World Applied Sciences Journal 5 (4): 441-448, 2008
ISSN 1818-4952

Application of Differential Transform Method in Free Vibration Analysis of Rotating Non-Prismatic Beams

Reza Attarnejad and Ahmad Shahba

School of Civil Engineering, University College of Engineering, University of Tehran, Iran

Abstract: Rotating beams are considerably used in different mechanical and aeronautical installations. In this paper, free vibration of non-prismatic rotating Euler-Bernoulli beams is studied. Dynamic stiffness matrix is evaluated by using differential transform method, a powerful numerical tool in solution of ordinary differential equations, for solving the governing equation of motion. The method is capable of modeling any beam whose cross-sectional area and moment of inertia vary along beam with any two arbitrary functions and any type of cross-section with just one or few elements so that it can be used in most of engineering applications. In order to verify the competency of the method, natural frequencies are obtained for two problems and the effects of rotational speed parameter and taper ratio on natural frequencies are investigated.

Key words: Dynamic stiffness matrix . Differential transform . Natural frequency . Non-prismatic . Rotating beam

INTRODUCTION

Non-prismatic beams, i.e. beams with cross-section varying continuously or discontinuously along their length, are widely used as structural elements in engineering field since they help the designers to meet the architectural and aesthetic requirements. On the other hand, application of non-prismatic beams in areas in which optimization of weight and strength is of great significance such as satellites, has magnified the importance of their analysis. Furthermore, rotating beams are widely used in special structures such as blades of helicopters, windmill turbines and aircraft propellers. Since vibration is one of the most important factors which is taken into account in design phase and moreover it poses restrictions and limitations in the performance, comprehension of vibration properties particularly natural frequency seems to be essential.

The governing equation of free vibration of non-prismatic beams is a fourth order differential equation with variable constants. Except for some special cases [1-3], there exists no exact explicit solution. Thus approximation techniques have played the most effective role in this problem. In technical literature, a great deal of work in this area has been devoted to either formulating novel elements or improving the existing elements. Some researchers have recently used numerical techniques such as Frobenius and Rayleigh-Ritz methods [4-7]. Among different numerical techniques, Finite Element Method (FEM) has acquired a dominant position due to its simplicity and generality. Conventional FEM requires discretizing the member into elements and indeed, the original beam is modeled as a stepped beam consisting of uniform elements whose cross-sectional area and moment of inertia is set to the average value of cross-sectional area (A) and moment of inertia (I) in both faces of the element respectively. It is evident that by increasing the number of elements, a more valid actual model is obtained while on the other hand, the analysis will be computationally expensive. Vinod *et al*. [8] introduced an approximate spectral finite element for rotating Euler-Bernoulli beams. Gunda and Ganguli [9] derived new rational shape functions which depend on the rotational speed and element position along the beam.

Zhou [10] was the first one to use differential transform method (DTM) in engineering applications. He employed DTM in solution of initial boundary value problems in electric circuit analysis. In recent years, concept of DTM has broadened to problems involving partial differential equations and systems of differential equations [11-13]. Some researchers have lately applied DTM for analysis of uniform and non-uniform beams [14-18]. Ozdemir and Kaya [15] calculated natural frequencies for non-prismatic beams whose cross-sectional area and moment of inertia vary in accordance to two arbitrary powers n and n+2, respectively. They

Corresponding Author: Mr. Ahmad Shahba, School of Civil Engineering, University College of Engineering, University of Tehran, Iran

derived a formulation for the special case of a cantilevered beam and n = 1. Ozgumus and Kaya [16] determined the first natural frequencies of a cantilevered double tapered rotating Euler-Bernoulli beams whose height and breadth vary linearly along beam axis. Mei [18] employed DTM for rotating uniform beams. In this paper, DTM is employed to obtain dynamic stiffness matrix for non-prismatic rotating beam elements with any type of variation of cross-sectional area and moment of inertia along the beam. Dynamic stiffness matrix can be used to determine natural frequencies of the beam to any desired accuracy since it is obtained from solution of differential equation of motion of the beam undergoing free vibration. Using just one element, dynamic stiffness matrix is able to determine any number of natural frequencies and it is regarded as an advantage which motivates the researches to derive exact dynamic stiffness matrix for non-prismatic members.

