Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

Application of BEM to elastoplastic contact problems

D. Martín[a], M.H. Aliabadi[a] and V.M.A. Leitão[b]
[a] *Wessex Institute of Technology, University of Portsmouth, Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK*
[b] *Departamento de Engenharia Civil, Instituto Superior Técnico, Lisbon, Portugal*

Abstract - In this paper, the contact between bodies with elastoplastic behaviour is studied. In order to solve the contact problem, a direct constraint technique is employed. Friction between the bodies is taken into account, and the materials may have different elastoplastic properties. An initial strain BEM formulation is used to study the elastoplastic problem. The material is assumed to obey the Von Mises yield criterion with its associated flow rule. Two numerical examples are presented, to demonstrate the efficiency of the proposed method.

Introduction

In engineering structures, the transfer of loads is usually achieved through contact. Where contact exists at small regions, high values of tractions and/or stress concentrations may occur, leading to the formation of localized plastic zones. It is of practical importance, thus, to be able to analyse contact problems in which the possibility of elastoplastic behaviour of the material is taken into account.

The BEM is particularly well suited to handle contact problems, since contact is inherent to the boundaries of the bodies involved. BEM has been applied to elastic contact problems by many researchers, for example Andersson [1], Karami and Fenner [2], Paris and Garrido [3] and Man, Aliabadi and Rooke [4]. The common feature of these formulations is the use of a direct approach to the problem, where the solution is obtained explicitly from equilibrium considerations and compatibility conditions, *i.e.*, a direct constraint technique.

The need to include elastoplasticity is important for some type of problems, and has received attention from researchers using BEM, [5],[6].

In this paper, a direct constraint approach is used to solve the contact problem. The contact areas are modelled using boundary elements with linear interpolation functions, and quadratic interpolation functions are used everywhere else. An initial strain approach, described in [7],[8] is employed to include the non linear behaviour of the materials, which was extended to allow the domain discretization by linear internal cells, compatible with the type of boundary elements used in the contact areas. The Von Mises yield criterion and its associated flow rule is employed, [9].

Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

BE Formulation for Elastoplastic Problems

For an homogeneous body of domain Ω, enclosed by a boundary Γ, the following integral equation can be written for the displacement rate $\dot{u}_j$ at a point $\mathbf{z}' \in \Gamma$:

$$c_{ij}(\mathbf{z}')\,\dot{u}_j(\mathbf{z}') + \fint_\Gamma p^*_{ij}(\mathbf{z}',\mathbf{x}')\,\dot{u}_j(\mathbf{x}')\,d\Gamma(\mathbf{x}') = \int_\Gamma u^*_{ij}(\mathbf{z}',\mathbf{x}')\,\dot{p}_j(\mathbf{x}')\,d\Gamma(\mathbf{x}') +$$

$$+ \int_\Omega \sigma^*_{ijk}(\mathbf{z}',\mathbf{x})\,\dot{\varepsilon}^p_{jk}(\mathbf{x})\,d\Omega(\mathbf{x}), \qquad (1)$$

in which the variable $\mathbf{x}'$ is used to refer to the boundary of the body, and the variable $\mathbf{x}$ to the domain; $\dot{u}_j$, $\dot{p}_j$ are the displacement and traction rates respectively; $\dot{\varepsilon}^p_{jk}$ is the plastic strain rate; p^*_{ij}, u^*_{ij} and σ^*_{ijk} are the fundamental solutions of elasticity. The symbol $\fint$ indicates a Cauchy principal value integral, while c_{ij} is a constant that depends on the geometry of the boundary at $\mathbf{z}'$. Although no time-dependent effects are studied in this paper, the rate notation is used to show that the magnitudes involved depend on the loading history.

