}{w_k(\zeta)} \frac{f(\zeta)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta,$$

where $d\zeta = d\zeta_1 \dots d\zeta_n$.

We can prove [11] that for every $\varepsilon_j > 0$ there exist $\lambda_j, r_j^{(1)}$ and $k_j^{(1)}$ such that

$$\left| \frac{1}{2\pi i} \int_{E_{r_j}^{(j)}} \left| \frac{(z_j - \eta_{j0}) \dots (z_j - \eta_{jk_j})}{(\zeta_j - \eta_{j0}) \dots (\zeta_j - \eta_{jk_j})} \frac{d\zeta_j}{\zeta_j - z_j} \right| \right| \leq \lambda_j \left(\frac{d_j e^{\varepsilon_j}}{r_j} \right)^{k_j}$$

for $r_j > r_j^{(1)}$, $k_j > k_j^{(1)}$.

Hence and from (6.3) we have

$$(6.4) \quad \|f - L_k\|_E \leq \lambda \frac{M_f(r)}{r^k} (de^\varepsilon)^k$$

$$\text{for } r > r^{(1)} = (r_1^{(1)}, \dots, r_n^{(1)}), \quad k > k^{(1)} = (k_1^{(1)}, \dots, k_n^{(1)}),$$

where $\lambda = \lambda_1 \dots \lambda_n$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $e^\varepsilon = (e^{\varepsilon_1}, \dots, e^{\varepsilon_n})$.

Let $\gamma = (\gamma_1, \dots, \gamma_n) > \sigma$. In consequence of the definition of the type system of f there exists an $r^{(2)} > r^{(1)}$ such that

$$M_f(r) \leq \exp(\gamma_1 r_1^{\rho_1} + \dots + \gamma_n r_n^{\rho_n}) \quad \text{for } r > r^{(2)}.$$

Let $k^{(2)} > k^{(1)}$ such that

$$\left(\frac{k_j}{r_j \rho_j}\right)^{1/\rho_j} > r_j \quad \text{for } j = 1, \dots, n, \quad k > k^{(2)}.$$

Putting

$$r = \left(\left(\frac{k_1}{\gamma_1 \rho_1}\right)^{1/\rho_1}, \dots, \left(\frac{k_n}{\gamma_n \rho_n}\right)^{1/\rho_n}\right)$$

in (6.4) and presenting γ in the form $\gamma = (\sigma_1 e^{\delta_1}, \dots, \sigma_n e^{\delta_n})$, we get

$$(6.5) \quad \|f - L_k\|_{\mathcal{E}} \leq \lambda (d e^\varepsilon)^k \left(\frac{e \gamma \rho}{k}\right)^{k/\rho} \leq \lambda d^k \left(\frac{e \sigma \rho}{k}\right)^{k/\rho} \left(e^{\left(\varepsilon + \frac{\delta}{\rho}\right)k}\right) \quad \text{for } k > k^{(2)},$$

where $\delta = (\delta_1, \dots, \delta_n)$.

Hence because of ε and γ being arbitrary we get

$$\limsup_{\min(k_j) \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_{\mathcal{E}} \left(\frac{k}{e \sigma \rho}\right)^{k/\rho}}{d^k}} \leq 1.$$

Suppose that

$$\limsup_{\min(k_j) \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_{\mathcal{E}} \left(\frac{k}{e \sigma \rho}\right)^{k/\rho}}{d^k}} = \eta_1 < 1$$

and take η such that $\eta < \eta_1 < 1$. From the definition of the limsup there would exist a $k^{(3)}$ such that

$$\|f - L_k\|_{\mathcal{E}} \leq \eta^{|k|} d^k \left(\frac{e \sigma \rho}{k}\right)^{k/\rho} \quad \text{for } k > k^{(3)}.$$