DIFFERENTIAL TRANSFORM METHOD

Differential transform method (DTM) is one of the most feasible numerical tools for solving differential equations. Jang *et al*. [11] state that "the differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations". The basic definition of DTM is as follows, If function $f(x)$ is analytic in domain D, let $x = x_0$ represent any point within domain D, thus differential transform of $f(x)$ is given by:

$$\overline{F}(k) = \frac{1}{k!}.\left(\frac{d^k f}{dx^k}\right)_{x=x_0} \qquad (1)$$

The function $f(x)$ can now be presented by power series using differential transform of $f(x)$ as:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}.\left(\frac{d^k f}{dx^k}\right)_{x=x_0}.(x-x_0)^k = \sum_{k=0}^{\infty} \overline{F}(k).(x-x_0)^k \qquad (2)$$

Equation (5) is known as inverse transform. In practical problems, $f(x)$ is presented as:

$$f(x) = \sum_{k=0}^{N} \overline{F}(k).(x-x_0)^k \qquad (3)$$

where N is selected such that

$$\sum_{k=N+1}^{\infty} \overline{F}(k).(x-x_0)^k$$

is infinitesimal. Fundamental theorems of differential transform are listed in Table 1.

Table 1: Fundamental theorems of differential transform

Original functions	Transformed functions
$f(x) = g(x) \pm h(x)$	$\overline{F}(k) = \overline{G}(k) \pm \overline{H}(k)$
$f(x) = c.g(x); c = cons$	$\overline{F}(k) = c.\overline{G}(k)$
$f(x) = g(x).h(x)$	$\overline{F}(k) = \sum_{i=0}^{k} \overline{G}(k-i).\overline{H}(i)$
$f(x) = \frac{d^n g(x)}{dx^n}$	$\overline{F}(k) = (k+1).(k+2)...(k+n).\overline{G}(k+n)$

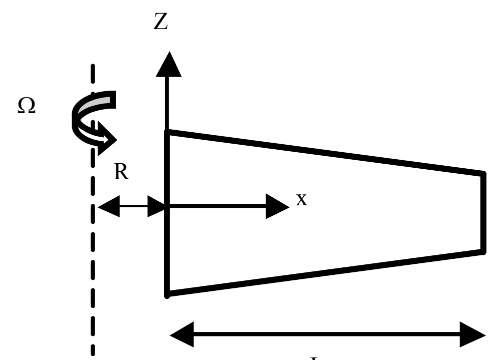

Fig. 1: Rotating tapered beam geometry

GOVERNING EQUATION OF MOTION

Consider the rotating beam shown in Fig.1. According to the Euler-Bernoulli beam theory, the governing partial differential equation of transverse vibration is:

$$\frac{\partial^2}{\partial x^2}\left(EI(x).\frac{\partial^2 w}{\partial x^2}\right) + \rho A(x).\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(T.\frac{\partial w}{\partial x}\right) = p_w \qquad (4)$$

where w and p_w are respectively vertical displacement and applied force per unit length, both in flapwise direction, EI is flexural rigidity, ρA is mass per unit length and $T(x)$ is the centrifugal force at distance x from the origin and given by:

$$T(x) = \int_x^L \rho A.\Omega^2.(R+x).dx \qquad (5)$$

in which L is length of the rotating beam, R is hub radius and Ω is angular rotational speed. Considering free vibration and sinusoidal variation of w with a circular natural frequency of ω, Equation (4) changes into

$$\frac{d^2}{dx^2}\left(EI(x).\frac{d^2 w}{dx^2}\right) - \omega^2.\rho A(x).w - \frac{d}{dx}\left(T.\frac{dw}{dx}\right) = 0 \qquad (6)$$