The following integral equation valid for the stress rates at an internal point $\mathbf{z}$ can be obtained differentiating eq. (1) with respect to the coordinates of $\mathbf{z}$, and applying the generalized Hooke's law to the elastic part of the total strain rate tensor:

$$\dot{\sigma}_{ij}(\mathbf{z}) = \int_\Gamma U^*_{ijk}(\mathbf{z},\mathbf{x}')\dot{p}_k(\mathbf{x}')d\Gamma(\mathbf{x}') - \int_\Gamma P^*_{ijk}(\mathbf{z},\mathbf{x}')\dot{u}_k(\mathbf{x}')d\Gamma(\mathbf{x}')$$

$$+ \fint_\Omega \Sigma^*_{ijkl}(\mathbf{z},\mathbf{x})\dot{\varepsilon}^p_{kl}(\mathbf{x})d\Omega(\mathbf{x}) + f_{ij}(\dot{\varepsilon}^p_{kl}(\mathbf{z})), \qquad (2)$$

in which U^*_{ijk}, P^*_{ijk} and Σ^*_{ijkl} are the fundamental solutions, and the free term f_{ij} results from the differentiation of the domain integral that appears in equation (1). After discretizing the boundary and those areas of the domain where yielding is expected to occur, and after a collocation procedure, equations (1) and (2) can be written in matrix form as:

$$\mathbf{H}\dot{\mathbf{u}} = \mathbf{G}\dot{\mathbf{p}} + \mathbf{D}\dot{\varepsilon}^\mathbf{P} \qquad (3)$$

$$\dot{\sigma} = \mathbf{G}'\dot{\mathbf{p}} - \mathbf{H}'\dot{\mathbf{u}} + (\mathbf{D}' + \mathbf{C}')\dot{\varepsilon}^\mathbf{P}. \qquad (4)$$

The vectors $\dot{\mathbf{u}}$ and $\dot{\mathbf{p}}$ contain the values of displacement and traction rates at all boundary nodes. Since equation (2) is only valid for internal points, the stress rates at boundary points must be computed from different expressions (ref. [8]). The resulting coefficients can be assembled into equation (4), and the stress rates at all points can be calculated in a unified way. Equation (3) can be rearranged according to the boundary conditions, which gives the final system of equations:

$$\mathbf{A}\dot{\mathbf{y}} = \dot{\mathbf{f}} + \mathbf{D}\dot{\varepsilon}^\mathbf{P}, \qquad (5)$$

where $\dot{\mathbf{y}}$ is the vector of the unknowns, and $\dot{\mathbf{f}}$ is the elastic part of the right hand side vector, which contains the contribution of the known boundary conditions.

In a similar fashion, equation (4) can also be manipulated to give:

$$\dot{\sigma} = -\mathbf{A}'\dot{\mathbf{y}} + \dot{\mathbf{f}}' + \mathbf{D}^*\dot{\varepsilon}^\mathbf{P}, \qquad (6)$$

where $\mathbf{D}^*=(\mathbf{D}'+\mathbf{C}')$, $\mathbf{A}'$ contains the corresponding columns of $\mathbf{H}'$ and $\mathbf{G}'$; and $\dot{\mathbf{f}}'$ includes the contribution of the prescribed values.

Equations (5) and (6) are solved in an incremental way, therefore can be rewritten as

$$\mathbf{A}\mathbf{y} = \mathbf{f} + \mathbf{D}(\varepsilon^\mathbf{P} + \Delta\varepsilon^\mathbf{P}) \qquad (7)$$

Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

$$\sigma = -\mathbf{A}'\mathbf{y} + \mathbf{f}' + \mathbf{D}^*(\varepsilon^{\mathbf{P}} + \Delta\varepsilon^{\mathbf{P}}), \tag{8}$$

where for a given load step, $\mathbf{y}$, $\mathbf{f}$, $\mathbf{f}'$ and σ contain the accumulated values of tractions, displacements and stresses; $\varepsilon^{\mathbf{P}}$ represents the accumulated plastic strains up to but not including the current load increment and $\Delta\varepsilon^{\mathbf{P}}$ stores the increment of the plastic strains due to this load increment, and must be determined through an iterative procedure. Once convergence is achieved at all control points, these increments are added to the total plastic strains, and a new load step is allowed.

Contact Problems

Solving a contact problem requires the determination of displacements and tractions that arise within the contact zone. The size of such zone may also be unknown, and therefore must be computed as part of the solution. In order to do this, a *potential contact zone* is chosen in advance. Within this region, node-pairs (a, b) are defined. A direct constraint technique (refs [1],[4]) is used here, in which compatibility and equilibrium conditions are enforced at every node-pair. These conditions take the form shown in table (1) according to any one of the three possible contact modes. Tractions and displacements in this table are referred to in local coordinates, tangential and normal directions.
The first mode is separation, where the nodes a, b are not in contact, and thus tractions in both directions are enforced to zero. In the presence of friction, two true contact modes are possible, namely slip, in which the node-pair may undergo tangential relative displacement, with the friction force opposing it, and stick, in which the relative displacement of the node-pair is zero. Frictional behaviour is considered to obey Coulomb's friction law, in which the tangential traction is related to the normal traction through the friction coefficient μ.

