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7/25/68

APPLICATION OF FINITE ELEMENT ANALYSIS IN FLUID MECHANICS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Mustafa Mehmet Aral

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Civil Engineering

Georgia Institute of Technology

September, 1971

APPLICATION OF FINITE ELEMENT ANALYSIS IN

FLUID MECHANICS

Approved:

Chairman

Date approved by Chairman: 8/5/71

ACKNOWLEDGMENTS

The gratitude of the author is extended to all who helped to make the present investigation possible. Special appreciation is extended to Dr. Paul G. Mayer, Chairman of the Thesis Reading Committee, for his interest and guidance during the preparation of this thesis. Dr. Mayer is also thanked for suggesting this area of research to the author.

Appreciation is also extended to Dr. C. V. Smith, Jr. of Aerospace Engineering Department for his help and encouragement throughout the preparation of the thesis. Dr. G. M. Slaughter is thanked for his continuous encouragement and for serving on the reading committee.

The School of Civil Engineering is acknowledged for the financial assistance provided by means of research and teaching assistantships. The School of Civil Engineering is also acknowledged for providing the computer time necessary for developing the computer program.

Appreciation is extended to the personnel of the Rich Electronic Computer Center for providing assistance with the computer program.

Last but not the least the author wishes to thank his wife, Sevgi Aral for her patience and encouragement during the course of this study.

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SUMMARY

Application of finite element analysis to problems in continuum mechanics is a relatively recent research area. The study presented here summarizes this area and the detailed mathematical and numerical analysis involved in the finite element method. More specifically, general mathematical schemes are developed for field problems that can be expressed in terms of selected elliptic and parabolic second order partial differential equations. In these formulations certain variational principles are used to arrive at the integral finite element equations. This is a customary approach for most of the studies in this field. Also the use of Galerkin's method is introduced to develop similar integral finite element equations for partial differential equations which do not have exact variational forms. This, of course, extends the use of finite element analyses to field problems which cannot be expressed in terms of exact variational forms.

A computer program is developed which solves the selected partial differential equations in cartesian and cylindrical coordinates. Solutions to twelve transport mechanics problems are presented to illustrate the use, accuracy and the efficiency of the method and program, at least for problems of the kind analyzed in this study.

CHAPTER I

INTRODUCTION

Historical Review

The development of general discrete methods of structural mechanics started in the early 1950's with the fundamental work of Argyris [2]*. As would be expected, this initiation was parallel to the advent and extended usage of high speed digital computers. These techniques were originally applied to the analysis of highly complex aircraft structures.

In 1956 Turner et al., [40] established the basis of the finite element method. The first application was in plane stress analysis. Since then it has been extended to axi-symmetric stress analysis, flat plate bending, three dimensional stress analysis and shell analysis in the field of structures.

However, it should be emphasized that the finite element method is by no means restricted to the solution of structural problems. The idea that structural methods can actually be interpreted in terms of variational procedures such as finding the stationary point of the total potential energy of the system (or the functional in field problem applications) has led to the application of finite element method to problems of continuum mechanics. The first paper in this area was published in 1965. In this paper Zienkiewicz and Cheung [50] utilized the method of variational calculus to formulate the integral finite element equations for steady temperature distribution problem.

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*Numbers in brackets refer to items in the Bibliography.

Since then various researchers have extended this method to the solution of unsteady heat transfer problems [48], steady confined and unconfined seepage problems [51], [25], and steady ideal flow of fluids [12].

Scope of the Problem

Models of linear solutions for field problems which use the finite element method have recently become an active research area. A whole group of studies involves the application of the finite element method to a class of problems that can be cast into a variational form, such as the ideal flow of fluids, flow of viscous incompressible fluids, heat conduction, electrical and magnetic field problems and seepage through porous media. These models exhibit many advantages over conventional methods of numerical analysis, mainly due to the simplicity with which boundary conditions can be applied and the ease with which complex domains can be approximated.

Thus, applying the finite element method in fields other than that of structural engineering, prior knowledge of the proper variational expression is useful. For example, in steady state seepage flow and heat conduction problems, such variational forms can be obtained for the well known Laplace and Poisson's equations. For problems where exact variational forms cannot be found, such as unsteady field problems, the finite element method can again be utilized either by using restricted variational forms or through the use of Galerkin's method. Thus, in application to field problems, the finite element method represents an approximate procedure for the solution of problems, utilizing methods of variational calculus or Galerkin's method. As will become apparent in the present work, finite element analysis furnishes a refined scheme for the application of the well-known Ritz method.

