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Filomat **20:2** (2006), 115–124

**APPLICATION OF CONVOLUTION AND
DZIOK-SRIVASTAVA LINEAR OPERATORS
ON ANALYTIC AND P -VALENT FUNCTIONS**

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Abstract

In this paper, we introduce a new class of multivalent functions defined by convolution and Dziok-Srivastava operator and study some properties of this class e.g. coefficient estimates, integral representation, distortion and closure theorems, convolution and integral operator.

1 Introduction

Let \mathcal{A}_p be the class of p -valent analytic functions with positive coefficients of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in \Delta = \{z : |z| < 1\} \quad (1)$$

For two functions $f(z)$ given by (1) and

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (2)$$

¹Received: March 16, 2006

²*Keywords and phrases.* p -valent and hypergeometric functions, convolution, distortion bounds, closure theorem.

³2000 *Mathematics Subject Classification.* 30C45, 30C50.

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ denoted by $(f * g)(z) = (g * f)(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \quad (3)$$

For $\{\alpha_1, \alpha_2, \dots, \alpha_m\} \subseteq \mathbb{C}$ and $\{\beta_1, \beta_2, \dots, \beta_n\} \subseteq \mathbb{C}$ the generalized hypergeometric function ${}_mF_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z)$ is defined by

$${}_mF_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k z^k}{(\beta_1)_k \cdots (\beta_n)_k k!} \quad (4)$$

$$(m \leq n + 1, m, n, \in \mathbb{N}_0 = \{0, 1, 2, \dots\})$$

where $(\lambda)_k$ is the pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & k = 0 \\ \lambda(\lambda + 1) \cdots (\lambda + k - 1) & k \in \mathbb{N} \end{cases} \quad (5)$$

We consider Dziok - Srivastava operator [2] on $f(z) \in \mathcal{A}_p$ that is defined by

$$\begin{aligned} \mathcal{DS}_p^{m,n} &= \mathcal{DS}_p^{(m,n)}(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n) f(z) \\ &= h_p(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z) * f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_m)_{k-p} a_k z^k}{(\beta_1)_{k-p} \cdots (\beta_n)_{k-p} (k-p)!} \end{aligned} \quad (6)$$

where

$$h_p(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z) = z^p {}_mF_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z).$$

Definition : Let $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ be a fixed p -valent analytic function in Δ . Define the class

$$\mathcal{A}_p(g(z), \alpha_1, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_n, \gamma) = \mathcal{A}_p^{g(z)}(m, n, \gamma)$$

by

$$\begin{aligned} \mathcal{A}_p^{g(z)}(m, n, \gamma) &= \{f(z) \in \mathcal{A}_p : \operatorname{Re} \left\{ 1 + \frac{z(\mathcal{DS}_p^{m,n}(f * g)(z))''}{(\mathcal{DS}_p^{m,n}(f * g)(z))'} \right\} < p\gamma, \\ &\quad (1 < \gamma < 1 + \frac{1}{2p}, z \in \Delta)\} \end{aligned} \quad (7)$$

For other subclasses of p -valent functions, we can see the recent works of authors in [1], [5], [7].

2 Main Results

In this section we first find a necessary and sufficient condition for functions to be in the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 2.1 : $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\gamma)}{p^2(\gamma-1)} \theta(k, p) a_k b_k \leq 1. \quad (8)$$

where

$$\theta(k, p) = \frac{(\alpha_1)_{k-p} \cdots (\alpha_m)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_n)_{k-p} (k-p)!}$$

Proof : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then by using (6) and (7) we obtain

$$Re \left\{ 1 + \frac{z(p(p-1)z^{p-2} + \sum_{k=p+1}^{\infty} \theta(k, p)k(k-1)a_k b_k z^{k-2})}{pz^{p-1} + \sum_{k=p+1}^{\infty} \theta(k, p)ka_k b_k z^{k-1}} \right\} < p\gamma$$

where

$$\theta(k, p) = \frac{(\alpha_1)_{k-p} \cdots (\alpha_m)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_n)_{k-p} (k-p)!}.$$

By letting $z \rightarrow 1^-$ through real values we have

$$\frac{p^2 + \sum_{k=p+1}^{\infty} \theta(k, p)k^2 a_k b_k}{p + \sum_{k=p+1}^{\infty} \theta(k, p)ka_k b_k} < p\gamma$$

or equivalently

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k, p)a_k b_k \leq p^2(\gamma-1).$$