Assuming cross-sectional area and moment of inertia to vary as:

$$A(x) = A_0.P_A(x) \quad I(x) = I_0.P_I(x) \tag{7}$$

in which A_0 and I_0 are cross-sectional area and moment of inertia at origin. Introducing the dimensionless parameters:

$$\xi = \frac{x}{L} \quad \delta = \frac{R}{L} \quad \eta^2 = \frac{\rho A_0 L^4 \Omega^2}{EI_0} \quad \mu^2 = \frac{\rho A_0 L^4 \omega^2}{EI_0} \tag{8}$$

Equation (6) can be rewritten in non-dimensional form as:

$$\frac{d^2}{d\xi^2}\left(P_I(\xi)\frac{d^2w}{d\xi^2}\right) - \mu^2 P_A(\xi)w - \eta^2\frac{d}{d\xi}\left(P_T(\xi)\frac{dw}{d\xi}\right) = 0 \tag{9}$$

where

$$P_T(\xi) = \int_{\xi}^{1} P_A(\xi).(\delta+\xi).d\xi \tag{10}$$

APPLICATION OF DTM

Applying DTM to Equation (9) besides the theorems presented in Table 1, the following recursive expression is obtained.

$$\begin{aligned}&\sum_{i=0}^{k}\overline{P_I}(k-i)(i+1)(i+2)(i+3)(i+4)\overline{W}(i+4)\\ &+2\sum_{i=0}^{k}(k-i+1)\overline{P_I}(k-i+1)(i+1)(i+2)(i+3)\overline{W}(i+3)\\ &+\sum_{i=0}^{k}(k-i+1)(k-i+2)\overline{P_I}(k-i+2)(i+1)(i+2)\overline{W}(i+2)\\ &=\mu^2\sum_{i=0}^{k}\overline{P_A}(k-i)\overline{W}(i)+\eta^2\left(\sum_{i=0}^{k}(k-i+1)\overline{P_T}(k-i+1)(i+1)\overline{W}(i+1)+\sum_{i=0}^{k}\overline{P_T}(k-i)(i+1)(i+2)\overline{W}(i+2)\right)\end{aligned} \tag{11}$$

It is assumed that each node has two degrees of freedom i.e. vertical displacement w and angle of rotation θ as shown in Fig. 2.

Assuming general geometric and force boundary conditions, we have at $x = x_1$ or $\xi = 0$,

$$W(x_1) = W_1 \quad \theta(x_1) = \theta_1 \quad V(x_1) = V_1 \quad M(x_1) = -M_1 \tag{12}$$

and at $x = x_2$ or $\xi = 1$

$$W(x_2) = W_2 \quad \theta(x_2) = \theta_2 \quad V(x_2) = -V_2 \quad M(x_2) = M_2 \tag{13}$$

Additionally, from elementary structural principles we have

$$\theta(x) = \frac{dW}{dx} \quad M(x) = EI(x)\frac{d^2W}{dx^2} \quad V(x) = \frac{d}{dx}\left(EI(x)\frac{d^2W}{dx^2}\right) \tag{14}$$

Using the recursive Equation (11), any $\overline{W}(k), k \geq 4$ is expressed in terms of the first four terms i.e. $\overline{W}(0)$, $\overline{W}(1)$, $\overline{W}(2)$ and $\overline{W}(3)$. It is simply possible to express nodal degrees of freedom {D} and forces {F} in terms of these four terms {C} using Equation (14) as,

$$\{D\} = [B]_{4\times4}.\{C\} \tag{15}$$

$$\{F\} = [G]_{4\times4}.\{C\} \tag{16}$$

Eliminating {C} in Equations (15) and (16), the dynamic stiffness matrix is obtained as:

$$[K] = [G].[B]^{-1} \tag{17}$$

Prescribing appropriate boundary conditions, the natural frequencies are obtained by equating the determinant of the dynamic stiffness matrix to zero. Another alternative can be used for calculation of natural frequencies. Once $\overline{W}(k), k \geq 4$ and consequently nodal degrees of freedom and forces obtained in terms of {C}, by defining the boundary conditions for the problem the following relation is obtained,

$$[A].\{C\} = \{0\} \tag{18}$$

from which it is possible to calculate the natural frequencies by setting the determinant of the coefficient matrix [A] to zero which gives the characteristic equation of the structure, as well.