Taking $\sigma'_j = \sigma \eta^{\rho_j}$ we would obtain $\sigma' = (\sigma'_1, \dots, \sigma'_n) < \sigma$ and

$$(6.6) \quad \|f - L_k\|_{\mathcal{E}} < d^k \left(\frac{e \sigma' \rho}{k}\right)^{k/\rho} \quad \text{for } k > k^{(3)}.$$

Let $\tilde{\nu} = (\nu, \dots, \nu) \in \mathbb{R}^n$, $\nu = 0, 1, \dots$. From (6.6), from the triangle inequality and after expanding the function f in the series

$$f(z) = L_{\tilde{\sigma}}(z) + \sum_{\nu=0}^{\infty} (L_{\tilde{\sigma}+\tilde{\nu}}(z) - L_{\tilde{\sigma}}(z)), \quad z \in C^n$$

we would obtain

$$\|L_{\tilde{\sigma}+\tilde{\nu}} - L_{\tilde{\sigma}}\|_{\mathcal{E}} \leq \|f - L_{\tilde{\sigma}+\tilde{\nu}}\|_{\mathcal{E}} + \|f - L_{\tilde{\sigma}}\|_{\mathcal{E}} \leq 2d^{\tilde{\nu}} \left(\frac{e \sigma' \rho}{\tilde{\nu}}\right)^{\tilde{\nu}/\rho}.$$

Hence and from Lemma 5.1 and property (W) of $T_f(\varrho)$ it would follow that $\sigma \in \text{int}T_f(\varrho)$, and this contradicts the assumption.

The other part of Theorem 6.1 is an immediate consequence of inequality (5.1).

Equality (6.1) or (6.2) is at the same time a necessary and sufficient condition for the systems ϱ, σ of positive numbers to be the order and the type systems of the entire function f , respectively. Before carrying out the proof of this fact we shall prove the necessary and sufficient condition for the system $\varrho = (\varrho_1, \dots, \varrho_n)$ of positive numbers to be an order system.

Under the same assumption on the compact set $E = E^{(1)} \times \dots \times E^{(n)}$ as in Theorem 6.1 the following is obtained:

THEOREM 6.2. *A system of n positive numbers $\varrho = (\varrho_1, \dots, \varrho_n)$ is an order system of the entire function f if and only if*

$$(a) \quad \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^{k/\varrho}}{-\ln \|f - L_k\|_E} = 1,$$

or equivalently if

$$(b) \quad \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^{k/\varrho}}{-\ln \mathcal{E}_k^*(f, E)} = 1.$$

Proof of the sufficient condition. Let δ be a real number such that

$$0 < 2\delta < \varrho_j, \quad \delta\varrho_j < 1 \quad \text{for } j = 1, \dots, n.$$

It follows from the definition of limsup that there exists a sequence $\{k^{(\nu)} = (k_1^{(\nu)}, \dots, k_n^{(\nu)})\}$ ($k^{(\nu+1)} > k^{(\nu)}$) convergent to infinity (i.e. $\lim_{\nu \rightarrow \infty} k_j^{(\nu)} = \infty$, $j = 1, \dots, n$) for which

$$\frac{\ln (k^{(\nu)})^{k^{(\nu)}/\varrho}}{-\ln \|f - L_{k^{(\nu)}}\|_E} > 1 - \delta, \quad \|f - L_{k^{(\nu)}}\| < 1 \quad \text{for } \nu = 1, 2, \dots$$

Hence

$$(6.7) \quad \ln \|f - L_{k^{(\nu)}}\|_E > -\frac{k_1^{(\nu)}}{\varrho_1 - \varepsilon_1} \ln k_1^{(\nu)} + \dots + \frac{k_n^{(\nu)}}{\varrho_n - \varepsilon_n} \ln k_n^{(\nu)}, \quad \nu = 1, 2, \dots,$$

where $\varepsilon_j = \delta\varrho_j$, $j = 1, \dots, n$.