In this dissertation the main consideration is given to the utilization of the finite element method in the solution of field problems in transport mechanics. Thus, the purpose of this study is to present the derivation of finite element equations which describe a discrete model and to develop a computer program for the solution of above mentioned field problems. More specifically, general mathematical schemes are developed for the partial differential equations summarized below:

a) $\frac{\partial}{\partial x}(K_x\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial \phi}{\partial y}) = 0$

b) $\frac{\partial}{\partial x}(K_x\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial \phi}{\partial y}) = g(x,y)$

c) $\frac{\partial}{\partial r}(K_r\frac{\partial \phi}{\partial r}) + \frac{K_r}{r}\frac{\partial \phi}{\partial r} + \frac{\partial}{\partial z}(K_z\frac{\partial \phi}{\partial z}) = 0$

d) $\frac{\partial}{\partial x}(K_x\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial \phi}{\partial y}) = K_t\frac{\partial \phi}{\partial t}$

e) $\frac{\partial}{\partial r}(K_r\frac{\partial \phi}{\partial r}) + \frac{K_r}{r}\frac{\partial \phi}{\partial r} + \frac{\partial}{\partial z}(K_z\frac{\partial \phi}{\partial z}) = K_t\frac{\partial \phi}{\partial t}$

f) $K_x\frac{\partial^2 \phi}{\partial x^2} + K_y\frac{\partial^2 \phi}{\partial y^2} - u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} = K_t\frac{\partial \phi}{\partial t}$

The first three partial differential equations model the steady diffusion problems in cartesian and cylindrical coordinates. For these partial differential equations exact variational forms are employed to arrive at integral finite element equations. Partial differential equations given in (d) and (e) model unsteady diffusion problems in cartesian and cylindrical coordinates respectively. For these partial differential equations restricted variational forms are utilized to arrive at the integral finite element equations. Also, for the same partial differential equations the use of Galerkin's method is demonstrated to yield the same integral finite element equations. This extension in the analysis, that is, the use of Galerkin's method, clearly increases the potential of the finite element method in application to problems in continuum mechanics. Using Galerkin's method, finite element analysis can be utilized in the solution of field problems which cannot be expressed in terms of exact variational forms. The last partial differential equation models convective diffusion of a certain concentration in a prescribed velocity field. For this partial differential equation again both the restricted variational form and Galerkin's method are utilized to derive the integral finite element equations. Analysis of this differential equation in itself is a major extension to fluid mechanics literature and illustrates the power of the finite element method in application to field problems.

As a result of this study a computer program is written which solves the above mentioned partial differential equations in cartesian and cylindrical coordinates. Twelve numerical examples are presented to illustrate the wide variety of the problems which can be solved.

Finite Element Concept

In the basic analysis of structural systems, frames, trusses, etc. have finite character. These systems consist of a finite number of elements interconnected at a finite number of joints or nodal points, as for example the truss system seen in Figure 1. This characteristic makes it possible to analyze such systems by means of simultaneous algebraic equations. Furthermore, this finite character is the basic difference between structural systems and continua. In a continuum, the true number of internodal points is infinite and therin lies the major difficulty to its solution. The concept of finite elements, as originally designated by Turner et al., [40] attempts to overcome this