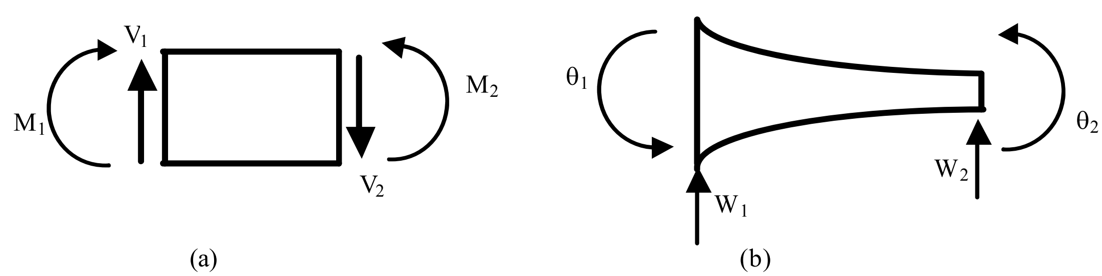

Fig. 2: A beam element: (a) nodal forces; (b) nodal degrees of freedom

RESULTS

Matlab is used to write a computer code for determination of natural frequencies for different problem configurations. Both cantilevered and hinged boundary conditions are considered.

Example 1 (uniform non-rotating beam): Natural dimensionless frequencies for a uniform non-rotating beam are reported in Table 2 for two different boundary conditions i.e. cantilevered and simply supported beam.

Example 2 (linear mass, linear stiffness, cantilevered beam): Assume that cross-sectional area and moment of inertia vary linearly along beam length as,

$$A(\xi) = A_0(1-\alpha\xi) \quad (19)$$

$$I(\xi) = I_0(1-\beta\xi) \quad (20)$$

where α and β are taper ratios and A_0 and I_0 correspond respectively to the values of cross-sectional area and moment of inertia at the thick end of the rotating beam at $\xi = 0$. The results for the special case of a cantilevered beam with $\alpha = 0.8$ and $\beta = 0.95$ which is used, as mentioned by Wright *et al*., in design of wind turbine blades are tabulated in Table 3 and 4.

Example 3 (Second order mass, fourth order stiffness, cantilevered beam): Assume that cross-sectional area and moment of inertia vary along beam as:

$$A(\xi) = A_0\left(1-c\xi\right)^n \quad (21)$$

$$I(\xi) = I_0\left(1-c\xi\right)^{n+2} \quad (22)$$

The results for the special case of n = 2 and c = 0.5 are tabulated in Table 5 for different values of rotational speed parameters. This case includes beams with circular cross-sections whose diameter vary linearly along beam length and moreover I-shaped beams with linearly varying breadth and height. In Table 6, the effects of rotational speed parameter and taper ratio on natural frequencies are shown.

The first four normalized mode shapes of the beams described in examples 2 and 3 for η = 10 are plotted and compared in Fig. 3.

DISCUSSION

It is observed that in all numerical examples the method could prove its competency in accuracy in comparison with other methods in literature. In all examples, increasing the rotational speed parameter

Table 2: Natural dimensionless frequencies for a non-rotating uniform beam

First mode		Second mode		Third mode		Fourth mode	
Present	Ref. [20]	Present	Ref. [20]	Present	Ref. [20]	Present	Ref. [20]
Cantilevered beam							
3.5160	3.5160	22.0345	22.0345	61.6972	61.6972	120.902	120.902
Simply supported beam							
9.8696	π^2	39.4784	$4\pi^2$	88.8264	$9\pi^2$	157.9137	$16\pi^2$