By a standard argument from (6.4) and (6.7) we obtain

$$(6.8) \quad \ln M_f(r) > \sum_{j=1}^n k_j^{(\nu)} \left(\ln \frac{r_j}{d_j} - \frac{\ln k_j^{(\nu)}}{\varrho_j - \varepsilon_j} \right) - \sum_{j=1}^n k_j^{(\nu)} \varepsilon_j - \ln \lambda$$

for $\nu > \nu_0$, $r > R^{(1)}$,

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$.

Let us take $\nu_1 > \nu_0$ so large that

$$d_j (ek_j^{(\nu)})^{\frac{1}{\varrho_j - \varepsilon_j}} > R_j^{(1)} \quad \text{for } j = 1, 2, \dots, n, \nu > \nu_1$$

and let us substitute in place of r_j/d_j (in (6.8))

$$\frac{r_j^{(\nu)}}{d_j} = (ek_j^{(\nu)})^{\frac{1}{\varrho_j - \varepsilon_j}}.$$

After this substitution we have

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^n k_j^{(\nu)} \frac{1}{\varrho_j - \varepsilon_j} - \sum_{j=1}^n k_j^{(\nu)} \varepsilon_j - \ln \lambda \quad \text{for } \nu > \nu_1.$$

Since $k_j^{(\nu)} = \frac{1}{e} \left(\frac{r_j^{(\nu)}}{d_j} \right)^{\varrho_j - \varepsilon_j}$, we have

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^n \left(\frac{1}{e(\varrho_j - \varepsilon_j) d_j^{\varrho_j - \varepsilon_j}} - \frac{\varepsilon_j}{e d_j^{\varrho_j - \varepsilon_j}} - \frac{\ln \lambda_j}{(r_j^{(\nu)})^{\varrho_j - \varepsilon_j}} \right) (r_j^{(\nu)})^{\varrho_j - \varepsilon_j}$$

for $\nu > \nu_1$.

Hence by a standard argument we conclude that

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^n (r_j^{(\nu)})^{\varrho_j - 2\varepsilon_j},$$

and this means that $\varrho - 2\varepsilon \notin P_f$.

Let ν be a natural number, and $\varepsilon > 0$. We shall put $\tilde{\nu} = (\nu, \dots, \nu) \in R^n$. By the definition of limsup it follows that for a sufficiently large ν , say $\nu > \nu_0$, the following is true:

$$(6.9) \quad \|f - L_{\tilde{\nu}}\|_E \leq (\tilde{\nu})^{\frac{-\tilde{\nu}}{\varrho + \varepsilon}},$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_j = \varrho_j \delta$, $j = 1, \dots, n$.

Hence

$$\|L_{\tilde{\nu}+1} - L_{\tilde{\nu}}\|_E \leq 2(\tilde{\nu})^{\frac{-\tilde{\nu}}{\varrho + \varepsilon}} \quad \text{for } \nu > \nu_0.$$

Let $K = (K_1, \dots, K_n)$ be such that

$$2(\tilde{\nu})^{\frac{-\tilde{\nu}}{\varrho + \varepsilon}} \leq d(\tilde{\nu}) \left(\frac{eK(\varrho + \varepsilon)}{\tilde{\nu}} \right)^{\frac{\tilde{\nu}}{\varrho + \varepsilon}} \quad \text{for } \nu = 1, 2, \dots$$

Since

$$\|L_{\tilde{\nu}+1} - L_{\tilde{\nu}}\|_E \leq d(\tilde{\nu}) \left(\frac{eK(\varrho + \varepsilon)}{\tilde{\nu}} \right)^{\frac{\tilde{\nu}}{\varrho + \varepsilon}} \quad \text{for } \nu > \nu_0,$$

by (6.9) and Lemma 5.1 we have

$$f(z) = L_{\bar{r}}(z) + \sum_{\nu=1}^{\infty} (L_{\bar{r}+\nu}(z) - L_{\bar{r}}(z)) \quad \text{for } z \in C^n.$$

Moreover, there exists an $r^{(0)} = (r_1^{(0)}, \dots, r_n^{(0)})$ such that

$$\ln M_f(r) \leq \sum_{j=1}^n (K_j + \varepsilon_j) r_j^{\rho_j + \varepsilon_j} \quad \text{for } r > r^{(0)}.$$

Hence it follows that $\rho + 2\varepsilon \in P_f$ for sufficiently small δ . Since we also had $\rho - 2\varepsilon \notin P_f$, this completes the proof of the sufficient condition.