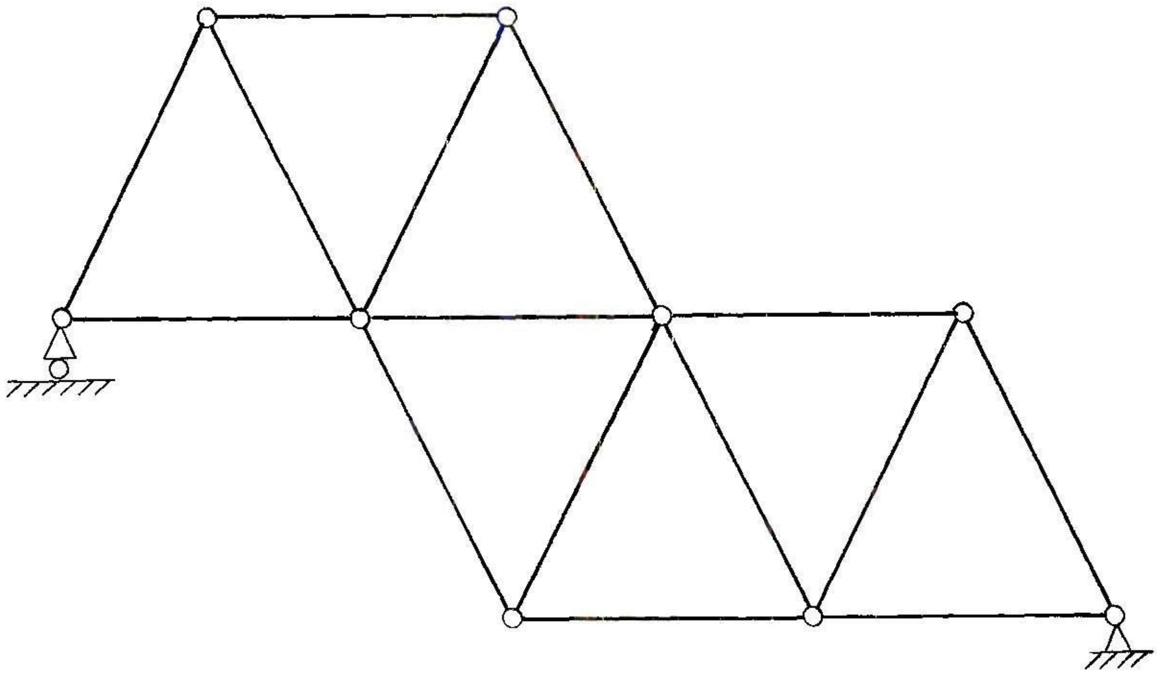

Figure 1. Simply Supported Truss

difficulty by conceptualizing the real continuum as divided into a set of two or three dimensional elements, interconnected only at a finite number of nodal points, at which some fictitious forces, representative of the distributed stresses actually acting on the element boundaries, are hypothesized to operate. If such an idealization is postulated, then the problem reduces to that of a conventional structural type, which in turn makes solution by simultaneous algebraic equations possible.

This idealization process can be shown schematically as seen below:

a) Basic structural analysis;

Frame Structure → (Idealization) → One dimensional structural elements

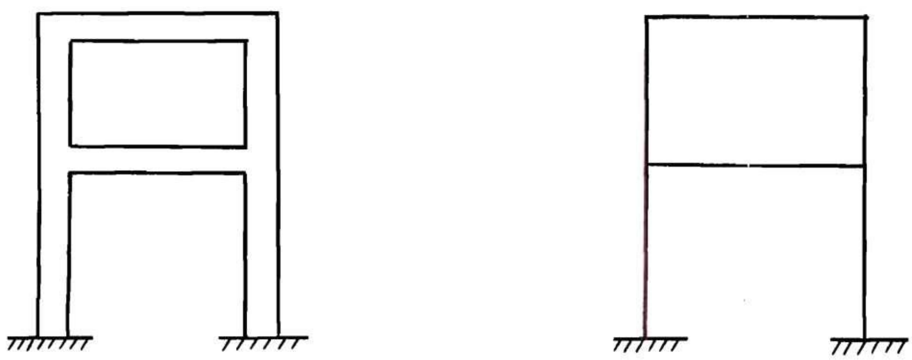

Figure 2. Idealization of a Frame Structure

b) Finite element analysis;

Continuum $\xrightarrow{\text{(Idealization)}}$ Two or three dimensional elements interconnected at a finite number of points.