To prove the “if” part, let (8) holds true, so

$$\begin{aligned} & \left| \frac{z(\mathcal{DS}_p^{m,n}(f * g)(z))'' - (p-1)(\mathcal{DS}_p^{m,n}(f * g)(z))'}{z(\mathcal{DS}_p^{m,n}(f * g)(z))'' - [2p(1-\gamma) - 1 + p](\mathcal{DS}_p^{m,n}(f * g)(z))'} \right| \\ & \leq \frac{\sum_{k=p+1}^{\infty} k(k-p)a_k b_k}{2p^2(\gamma-1) - \sum_{k=p+1}^{\infty} [k(k-p)(1-2(1-\gamma))]a_k b_k} \leq 1, \end{aligned}$$

so by maximum principal theorem the proof is complete. \square

Theorem 2.2 : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then

$$a_k \leq \frac{p^2(\gamma - 1)}{k(k - p\gamma)b_k\theta(k, p)}. \quad (9)$$

the result is sharp for functions of the form

$$f_k(z) = z^p + \frac{p^2(\gamma - 1)}{k(k - p\gamma)b_k\theta(k, p)}z^k \quad k = p + 1, p + 2, \dots$$

Proof : Let $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$. By (8), we have

$$k(k - p\gamma)\theta(k, p)a_k b_k \leq \sum_{k=p+1}^{\infty} k(k - p\gamma)\theta(k, p)a_k b_k \leq p^2(\gamma - 1)$$

or

$$a_k \leq \frac{p^2(\gamma - 1)}{k(k - p\gamma)\theta(k, p)b_k}$$

The sharpness is trivial and so omitted.

3 Distortion Bounds

In this section we obtain the distortion bounds for $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 3.1 : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then

$$\begin{aligned} r^p - \frac{p^2(\gamma - 1)}{(p + 1)(p + 1 - p\gamma)\theta(p + 1, p)b_{p+1}}r^{p+1} &\leq |f(z)| \\ &\leq r^p + \frac{p^2(\gamma - 1)}{(p + 1)(p + 1 - p\gamma)\theta(p + 1, p)b_{p+1}}r^{p+1} \end{aligned} \quad (10)$$

where

$$\theta(p + 1, p) = \frac{\prod_{i=1}^m \alpha_i}{\prod_{j=1}^n \beta_j}, \quad |z| = r < 1.$$

The result is sharp for function defined by

$$f(z) = z^p + \frac{p^2(\gamma - 1)}{(p + 1)(p + 1 - p\gamma)\theta(p + 1, p)b_{p+1}}z^{p+1}. \quad (11)$$

Proof : By using (8), (9) we obtain

$$b_{p+1}\theta(p+1, p)(p+1)(p+1-p\gamma) \sum_{k=p+1}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k, p)a_k b_k \leq p^2(\gamma-1)$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1, p)b_{p+1}} \quad (12)$$

For the function $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and using (12), for $|z| = r$ we have

$$\begin{aligned} |f(z)| &\leq r^p + \sum_{k=p+1}^{\infty} a_k r^k \\ &< r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq r^p + \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1, p)b_{p+1}} r^{p+1}, \end{aligned}$$

also

$$\begin{aligned} |f(z)| &\geq r^p - \sum_{k=p+1}^{\infty} a_k r^k \\ &\geq r^p - \frac{p^2(\gamma-1)}{(p+1)(p+1-p\gamma)\theta(p+1, p)b_{p+1}} r^{p+1}. \end{aligned}$$

Now the proof is complete. □

Corollary : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then

$$\begin{aligned} pr^{p-1} - \frac{p^2(\gamma-1)}{(p+1-p\gamma)\theta(p+1, p)b_{p+1}} r^p &\leq |f'(z)| \\ &\leq pr^{p-1} + \frac{p^2(\gamma-1)}{(p+1-p\gamma)\theta(p+1, p)b_{p+1}} r^p. \end{aligned}$$

The result is sharp for the function given by (11).

4 Integral Representation

In this section we obtain integral representation for $\mathcal{DS}_p^{m,n}(f * g)(z)$ with suitable ranges on variables.

Theorem 4.1 : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ then

$$\mathcal{DS}_p^{m,n}(f * g)(z) = \int_{\epsilon_2}^z \exp \left(\int_{\epsilon_1}^z \frac{p\gamma Q|t| - 1}{t} dt \right) ds$$

where $\epsilon_1 \rightarrow 0, \epsilon_2 \rightarrow 0$ and $|Q(z)| < 1, z \in \Delta$.