Proof of the necessary condition. Let $\rho' = (\rho'_1, \dots, \rho'_n) > \rho$. Applying (6.4) and the definition of the order system of f in the same way as in the proof of (6.5), we get

$$(6.10) \quad \|f - L_k\|_E \leq \lambda (d e^{\rho})^k \left(\frac{e \rho'}{k} \right)^{k/\rho'} \quad \text{for } k > k^{(\rho)}.$$

After calculating the logarithm of (6.10) bilaterally and dividing both sides by $\ln \|f - L_k\|_E$ we get

$$(6.11) \quad 1 \geq \frac{\ln k^{k/\rho'}}{-\ln \|f - L_k\|_E} + \frac{\ln \lambda}{\ln \|f - L_k\|_E} + \\ + \sum_{j=1}^n \frac{\ln d_j + \varepsilon_j + \frac{1}{\rho'_j} + \frac{\ln \rho'_j}{\rho'_j}}{\frac{1}{k_j} \ln \|f - L_k\|_E} \quad \text{for } k > k^{(\rho)}.$$

It follows immediately from this that

$$\delta = \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^{k/\rho'}}{-\ln \|f - L_k\|_E} \leq 1.$$

Suppose that $\delta < 1$. Taking ε_0 so small that

$$\delta \rho + 2\varepsilon < \rho, \quad \text{where } \varepsilon = (\varepsilon_0 \rho_1, \dots, \varepsilon_0 \rho_n),$$

we would have

$$\|f - L_{k^{(\nu)}}\|_E \leq (k^{(\nu)})^{\frac{k^{(\nu)}}{\delta \rho + \varepsilon}}$$

for any sequence $\{k^{(\nu)}\}$ convergent to infinity. Now, from Lemma 5.1 it would follow that $(\delta \rho + 2\varepsilon) \in P_f$ and consequently $\rho \in \text{int} P_f$, and this contradicts the assumption.

Condition (b) follows immediately from (a) and (5.1).

Let a compact set $K \subset C^n$ be such that there exists a compact set $E \subset K$ satisfying the assumptions of Theorem 6.2.

Theorem 6.2 implies

THEOREM 6.2a. A system of n positive real numbers $\varrho = (\varrho_1, \dots, \varrho_n)$ is an order system of an entire function f if and only if

$$\limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^{k/\varrho}}{-\ln \delta_k^*(f, K)} = 1.$$

Keeping the notation of Theorem 6.2 we shall prove the following

THEOREM 6.3. Let $f: E \rightarrow B$ be a function defined and bounded on E , where E is as in Theorem 6.2. Let $\varrho = (\varrho_1, \dots, \varrho_n)$, $\sigma = (\sigma_1, \dots, \sigma_n)$ be systems of n positive numbers.

If

$$(i) \quad \limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_E}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1,$$

or equivalently if

$$(ii) \quad \limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\delta_k^*(f, E)}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1,$$

then there exists an entire function \tilde{f} such that

1° $\tilde{f}(z) = f(z)$ for $z \in E$;

2° ϱ is an order system of \tilde{f} , and σ is a type system of \tilde{f} corresponding to ϱ .

Proof. Let us take $\varepsilon > 0$ and $K^{(0)} = (K_1^{(0)}, \dots, K_n^{(0)})$ such that

$$(6.12) \quad \|f - L_k\|_E \leq d^k \left(\frac{e\sigma\varrho}{k}\right)^{k/\varrho} e^{-|k|\varepsilon} \quad \text{for } k > K^{(0)}.$$

From this and by Lemma 5.1 it follows that a function

$$(6.13) \quad \tilde{f}(z) = L_{\tilde{\nu}}(z) + \sum_{\nu=1}^{\infty} (L_{\tilde{\nu}+\nu}(z) - L_{\tilde{\nu}}(z)) \quad \text{for } z \in C^n,$$

where $\tilde{\nu} = (\nu, \dots, \nu) \in E^n$, is an entire function satisfying condition 1°.