Separation	**Slip**	**Stick**
$p_t^a + p_t^b = 0$	$p_t^a + p_t^b = 0$	$p_t^a + p_t^b = 0$
$p_n^a + p_n^b = 0$	$p_n^a + p_n^b = 0$	$p_n^a + p_n^b = 0$
$p_t^a = 0$	$p_t^a \pm \mu p_n^a = 0$	$u_t^a - u_t^b = 0$
$p_n^a = 0$	$u_n^a - u_n^b = 0$	$u_n^a - u_n^b = 0$

Table 1: Modes of contact

In order to solve the elastoplastic contact problem, it is necessary to write equation (7) for all the bodies involved and couple them by enforcing the compatibility and equilibrium conditions described above. For example, for two bodies in contact, and using a superscript for each of them:

$$\left[\begin{array}{c|c} \mathbf{A}^1 & 0 \\ \hline 0 & \mathbf{A}^2 \\ \hline \multicolumn{2}{c}{\text{Contact Conditions}} \end{array}\right] \left[\begin{array}{c} \mathbf{y}^1 \\ \hline \mathbf{y}^2 \end{array}\right] = \left[\begin{array}{c} \mathbf{f}^1 \\ \hline \mathbf{f}^2 \\ \hline 0 \end{array}\right] + \left[\begin{array}{c|c} \mathbf{D}^1 & 0 \\ \hline 0 & \mathbf{D}^2 \\ \hline \multicolumn{2}{c}{0} \end{array}\right] \left[\begin{array}{c} \varepsilon^{p^1} + \Delta\varepsilon^{p^1} \\ \hline \varepsilon^{p^2} + \Delta\varepsilon^{p^2} \end{array}\right] \tag{9}$$

In order to compute the internal stresses, it is enough to write equation (8) for each body separately.

$$\sigma^1 = -\mathbf{A}'^1\mathbf{y}^1 + \mathbf{f}'^1 + \mathbf{D}^{*1}(\varepsilon\mathbf{P}^1 + \Delta\varepsilon\mathbf{P}^1) \tag{10}$$

$$\sigma^2 = -\mathbf{A}'^2\mathbf{y}^2 + \mathbf{f}'^2 + \mathbf{D}^{*2}(\varepsilon\mathbf{P}^2 + \Delta\varepsilon\mathbf{P}^2). \tag{11}$$

The contact modes at each node-pair are not known in advance, and in consequence they are first set arbitrarily to slip . Equations (9), (10) and (11) are then solved for the current load step, where the iterative procedure to determine $\Delta\varepsilon\mathbf{P}$ must be carried out. Subsequently the assumed contact modes are checked, and if they are found to be incorrect, they are updated and the system of equations is solved again, until no such violations are detected. The solution of equation (9) requires the computation of the inverse of matrix **A**. Inverting it every time the contact conditions change would be a very expensive task. Special schemes, such as the Sherman- Morrison-Woodbury formula (ref. [10]) can be used to obtain the inverse of a matrix, after small changes happen to the original one. This is true for contact problems, where usually the contact area is small compared to the boundaries of the bodies involved.

Numerical examples

Flat punch on an elastoplastic foundation

This is a conforming type of problem, where the contact area is known in advance. However, in the presence of friction the partition between sticking and sliding zones must be found iteratively.

Consider the geometry shown in figure (1a), where the foundation has a width W and a height H. The dimensions of the punch are w and h, respectively. The ratios between them are assumed to be $h/w = 2$, $H/W = 1$ and $w/W = 1/4$. It is taken $W = 160$ mm. Figure (1b) shows the domain discretization, in the neighbourhood of the corner of the punch. A uniform compressive load per unit thickness, t_0, is applied on the upper face of the punch.

