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Application of $Exp(-\varphi(\xi))$-expansion Method to Find the Exact Solutions of Nonlinear Evolution Equations

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Abstract: *In this paper, we explore new applications of the* $\exp(-\varphi(\xi))$*-expansion method for finding exact traveling wave solutions of generalized Klein-Gordon Equation and right-handed nc-Burgers equation. By means of this method three new solutions of each equations is obtained including the hyperbolic functions, exponential functions and rational function solutions. The proposed method is very effective, efficient and applicable mathematical tools for nonlinear evolution equations (NLEEs). So this method can be used for many other nonlinear evolution equations.*

Keywords: *The* $\exp(-\varphi(\xi))$*-expansion method; generalized Klein-Gordon Equation and right-handed nc-Burgers equation; nonlinear partial differential equation; traveling wave solutions.*
Mathematics Subject Classification*: 35K99, 35P05, 35P99.*

I. Introduction

Nonlinear evolution equations (NLEEs) attracted much attention in a variety of applied science and the essential features of these equations are of wide applicability such as in fluid mechanics, aero dynamics, nonlinear optics, plasma physics, hydrodynamics, chemistry and biology. In recent years, both mathematicians and physicist have devoted considerable effort to study of exact solution of the nonlinear ordinary or partial differential equation and many powerful methods have been presented. For instance the inverse scattering transform [1], the complex hyperbolic function method [2, 3], the rank analysis method [4], the ansatz method [5, 6], the (G'/G)-expansion method [7-14], the modified simple equation method [15, 16], the exp-functions method[17], the sine-cosine method [18], the Jacobi elliptic function expansion method [19, 20], the F-expansion method[21, 22], the Backlund transformation method [23], the Darboux transformation method [24], the homogeneous balance method [25-27], the Adomian decomposition method [28, 29], the auxiliary equation method[30, 31], the $\exp(-\varphi(\xi))$-expansion method [32, 33] and so on.

In this paper, we use the $\exp(-\varphi(\xi))$-expansion method to seek the traveling wave solutions of generalized Klein-Gordon equation [34] of the form

$$U_{tt} + \alpha U_{xx} + \beta_1 U + \beta_2 U^3 = 0 \tag{1}$$

where α, β_1 and β_2 are arbitrary constants.

It contains some particular important equations, such as Duffing equation, Landau-Ginzburg-Higgs equation and so on.

When $\alpha = -1, \beta_1 = -m^2, \beta_2 = g^2$ Eq (1) becomes Landau-Ginzburg-Higgs equation [1]

$$U_{tt} - U_{xx} - m^2 U + g^2 U^3 = 0 \tag{2}$$

When $\alpha = 0$ Eq(1) becomes Duffing equation [35]

$$U_{tt} + \beta_1 U + \beta_2 U^3 = 0 \tag{3}$$

We also seek traveling wave solution of the right-handed nc-Burgers equation using the $\exp(-\varphi(\xi))$-expansion method.

The paper is organized as follows: In section 2, we make an introduction of the $\exp(-\varphi(\xi))$-expansion method. By this method, we obtain the exact solutions of the generalized Klien-Gordon equation and right-handed nc-Burgers equation in section 3. In section 4, some conclusions are given.

II. Description of the $\exp(-\varphi(\xi))$-expansion method

In the following, we will summarize the main steps of $\exp(-\varphi(\xi))$-expansion method. Consider a nonlinear equation, say in two independent variable x and t, is given by

$$P(U,U_x,U_t,U_{xx},U_{xt},U_{tt},\ldots\ldots)=0 \tag{4}$$

where $U=U(x,t)$ is an unknown function, P is a polynomial in $U=U(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Combining the independent variable x and t into one variable $\xi=x\pm Vt$, we suppose that

$$U(x,t)=u(\xi),\ \xi=x\pm Vt, \tag{5}$$

the travelling wave variable (5) permits us reducing Eq.(4) to an ODE for $u=u(\xi)$

$$P(u,u',u'',\cdots\ \cdots\cdots)=0 \tag{6}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $\exp(-\varphi(\xi))$ as follows

$$u=\sum_{i=0}^{m}a_i\exp(-\varphi(\xi))^i \tag{7}$$

where $\varphi'(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi)=\exp(-\varphi(\xi))+\mu\exp(\varphi(\xi))+\lambda, \tag{8}$$

then the solutions of ODE (8) are

when $\lambda^2-4\mu>0,\mu\neq 0$, then $\varphi(\xi)=\ln\left(\dfrac{-\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C))-\lambda}{2\mu}\right)$ (9)