Calculating the logarithms of both sides in (6.12) and applying an analogous reasoning to that used in the proof of the necessary condition of Theorem 6.2, we get

$$(6.14) \quad \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^{k/\varrho}}{-\ln \|f - L_k\|_E} \leq 1.$$

From the definition of limsup it follows that there exists a sequence $\{k^{(\nu)} = (k_1^{(\nu)}, \dots, k_n^{(\nu)})\}$ such that $\min\{k_j^{(\nu)}: j = 1, \dots, n\} \rightarrow \infty$ and

$$\|f - L_{k^{(\nu)}}\|_E > d^{k^{(\nu)}} \left(\frac{e\sigma\varrho}{k^{(\nu)}}\right)^{k^{(\nu)}/\varrho} e^{-|k^{(\nu)}|\varepsilon} \quad \text{for } \nu = 1, 2, \dots$$

By a similar reasoning we shall obtain the inequality opposite to that in (6.14). Thus from Theorem 6.2 it follows that ϱ is an order system of \tilde{f} .

Inequality (6.12) may be written in the form

$$(6.15) \quad \|f - L_k\|_{\mathcal{E}} \leq d^k \left(\frac{e\sigma' \varrho}{k} \right)^{k/\varrho} \quad \text{for } k > K^{(\sigma)},$$

where $\sigma' = (\sigma'_1, \dots, \sigma'_n)$, $\sigma'_j = \sigma_j e^{\varepsilon_j}$ for $j = 1, \dots, n$.

It follows from Lemma 5.1 that $(\sigma' + \tilde{\varepsilon}) \in T_{\tilde{f}}(\varrho)$, where $\tilde{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in R^n$. Therefore $\sigma \in T_{\tilde{f}}(\varrho)$.

Suppose that $\sigma \in \text{int } T_{\tilde{f}}(\varrho)$. Since inequality (6.15) holds true for every $\gamma \in \text{int } T_{\tilde{f}}(\varrho)$, taking a sufficiently small $\varepsilon > 0$ such that $\gamma = e^{-\varepsilon} \sigma \in \text{int } T_{\tilde{f}}(\varrho)$ we would have

$$\limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_{\mathcal{E}}}{d^k} \left(\frac{k}{e\gamma\varrho} \right)^{k/\varrho}} \leq 1.$$

On the other hand, on the strength of the assumption

$$\limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_{\mathcal{E}}}{d^k} \left(\frac{k}{e\gamma\varrho} \right)^{k/\varrho}} > e^\varepsilon > 1,$$

which contradicts the previously obtained inequality. Thus σ is a type system of \tilde{f} . From Theorem 6.1-6.3 we get as a conclusion the following generalization of the results obtained in [2], [9] and [11].

THEOREM 6.4. *Function $f: E \rightarrow B$, where E is as in Theorem 6.2, is prolongable to an entire function \tilde{f} of the order p if and only if*

$$p = \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^k}{-\ln \mathcal{E}_k^*(f, E)},$$

or, equivalently, if and only if

$$p = \limsup_{\min\{k_j\} \rightarrow \infty} \frac{\ln k^k}{-\ln \|f - L_k\|_{\mathcal{E}}}.$$

Moreover, if $0 < p < \infty$, then the adjoint type q of \tilde{f} is given by

$$\limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\mathcal{E}_k^*(f, E)}{d^k} k^{k/p}} = (epq)^{1/p},$$

or equivalently by

$$\limsup_{\min\{k_j\} \rightarrow \infty} \sqrt[|k|]{\frac{\|f - L_k\|_{\mathcal{E}}}{d^k} k^{k/p}} = (epq)^{1/p}.$$

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