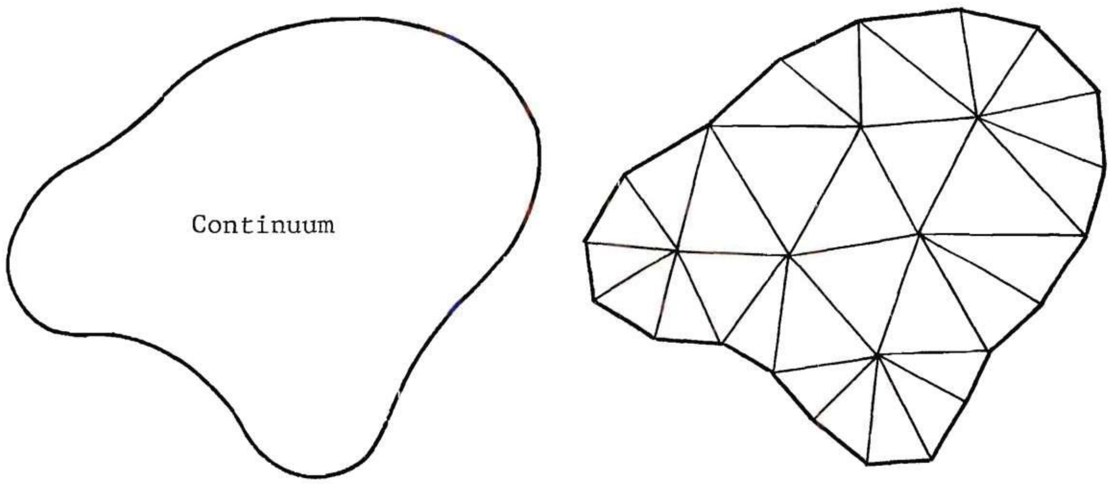

Figure 3. Idealization of a Continuum

The basic idea of discrete element techniques of structural analyses lies in the physical idealization of structures as assemblies of structural components. Thus, the formulative effort is reduced to the development of force-displacement (or stiffness) equations for the discrete elements that make up the structure to be analyzed.

Once the element relationships have been evaluated, the elements are assembled to form the complete analytical model of the structure by joining all elements at their respective junctures and applying, in the process, the requirements of point equilibrium and compatibility. Thus the stiffness matrix for the complete structure can be assembled merely by adding stiffness coefficients.

The idealization of a continuum is not a new idea. It has been used at various phases of structural analysis (plate bending problems, idealization of shells as space truss systems, etc.). Thus, the contribution of the finite element method is not the structural approximation of the continuum, but rather the use of two or three dimensional elements in this approximation. Idealization is obtained, as seen in Figure 3, merely by dividing the original continuum into segments of appropriate sizes and shapes, all of the properties such as elasticity, conductivity, permeability of the original system being retained in the individual elements. This in turn makes the treatment of arbitrary material properties possible, which by itself, is one of the principal advantages of the finite element method.

The finite element concept has been described intuitively above. A detailed explanation of the application of this method to structural systems can be found in references [2], [40], [26], [27] and [52]. What, in fact, has been suggested in the finite element approach to force-displacement analyses of structural systems is equivalent to finding the stationary point of the total potential energy of the system in terms of a prescribed displacement field.

Let U represent the strain energy of a structure under a system of distributed loads $\{P\}$ (per unit volume) and concentrated loads $\{R\}$ subject to displacements $\{f\}$ and $\{\beta\}$ respectively. If the system is in equilibrium both internally and externally, and if an infinitesimal change of displacements $\delta\{\beta\}$ and $\delta\{f\}$ is then imposed, the external work done by the loads must equal the change in strain energy, i.e.

$$\delta U = \{R\}^T \delta\{\beta\} + \int_V \{P\}^T \delta\{f\} dV \qquad (1.1)$$

If the external forces are conservative, Equation (1.1) can usually be written as follows:

$$\delta[U - \{R\}^T \{\beta\} - \int_V \{P\}^T \{f\} dV] = 0 \qquad (1.2)$$

The last two terms of the quantity in brackets above represent the potential energy of the load system, W. Hence

$$\emptyset = U + W \qquad (1.3)$$

represents the total potential energy of the system, and

$$\delta[\emptyset] = 0 \qquad (1.4)$$

This is simply the statement that the total potential energy must be stationary at the equilibrium configuration.

Thus, originating from structural methods interpreted in terms of variational procedures as the stationary points of certain functionals, the finite element method leads, in its extension, to other boundary value problems where a variational formulation is possible. A typical variational problem is one in which an energy expression

$$I(f,g) = \int_V F(f(x,y,z),\ g(x,y,z))dV \qquad (1.5)$$

must be stationary for variations δf and δg. That is,

$$\delta I(f, g) = 0 \tag{1.6}$$

Throughout the previous discussion, the analysis of problems have been treated from a structural analysis point of view. The calculus of variations is now introduced for the development of variational forms. This is the mathematical basis of the application of finite element analysis to field problems in fluid mechanics that will be considered in this dissertation. In fact, physical rather than mathematical reasoning can be applied to arrive at a definition of variational forms. To illustrate, a steady seepage problem will be analyzed below.