When $\lambda^2-4\mu>0,\mu=0$, then $\varphi(\xi)=-\ln\left(\dfrac{\lambda}{\exp(\lambda(\xi+C))-1}\right)$ (10)

When $\lambda^2-4\mu=0,\mu\neq 0,\lambda\neq 0$, then $\varphi(\xi)=\ln\left(-\dfrac{2(\lambda(\xi+C)+2)}{\lambda^2(\xi+C)}\right)$ (11)

When $\lambda^2-4\mu=0,\mu=\lambda=0$, then $\varphi(\xi)=\ln(\xi+C)$ (12)

When $\lambda^2-4\mu<0$, then $\varphi(\xi)=\ln\left(\dfrac{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C))-\lambda}{2\mu}\right)$ (13)

$a_i,V,\lambda;\ i=0,\cdots\cdots,m$ and μ are constants to be determined later, $a_m\neq 0$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE(6).

Step 3. By substituting (7) into Eq.(6) and using the ODE (8), collecting all terms with the same order of $\exp(-\varphi(\xi))$ together, the left hand side of Eq.(6) is converted into another polynomial in $\exp(-\varphi(\xi))$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i,\cdots V,\lambda;\ i=0,\cdots\cdots,m$ and μ.

Step 4. Assuming that the constants $a_i,\cdots V,\lambda;\ i=0,\cdots\cdots,m$ and μ can be obtained by solving the algebraic equations in step 3, since the general solutions of ODE (8) have been well known for us, then substituting $a_i,\cdots V;\ i=0,\cdots\cdots,m$, and the general solutions of Eq.(8) into (7). We have more traveling wave solutions of nonlinear evolution equation (1).

In the subsequent sections we will illustrate the proposed method in detail with various nonlinear evolution equations in mathematical physics.

III. Application

3.1. The generalized Klein-Gordon equation

In this sub-section, first we will exert the $\exp(-\varphi(\xi))$-expansion method to seek the traveling wave solutions of the generalized Klein-Gordon equation (1).
Substitute (5) into (1) we change Eq. (1) into the ODE:

$$(V^2+\alpha)u''+\beta_1 u+\beta_2 u^3=0 \qquad (14)$$

Balance the highest order derivate and the highest nonlinear terms in equation (14), we get $m=1$, so assume the equation (7) has the following solution

$$u(\xi)=a_0+a_1\exp(-\varphi(\xi));\ a_1\neq 0 \qquad (15)$$

where $u(x,t)=u(\xi),\ \xi=x-Vt$

Substitute (5), (8) and(15) into (14), let the coefficient of $(\exp(-\varphi(\xi)))^i,\ (i=0,1,2,\cdots)$ be zero, yields a set of algebraic equations about a_i, V as follows:

$$\begin{cases} 2a_1\alpha+2a_1V^2+\beta_2 a_1^3=0 \\ 3a_1V^2\lambda+3a_1\lambda\alpha+3\beta_2 a_0 a_1^2=0 \\ a_1\alpha\lambda^2+3\beta_2 a_0^2 a_1+2a_1V^2\mu+a_1V^2\mu+a_1V^2\lambda^2+\beta_1 a_1+2a_1\alpha\mu=0 \\ \beta_2 a_0^3+a_1V^2\mu\lambda+a_1\mu\lambda\alpha+\beta_1 a_0=0 \end{cases}$$

Then the solution of the equations is

$$V=\pm\sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta_1}{4\mu-\lambda^2}},\ a_0=\pm\lambda\sqrt{\frac{\beta_1}{(4\mu-\lambda^2)\beta_2}},\ a_1=\pm 2\sqrt{\frac{\beta_1}{(4\mu-\lambda^2)\beta_2}} \qquad (16)$$

Substituting (16) into (15) , we have

$$u=\pm\sqrt{\frac{\beta_1}{(4\mu-\lambda^2)\beta_2}}(\lambda+2\times\exp(-\varphi(\xi))) \qquad (17)$$

where $\xi=x\pm\sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta_1}{4\mu-\lambda^2}}t$

Respectively substituting (9), (10) and (13) into the formula (17), we have three types of traveling wave solutions of the generalized Klien-Gordon equations (1) as followings
when $\lambda^2-4\mu>0, \mu\neq 0$, then

$$u_{1,2}=\pm\sqrt{\frac{\beta_1}{(4\mu-\lambda^2)\beta_2}}\left(\lambda-\frac{4\mu}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C))+\lambda}\right) \qquad (18)$$

where $\xi=x\pm\sqrt{\frac{\alpha\lambda^2-4\alpha\mu-2\beta_1}{4\mu-\lambda^2}}t$ and C is arbitrary constant.