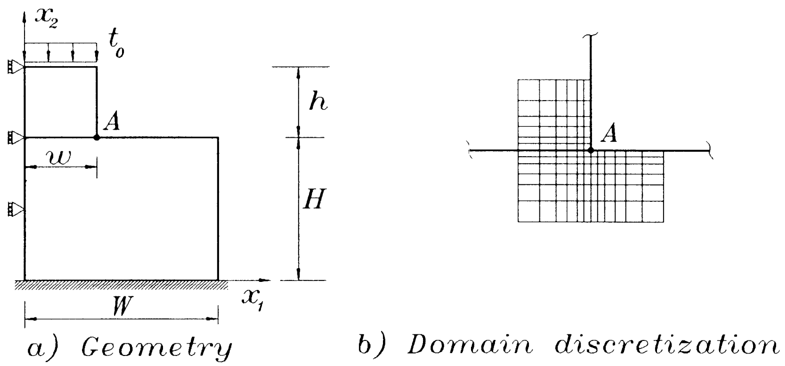


Figure 1: Flat punch

Both punch and foundation are assumed to have the same material properties: elastic modulus $E = 210$ GPa; Poisson's ratio $\nu = 0.3$; yield stress $\sigma_Y = 196$ MPa; plastic modulus $H' = 0$ (elastic-perfectly plastic material). The friction coefficient between the bodies is taken as $\mu = 0.2$. The problem is considered under plane strain condition.

This example is discretized using 85 boundary elements, of which 26 are linear, and 108 internal cells.

The first node to become plastic is the lower right corner of the punch (point A), and the value of the load for which this happen (or *load at first yield*), is $t_{oY} = 47.4$ MN/m. Figure (2) shows the distribution of normal contact tractions, normalized with respect to t_{oY}. It can be seen that as the material near the edge of the punch yields, the traction gradient decreases, and the maximum value is found towards the interior of the contact zone. This fact has also been observed in ref[5]. Figure (3) shows the distribution of tangential contact tractions, normalized with respect of (μt_{oY}). The partition of sticking and sliding zones is found at $x/w = 0.58$ during all the loading process. The sticking zone is situated between $0 < x/w < 0.58$.

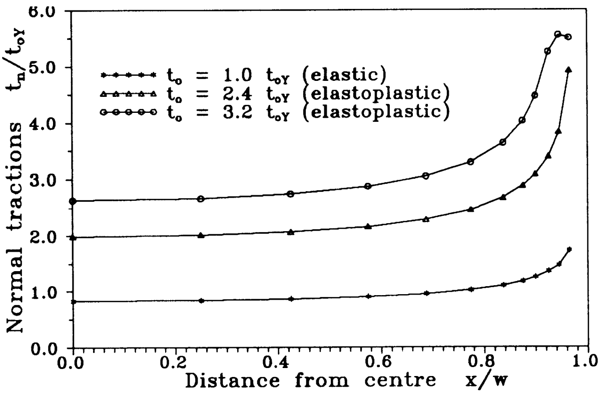


Figure 2: Flat punch: Normal contact tractions

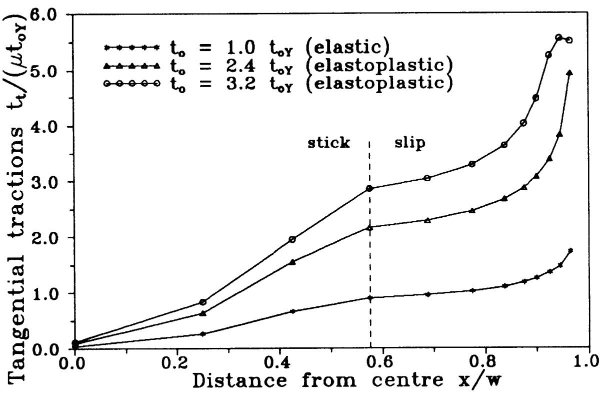


Figure 3: Flat punch: Tangential contact tractions

Perfect-fit pin in an infinite plate

This is still a conforming type of problem, because both contacting surfaces fit together in the unloaded state. However, the size of the contact area is no longer known after the

Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

application of the load. The problem may also be classified as a receding type, because the size of the contact area decreases as the applied load increases. Figure (4a) shows the geometry of this example. A pin of radius $R = 25$ mm is embedded in a plate of semi-width W and semi-height H. The ratios between them are $H/W = 1$ and $W/R = 20$, the latter value is considered appropriate to model an infinite plate. Figure (4b) shows the domain discretization of the area expected to yield. The plate is loaded in the x_2 direction with a uniform traction per unit thickness, t_0.