Let the total head for a steady seepage problem be denoted by "h". For the two dimensional case, the following can be written:

$$h = h\ (x,y) \tag{1.7}$$

If the flow is through an isotropic medium, Darcy's law indicates that the velocity of flow in the x-direction, u, is

$$u = -\ K \frac{\partial h}{\partial x} \tag{1.8}$$

and the velocity in the y-direction, v, is

$$v = -\ K \frac{\partial h}{\partial y} \tag{1.9}$$

where K is the permeability of the medium. The pressure drop, Δp, per unit volume over a distance Δx is

$$\frac{\partial p}{\partial x} = -\gamma_w \frac{\partial h}{\partial x} \tag{1.10}$$

Thus, the rate of dissipation of energy due to the flow in the x-direction is

$$I_x = \frac{\partial p}{\partial x} \cdot u \tag{1.11}$$

or

$$I_x = \gamma_w K(\frac{\partial h}{\partial x})^2 \tag{1.12}$$

Hence, the total rate of energy dissipation over the entire region of the problem is

$$I = \iint_A \gamma_w K[(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2]dxdy \tag{1.13}$$

Physical reasoning indicates that a steady state case must be one in which the rate of energy dissipation is a minimum; thus, the first variation of I must vanish. Taking the first variation of Equation (1.13),

$$\delta I = \delta\iint_A \gamma_w K[(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2]dxdy = 0 \tag{1.14}$$

or

$$\delta\iint_A [(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2]dxdy = 0 \tag{1.15}$$

From this variational form, equation (1.16) can be obtained by the application of Euler's theorem of the calculus of variations [23], [44], [10], [4]. Accordingly,

$$\frac{\partial}{\partial x}(2\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(2\frac{\partial h}{\partial y}) = 0 \tag{1.16}$$

or

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (1.17)$$

This is the well known Laplace equation, which is the differential statement of the steady seepage problem. As can be seen from the above example, the differential equation obtained by the application of the variational approach using either physical or mathematical reasoning is identical to that derived from considerations of continuity and equilibrium. This means that for a well-posed physical problem, the variational approach and the equilibrium-continuity approach must give the same results. Thus, the choice between the two methods would be reduced to considerations of convenience.

Generally speaking, it can be stated that there are two alternate procedures that can be used in the numerical solution of field problems, provided that a solution is possible in the first place.

a) A partial differential equation can be written and its direct solution can be attempted, or

b) the problem can possibly be transformed into a variational form. The solution in this case involves finding a function which will make a certain functional stationary over the whole field.

Finite difference approximation is utilized in the first case. Finite element approximation, as treated in this thesis, can be utilized in both cases. It can be expected that the two approaches result, occasionally, in identical equations for simple problems. The freedom of shape of element and the ease with which non-homogeneity and boundary conditions can be treated make finite element analyses advantageous.

Application of the finite element method to field problems requires different formulations for different types of problems. However, once formulated the mathematical model has considerable generality; it is applicable to different boundary conditions, and to different geometrical and physical properties of the field. The restriction as to the type of problem is of no great concern, since many different phenomena have a common form with regard to the differential equation model. For example, Laplace's equation provides a model for flow of ideal fluids, heat conduction, or seepage flows.

As to the mathematical preliminaries, knowledge of differential calculus and of matrix algebra are the basic requirements of finite element analysis. Matrix methods provide the most practical means of organizing the computations. It is a characteristic of the finite element method, whether used in a structural context or to describe other phenomena, that the essential properties of an element will be of the form encountered in structural analysis. Hence, after the formulation of matrix equations, the general procedures of assembly and solution will follow a pattern based on structural analogy and, needless to say, high speed calculators are an essential element of the whole solution process.

To summarize, the first approximation applied to the continua might be of a physical nature; a modified system is substituted for an actual continuum. There is no need for approximations in the differential equations of the substitute system. Approximations are introduced, however, in the assumed distributions of the unknowns. These distributions are not restricted to a certain type or form of a function. Thus, the steps involved in the analysis as studied in this dissertation are:

a) Formulation of the differential equation.

b) Derivation of the associated variational problem (if such exists).

c) Division of the continuum by imaginary lines or surfaces into a number of finite elements.

d) Formulation of the integral equations within the element.

e) Formulation of the finite element equations within an element.

f) Assembly of the equations for all elements.

g) Substitution of the boundary conditions into the resultant equations.

h) Solution of the resultant system of simultaneous algebraic equations.