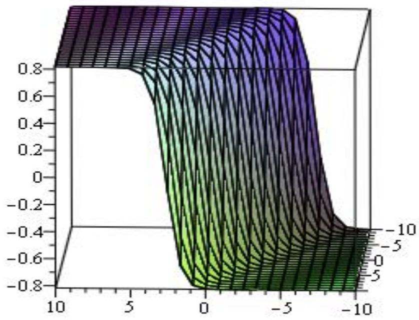

Fig. 1: Kink solution, the shape of solution of $u_{1,2}(\varphi)$ with $\beta_1=-2$, $\beta_2=3$, $\lambda=3$, $\mu=1$, $\alpha=-1$, $c=1$ with $-10\le x,t\le 10$.

When $\lambda^2-4\mu>0, \mu=0$, then

$$u_{3,4}=\pm\sqrt{\frac{-\beta_1}{\beta_2}}\left(1+\left(\frac{2}{\exp(\lambda(\xi+C))-1}\right)\right), \tag{19}$$

where $\xi=x\pm\sqrt{\dfrac{\alpha\lambda^2-4\alpha\mu-2\beta_1}{4\mu-\lambda^2}}t$ and C is arbitrary constant.

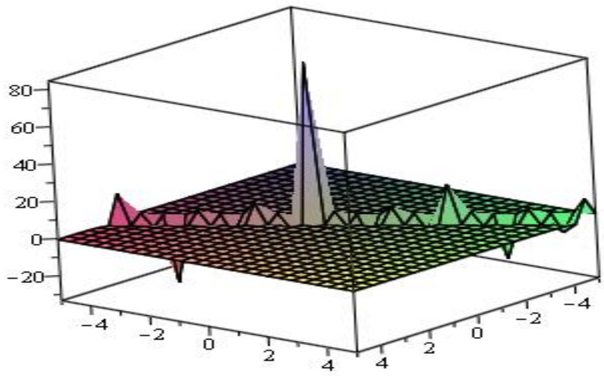

Fig. 2: Singular Kink solution, the shape of solution of $u_{3,4}(\varphi)$ with $\beta_1=-1$, $\beta_2=4$, $\lambda=2$, $\mu=0$, $\alpha=-1$, $c=1$ with $-5\le x,t\le 5$.

When $\lambda^2-4\mu<0$, then

$$u_{5,6}=\pm\sqrt{\frac{\beta_1}{(4\mu-\lambda^2)\beta_2}}\left(\lambda+\frac{4\mu}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C))-\lambda}\right), \tag{20}$$

where $\xi = x \pm \sqrt{\dfrac{\alpha\lambda^2 - 4\alpha\mu - 2\beta_1}{4\mu - \lambda^2}}t$ and C is arbitrary constant.

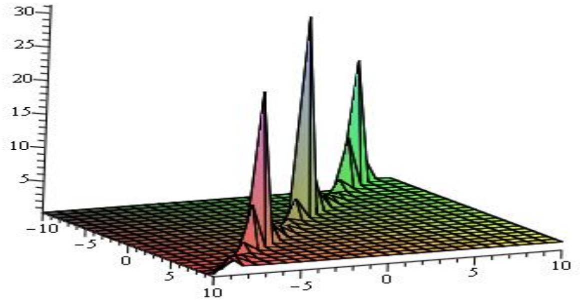

Fig. 3: Singular Kink solution, the shape of solution of $u_{5,6}(\varphi)$ with $\beta_1 = -1$, $\beta_2 = 2$, $\lambda = 1$, $\mu = 2$, $\alpha = -1$, $c = 1$ with $-10 \le x,t \le 10$.

3.2. The right-handed nc-Burgers equation

In this sub-section, we will bring to bear the $\exp(-\varphi(\xi))$-expansion method to find the traveling wave solutions to the right-handed nc-Burgers equation

$$U_t = U_{xx} + 2UU_x \tag{21}$$

Using traveling wave transformation (5), (21) is reduced to the following ODE:

$$u'' + 2uu' + Vu' = 0 \tag{22}$$

Integrating (9) with respect to ξ and setting the constant of integration to zero, we obtain

$$u' + u^2 + Vu = 0 \tag{23}$$

Balancing the highest order derivative and nonlinear term, we obtain $m = 1$.
Now, the solutions of Eq.(23), according to Eq(7), is

$$u(\xi) = a_0 + a_1 \exp(-\varphi(\xi));\ a_1 \neq 0 \tag{24}$$

where $u(x,t) = u(\xi),\ \xi = x - Vt$

Substitute (5),(8) and (24) into (23), let the coefficient of $(\exp(-\varphi(\xi)))^i$, $(i = 0,1,2,\cdots)$ be zero, yields a set of algebraic equations about a_i, V as follows:

$$\begin{cases} a_1^2 - a_1 = 0 \\ 2a_0a_1 - a_1\lambda + Va_1 = 0 \\ a_0^2 + Va_0 - a_1\mu = 0 \end{cases}$$