Table 3: The first and second non-dimensional natural frequencies of rotating tapered cantilevered beam under different rotational speed parameters η with $\delta = 0$ (Example 2)

	First mode			Second mode		
η	Present	Ref. [9]	Ref. [19]	Present	Ref. [9]	Ref. [19]
0	5.2738	5.2738	5.2738	24.0041	24.0041	24.0041
1	5.3903	5.3903	5.3903	24.1070	24.1069	24.1069
2	5.7249	5.7249	5.7249	24.4130	24.4130	24.4130
4	6.8928	6.8928	6.8928	25.6013	25.6013	25.6013
6	8.4653	8.4653	8.4653	27.4693	27.4693	27.4693
8	10.2379	10.2379	10.2379	29.8894	29.8894	29.8894
10	12.1092	12.1092	12.1092	32.7369	32.7370	32.7369

Table 4: The third and fourth non-dimensional natural frequencies of rotating tapered cantilevered beam under different rotational speed parameters η with δ = 0 (Example 2)

	Third mode			Fourth mode		
η	Present	Ref. [9]	Ref. [19]	Present	Ref. [9]	Ref. [19]
0	59.9702	59.9702	59.9701	112.9095	112.910	112.909
1	60.0696	60.0696	60.0696	113.0091	113.009	113.009
2	60.3669	60.3670	60.3669	113.3073	113.308	113.307
4	61.5412	61.5412	61.5412	114.4922	114.493	114.492
6	62.4483	63.4483	63.4483	116.4385	116.439	116.439
8	66.0222	66.0223	66.0222	119.1067	119.107	119.107
10	69.1851	69.1853	69.1851	122.4464	122.447	122.446

Table 5: The first four non-dimensional natural frequencies of rotating tapered cantilevered beam under different rotational speed parameters η with δ = 0 (Example 3)

	First mode		Second mode		Third mode		Fourth mode	
η	Present	Ref. [7]	Present	Ref. [7]	Present	Ref. [7]	Present	Ref. [7]
0	4.62516	4.62515	19.5476	19.5476	48.5789	48.5789	91.8128	91.8128
2	5.15642	5.15641	20.0734	20.0733	49.0906	49.0906	92.3244	92.3243
4	6.47262	6.47262	21.5749	21.5749	50.5939	50.5938	93.8415	93.8415
6	8.16630	8.16630	23.8684	23.8684	53.0019	53.0018	96.3142	96.3142
8	10.0193	10.0192	26.7454	26.7454	56.1941	56.1941	99.6673	99.6673
10	11.9415	11.9415	30.0297	30.0299	60.0396	60.0399	103.810	103.810

Table 6: The effects of taper ratio (c) and rotational speed parameter (η) on non-dimensional natural frequencies of a rotating cantilevered beam with n = 2 and δ = 0 (Example 3)

	η = 0					
	First mode		Second mode		Third mode	
c	Present	Ref. [7]	Present	Ref. [7]	Present	Ref. [7]
0.1	3.67371	3.67370	21.5503	21.5503	59.1886	59.1886
0.2	3.85512	3.85511	21.0568	21.0568	56.6304	56.6303
0.3	4.06694	4.06694	20.5555	20.5555	54.0152	54.0152
0.4	4.31878	4.31878	20.0500	20.0500	51.3346	51.3346
0.5	4.62516	4.62515	19.5476	19.5476	48.5789	48.5789
c	η = 5					
0.1	6.56331	6.56330	24.9029	24.9029	62.6153	62.6152
0.2	6.69694	6.69693	24.3478	24.3478	59.9763	59.9763
0.3	6.85629	6.85629	23.7820	23.7820	57.2816	57.2815
0.4	7.04981	7.04980	23.2089	23.2088	54.5230	54.5230
0.5	7.29015	7.29014	22.6360	22.6360	51.6918	51.6918

gives rise to all natural frequencies which is on account of the increasing of the centrifugal tension force. On the other hand, it is concluded from Table 6 that the first four natural frequencies except for the first one decrease with the taper ratio. This might be due to softening effect resulting from the decrease of cross-sectional area.