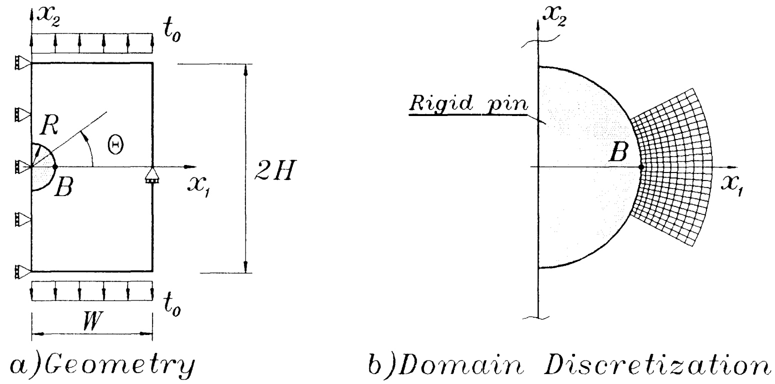


Figure 4: Perfect fit pin in an infinite plate

The pin is supposed to be rigid, and the plate has an elastic modulus $E = 73$ GPa, Poisson's ratio $\nu = 0.33$, yield stress $\sigma_Y = 380$ MPa and a plastic modulus $H' = 600$ MPa. The problem is analyzed assuming plane strain state. The discretization consists of 106 boundary elements, of which 72 are linear, and 240 internal cells.

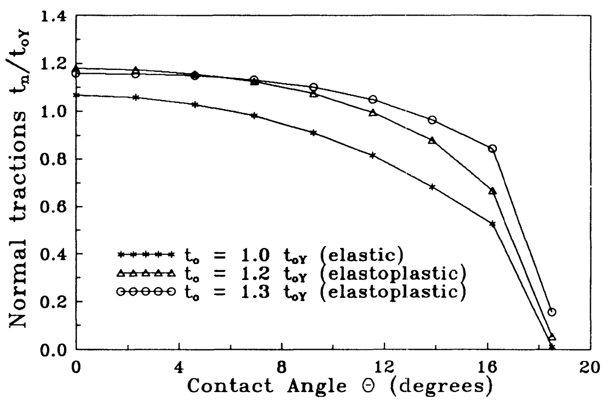


Figure 5: Normal tractions vs contact angle

The load at first yield is found to be $t_{oY} = 130.7$ MN/m, and the node where the plastic zone starts is point B. Figure (5) shows the distribution of normal contact tractions as a function of the angle Θ, normalized with respect to t_{oY}. The contact area develops between $-18.7^o < \Theta < 18.7^o$, and does not change significantly during the loading process. The distribution of hoop stresses $\sigma_{\theta\theta}$, also normalized with respect to t_{oY}, can

Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

be seen in figure (6). The elastic response compares favourably with the analytical results obtained in ref [11]. Finally the extent of the plastic zone for $t_o = 1.3t_{oY}$ is shown in figure (7). Also shown is the distribution of the equivalent plastic strain for the same load. This results agree with those presented in refs.[6],[12].

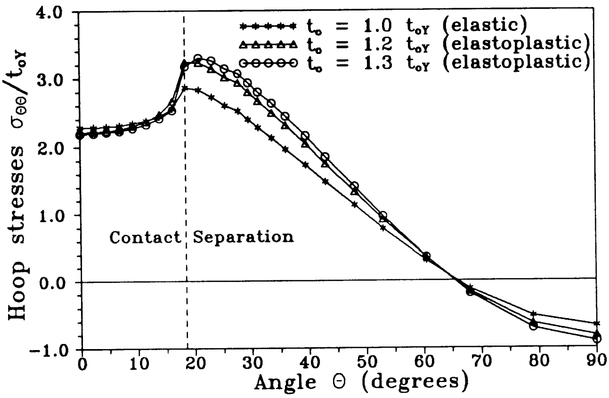
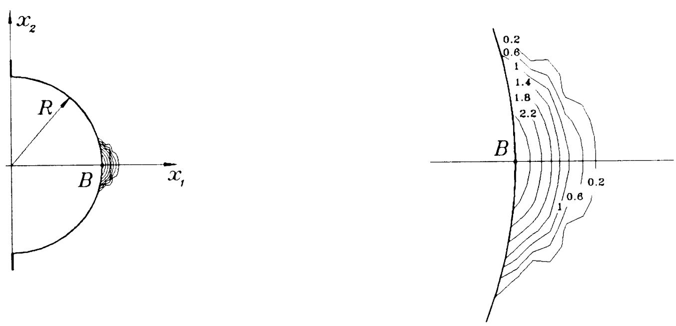