Then the solution of the equations is

$$V = \pm\sqrt{\lambda^2 - 4\mu},\ a_0 = \frac{\lambda}{2} \mp \frac{\sqrt{\lambda^2 - 4\mu}}{2},\ a_1 = 1 \tag{25}$$

Substituting (16) into (15) , we have

$$u = \frac{\lambda}{2} \mp \frac{\sqrt{\lambda^2 - 4\mu}}{2} + \exp(-\varphi(\xi))) \tag{26}$$

where $\xi = x \pm \sqrt{\lambda^2 - 4\mu t}$

Respectively substituting (9), (10) and (13) into the formula (26), we have three types of traveling wave solutions of the right-handed nc-Burgers equations (21) as followings

when $\lambda^2 - 4\mu > 0, \mu \neq 0$, then

$$U=\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}-\frac{2\mu}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C))+\lambda} \tag{27}$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

$$U=\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4\mu}}{2}-\frac{2\mu}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C))+\lambda} \tag{28}$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

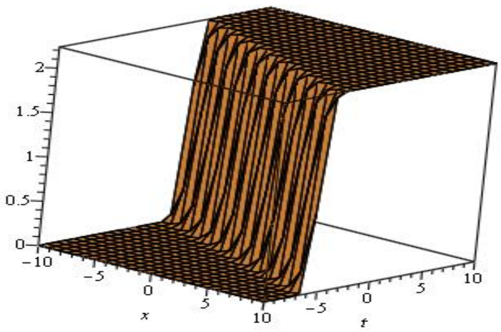

Fig. 4: Kink solution, the shape of solution of Eq. (27) and (28) with $\lambda=3,\ \mu=1,\ C=2$ with $-10\le x,t\le 10$.

When $\lambda^2-4\mu>0,\mu=0$, then

$$U=\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}+\left(\frac{\lambda}{\exp(\lambda(\xi+C))-1}\right), \tag{29}$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

$$U=\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4\mu}}{2}+\left(\frac{\lambda}{\exp(\lambda(\xi+C))-1}\right), \tag{30}$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

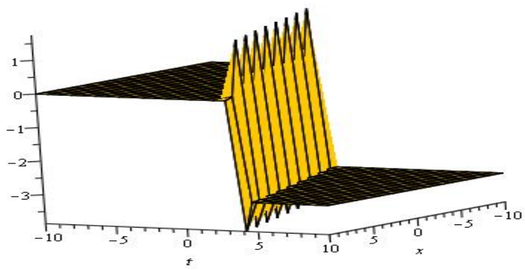

Fig. 5: Singular Kink solution, the shape of solution of Eq. (29) and (30) with $\lambda=3,\ \mu=0,\ C=2$ with $-10\le x,t\le 10$.

When $\lambda^2-4\mu<0$, then

$$U=\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}+\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C))-\lambda}, \quad (31)$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

$$U=\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4\mu}}{2}+\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C))-\lambda}, \quad (32)$$

where $\xi=x+\sqrt{\lambda^2-4\mu}t$ and C is arbitrary constant.

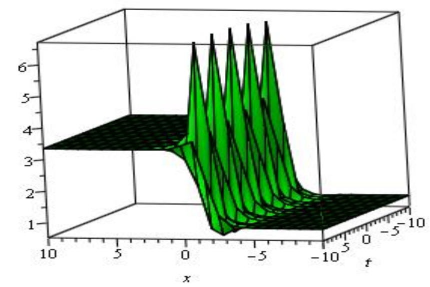

Fig. 6: Singular Kink solution, the shape of solution of Eq. (31) and (32) with $\lambda=1,\ \mu=1,\ C=2$ with $-10\le x,t\le 10$.

Remark: Some of these solutions presented in this latter have been checked with Maple by putting them back into the original equations.

IV. Conclusion:

This paper, we get new exact traveling wave solutions for the generalized Klein-Gordon equations and the right-handed nc-Burgers equation, including the hyperbolic functions, exponential functions and rational function solutions. Otherwise, the general solutions of the ODE have been well known for the researchers. Furthermore, the proposed method appears to be easier, faster and can be handle by hand. This study confirms that the method is direct, concise and more effective; it can be used in many other nonlinear evolution equations. This will have a good sense to promote the extensive application of the equations.

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