Proof : By letting $\mathcal{DS}_p^{m,n}(f * g)(z) = M(z)$ in (7) we have

$$\operatorname{Re} \left\{ 1 + \frac{zM''(z)}{M'} \right\} < p\gamma.$$

Since for all z ,

$$\operatorname{Re} \left\{ 1 + \frac{zM''}{M'} \right\} < \left| 1 + \frac{zM''}{M'} \right|,$$

hence by choosing the values of z such that

$$\left| 1 + \frac{zM''}{M'} \right| < p\gamma$$

we conclude

$$1 + \frac{zM''}{M'} = p\gamma Q(z) \quad \text{or} \quad \frac{M''}{M'} = \frac{p\gamma Q(z) - 1}{z}.$$

After integration we obtain

$$\log(M'(z)) = \int_{\epsilon_1}^z \frac{p\gamma Q(t) - 1}{t} dt \quad (\epsilon_1 \rightarrow 0),$$

thus

$$M'(z) = \exp \int_{\epsilon_1}^z \frac{p\gamma Q(t) - 1}{t} dt \quad (\epsilon_1 \rightarrow 0),$$

after integration we obtain the result. □

5 Some Properties of the Class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$

In this section, we prove the closure theorems for the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 5.1 : Let $F_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k$ ($j = 1, 2, \dots, q$) be in the

class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$ and $\eta_j \geq 0$ for $j = 1, 2, \dots, q$ and $\sum_{j=1}^q \eta_j \leq 1$ then the function

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\sum_{j=1}^q \eta_j a_{k,j} \right) z^k$$

belongs to $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Proof : Since $F_j(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then from Theorem 2.1 for every $j = 1, 2, \dots, q$ we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k, p)b_k a_{k,j} \leq p^2(\gamma-1)$$

Also

$$\begin{aligned} & \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k, p)b_k \left(\sum_{j=1}^q \eta_j a_{k,j} \right) \\ &= \sum_{j=1}^q \eta_j \left(\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k, p)b_k a_{k,j} \right) \\ &\leq \sum_{j=1}^q \eta_j p^2(\gamma-1) \\ &\leq p^2(\gamma-1). \end{aligned}$$

So by Theorem 2.1, $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 5.2 : Let $F_p(z) = z^p$ and

$$F_k(z) = z^p + \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k, p)b_k} z^k, \quad (k = p+1, \dots).$$

Then $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ if and only if

$$f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k F_k(z)$$

where $\sum_{k=p}^{\infty} \eta_k = 1$ and $\eta_k \geq 0$.

Proof : Let $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then from Theorem 2.2, we have

$$a_k \leq \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k, p)b_k} \quad (k = p+1, p+2, \dots)$$

therefore by letting

$$\eta_k = \frac{k(k-p\gamma)\theta(k, p)b_k a_k}{p^2(\gamma-1)} \quad (k = p+1, p+2, \dots)$$

and $\eta_p = 1 - \sum_{k=p+1}^{\infty} \eta_k$. We conclude the required result. Conversely, let $f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k F_k(z)$, then

$$\begin{aligned} f(z) &= \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k \left(z^p + \frac{p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k \right) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} \frac{k(k-p\gamma)}{p^2(\gamma-1)} \theta(k,p)b_k \\ &= \sum_{k=p+1}^{\infty} \eta_k = 1 - \eta_p \leq 1. \end{aligned}$$

Hence by Theorem 2.1, we have $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

6 Convolution Property and Integral Operator

In this section we show that the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$ is closed under convolution and one special operator.

Theorem 6.1 : Let $h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k$ be analytic in unit disk Δ and

$0 \leq c_k \leq 1$. If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then $(f * h)(z)$ is also in the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Proof : Since $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ then by Theorem 2.1 we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1)$$

By using the last inequality and the fact that

$$(f * h)(z) = z^p + \sum_{k=p+1}^{\infty} a_k c_k z^k$$

we have

$$\begin{aligned} & \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k c_k b_k \\ & \leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1) \end{aligned}$$

and this by Theorem 2.1 gives the result. □

Theorem 6.2 : If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then

$$F(z) = \frac{\lambda+p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt \quad (\lambda > -1; \quad z \in \Delta)$$

is also in the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$. See [3], [5].

Proof : Since $F(z) = f(z) * \left(z^p + \sum_{k=p+1}^{\infty} \frac{\lambda+p}{\lambda+k} z^k \right)$ and $\frac{\lambda+p}{\lambda+k} \leq 1$, by Theorem 6.1, the proof is trivial.

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