Another point concerns the convergence of DTM. If cross-sectional area and moment of inertia are assumed to vary along beam as:

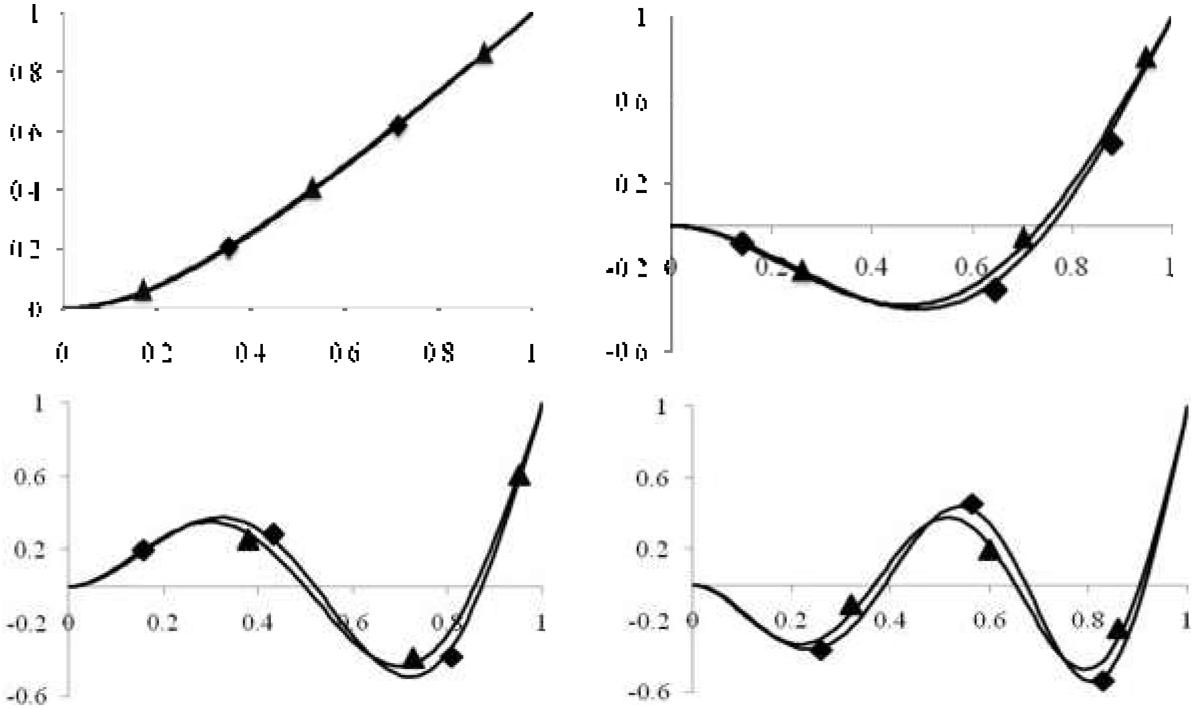

Fig. 3: First four normalized mode shapes of beams described in examples 2 ( ) and 3 ( ) for $\eta = 10$