Figure 6: Hoop stresses vs angle



a) Plastic zone *b) Equivalent plastic strains (%)*

Figure 7: Plastic zone for $t_o = 1.3t_{oY}$

Conclusions

A study of contact problem between elastoplastic solids has been presented. A direct constraint technique is used to handle the contact problem, in which the equilibrium and compatibility conditions in the contact zones are directly enforced. These conditions are specified according to the contact mode existent at each node-pair, namely, separation, stick or slip mode, and they are determined iteratively, since they are not kown *a priori*. Frictional behaviour can be taken into account.

The elastoplastic response of the material is solved by a BEM initial strain approach, capable of handling elastic- perfectly plastic as well as work hardening constitutive relationships. The Von Mises yield criterion with its associated flow rule is adopted. Two examples were presented in which good agreement with published results are shown.

References

1. T. Andersson, *Boundary Elements in two-dimensional Contact and Friction. Dissertations. No. 85*, Linköping University, 1982.
2. G. Karami, R.T. Fenner, A Two-Dimensional BEM Method for Thermo-Elastic Body Forces Contact Problems, in *Boundary Elements IX, Vol 2: Stress Analysis Applications*, C.A. Brebbia, W.L. Wendland and G. Kuhn, Eds., Computational Mechanics Publications, Southampton, Springer-Verlag, Berlin, pp. 417-437, 1987.
3. F. Paris, J.A. Garrido, On the Use of Discontinuous Elements in 2D Contact Problems. *Boundary Elements VII*, pp 13-27 to 13-39. Computational Mechanics Publications, Southampton, 1985.
4. K.W. Man, M.H. Aliabadi, D.P. Rooke, Analysis of Contact Friction using the Boundary Element Method, in *Computational Methods in Contact Mechanics*, M.H. Aliabadi and C.A. Brebbia, Eds., Computational Mechanics Publications, Southampton, Elsevier Applied Science, London, pp. 1-60, 1993.
5. M. Alcantud, M. Doblaré, L. Gracia, J. Dominguez, Analysis of the Contact Problems Between Elastoplastic 2-D Bodies by Means of the BEM, in *Boundary Elements in Mechanical and Electrical Engineering*, C.A. Brebbia and A. Chaudouet-Miranda, Eds., Computational Mechanics Publications, Southampton, Springer-Verlag, Berlin, pp. 15-30, 1990.
6. G. Karami, Boundary Element Analysis of Elasto-plastic Contact Problems, *Computers & Structures*, 41, pp. 927-935, 1991.
7. V.M.A. Leitão, M.H. Aliabadi, D.P. Rooke, R. Cook, Residual stress fields effect on fatigue crack growth, in *Boundary Elements XIV, Vol. 2: Stress Analysis and Computational Aspects*, C.A. Breebia, J. Dominguez, F. Paris, Eds., Computational Mechanics Publications, Southampton, Elsevier, Applied Science, London, pp. 331-350, 1992.
8. C.A. Brebbia, J.C.F. Telles, L.C. Wrobel, *Boundary Element Techniques*, Springer-Verlag, Berlin, 1984.
9. A. Mendelson, *Plasticity: Theory and Application*, Macmillan, New York, 1968.
10. W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in FORTRAN*, Cambridge University Press, 1992.
11. H.B. Wilson, Approximate Determination of Contact Stresses in an Infinite Plate with a Smooth Circular Indent, *Proc. Second South Eastern Conference on Theoretical and Applied Mechanics*, pp. 147-163, 1964.
12. G.Z Voyiadjis, S. Navaee, Finite Strain Contact Problem of Cylinder Embedded in Body, *J. of Engineering Mechanics*, ASCE 110, pp. 1597-1609, 1984.