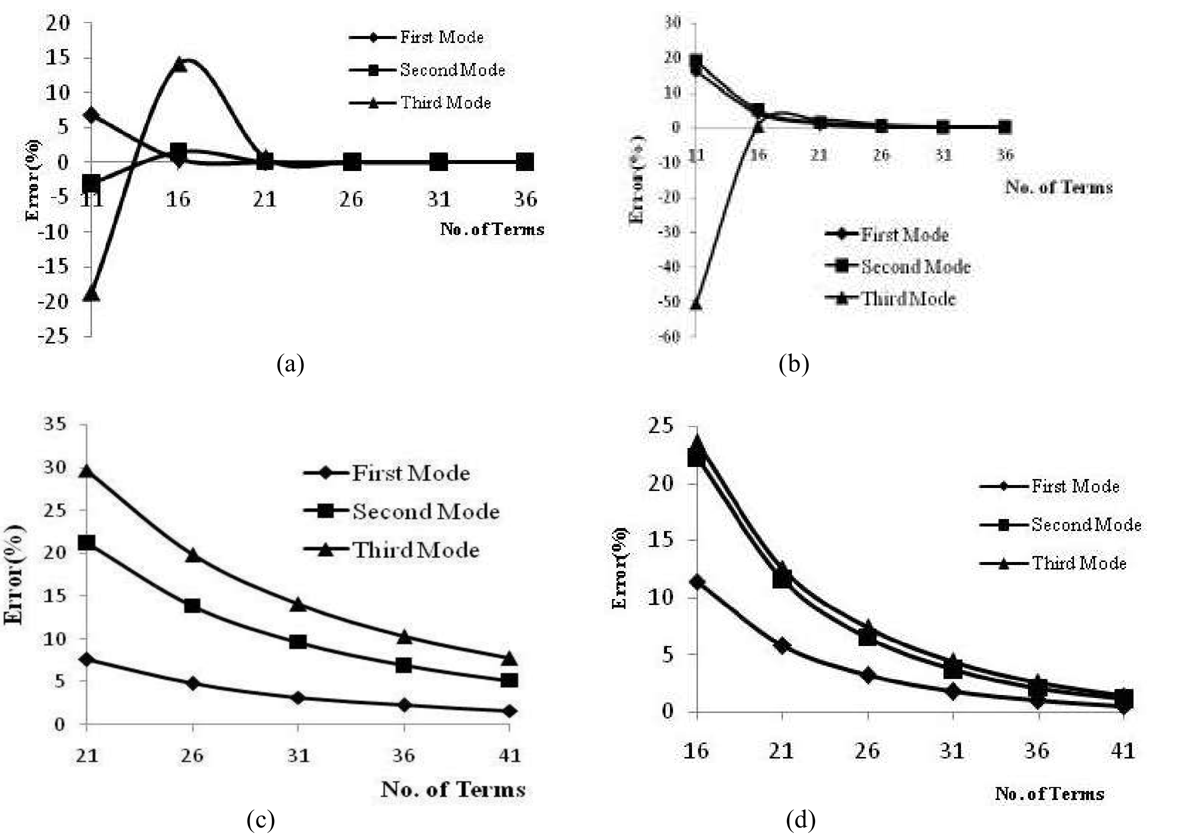

Fig. 4: Effect of No. of terms on accuracy of frequency computations when $\eta = 0$ for (a) $c = 0.4$; (b) $c = 0.6$; (c) $c = 0.8$; (d) $c = 0.9$

$$A(\xi) = A_0(1 - c\xi)^n \quad (23)$$

$$I(\xi) = I_0(1 - c\xi)^m \quad (24)$$

then c is such that $0 \leq c < 1$. Obviously c = 0 corresponds to the uniform beam and if $c \geq 1$, the beam tapers to zero between its ends or at one of its ends which is not practically possible. The authors observed that DTM has low rate of convergence as taper ratio increases towards unity and it is required to use more terms to obtain satisfactory results. This concept is illustrated in Fig. 4 for numerical Example 3. The results from Ref. [7] are regarded as the basis for error calculations. It is evident that by using more of terms, more accurate results are obtained and the error converges to zero. For higher values of taper ratio, errors concerning higher frequencies are greater than lower frequencies.

CONCLUSIONS

Dynamic Stiffness matrix was derived for general non-prismatic rotating Euler-Bernoulli beams using differential transform method. The method is applicable for beam elements with any type of cross-section and profile changes. It seems that the method is capable of modeling most of practical engineering problems concerning non-prismatic members. The natural dimensionless frequencies were determined and the effects of taper ratio and rotational speed parameter on variation of natural frequencies were studied. It was revealed that the convergence of DTM greatly depends on taper ratio especially as it increases towards unity. Comparing the results with those existing in literature, its competency in accuracy was verified.

REFERENCES

1. Kolousek, V., 1973. Dynamics in engineering structures. Butterworth, London.
2. Carlson, R.M. and J.T. Wang, 1978. An exact solution for the static bending of uniform rotating beams. Journal of American. Helicopter Society, 23 (10): 36-38.
3. Attarnejad, R. and A.K. Ghorbani-Tanha, 2007. A general fourth order ordinary differential equation with solution in terms of Bessel functions: Theory and engineering applications. Structural Engineering and Mechanics, 27: 773-776.
4. Banerjee, J.R., 2000. Free vibration of centrifugally stiffened uniform and tapered beams using the dynamic stiffness method. Journal of Sound and Vibration, 233: 857-875.
5. Wang, G. and N.M. Wereley, 2004. Free vibration analysis of rotating blades with uniform tapers. American Institute of Aeronautics and Astronautics Journal, 42 (12): 2429-2437.
6. Auciello, N.M. and A. Ercolano, 2004. A general solution for dynamic response of axially loaded non-uniform Timoshenko beams. International Journal of Solids and Structures, 41: 4861-4874.
7. Banerjee, J.R., H. Su and D.R. Jackson, 2006. Free vibration of rotating tapered beams using the dynamic stiffness method. Journal of Sound and Vibration, 298: 1034-1054.
8. Vinod, K.G., S. Gopalakrishnan and R. Ganguli, 2007. Free vibration and wave propagation analysis of uniform and tapered rotating beams using spectrally formulated finite elements. International Journal of Solids and Structures, 44: 5875-5893.
9. Gunda, J.B. and R. Ganguli, 2008. New rational interpolation functions for finite element analysis of rotating beams. International Journal of Mechanical Sciences, 50: 578-588.
10. Zhou, J.K., 1986. Differential transform and its application for electrical circuits. Huazhong University Press, China.
11. Jang, M.J., C.L. Chen and Y.C. Liu, 2001. Two-Dimensional Differential Transform for Partial Differential Equations. Applied Mathematics and Computation, 121: 261-270.
12. Ayaz, Fatma, 2003. On the two-Dimensional Differential Transform method. Applied Mathematics and Computation, 143: 361-374.
13. Ayaz, Fatma, 2004. Solutions of the system of differential equations using differential transform method. Applied Mathematics and Computation, 147: 547-567.
14. Ho, S.H. and C.K. Chen, 1998. Analysis of general elastically end restrained non-uniform beams using differential transform. Applied Mathematical Modelling, 22: 219-234.
15. Ozdemir, O and M.O. Kaya, 2006. Flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by differential transform method. Journal of Sound and Vibration, 289: 413-420.
16. Ozgumus, O.O. and M.O. Kaya, 2006. Flapwise bending vibration analysis of double tapered rotating Euler-Bernoulli beam by using the differential transform method. Meccanica, 41: 661-670.
17. Seval, Çatal, 2008. Solution of free vibration equations of beam on elastic soil by using differential transform method. Applied Mathematical Modelling, 32: 1744-1757.

18. Mei, C., 2008. Application of differential transformation technique to free vibration analysis of a centrifugally stiffened beam. Computers and Structures, 86: 1280-1284.
19. Wright, A.D., C.E. Smith, R.W. Thresher and J.L.C. Wang, 1982. Vibration modes of centrifugally stiffened beams. Journal of Applied Mechanics, 49 (1): 197-202.
20. Paz, Mario, 1991. Structural dynamics: Theory and Computation. Van Nostrand Reinhold, New York.
21. Attarnejad, R., 2000. On the derivation of the geometric stiffness and consistent mass matrices for non-prismatic Euler-Bernoulli beam elements. In the Proceedings of European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2000).
22. Attarnejad, R., 2002. Free Vibration of Non-Prismatic Beams. In the Proceedings of the 15th ASCE Engineering Mechanics Conference.