CHAPTER II

MATHEMATICAL ANALYSIS

Elements

As indicated in the previous chapter, the finite element analysis of a field problem can be divided into three phases, namely:

a) Idealization of the continuum,

b) Formulation of element equations,

c) Assembly of elements and solution of the resultant set of simultaneous algebraic equations.

In the finite element method, the continuum is assumed to be divided into volume or surface elements which have finite dimensions. Such an idealization is dictated by the need to find an alternative form of the equilibrium equations which will be easier to solve than the governing equations of the continuum. This modified conceptualization of the system results in a set of algebraic equations rather than differential equations, thus simplifying the solution considerably.

The shape, size and distribution of the elements are arbitrary. Rectangular, triangular and three-dimensional elements have been used efficiently in various studies. However, due to their sufficient accuracy and greater adaptability in fitting arbitrary boundary geometries, triangular elements will be utilized throughout this dissertation. It is also assumed that the value of the dependent variable varies linearly over each element. This means that the value of the dependent variable at any point within the element can be expressed uniquely by the values of the variable at the nodes of the element and the position of the point under consideration in the element.

In developing element equations, to simplify the resultant equations, the origin of the local axes which define the local coordinates of the element nodes is placed at the centroid of the element and the principal axes are inclined in the direction of local anisotropy. Consider the triangular element shown in Figure 4. Consider further any dependent variable $\emptyset$, which is assumed to vary linearly over the element

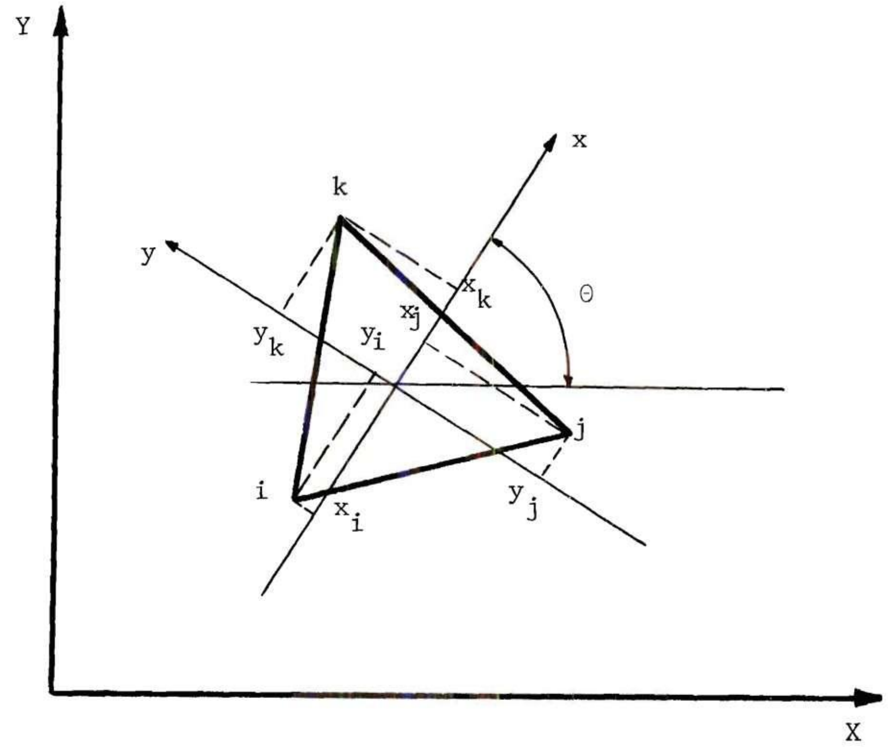

Figure 4. A Triangular Element

such that

$$\emptyset = a + bx + cy \tag{2.1}$$

The values of $\emptyset$ at the nodes of the triangular element (i,j, and k of Figure 4) can be written as:

$$\begin{aligned} \emptyset_i &= a + bx_i + cy_i \\ \emptyset_j &= a + bx_j + cy_j \\ \emptyset_k &= a + bx_k + cy_k \end{aligned} \tag{2.2}$$

Given the Equation (2.2), the constants a, b and c can be solved for in terms of $\emptyset_i$, $\emptyset_j$, $\emptyset_k$ and the coordinates of the nodes of the triangle. Using Cramer's rule,

$$a = \frac{1}{2A}\left[\emptyset_i(x_j y_k - x_k y_j) + \emptyset_j(x_k y_i - x_i y_k) + \emptyset_k(x_i y_j - x_j y_i)\right] \tag{2.3}$$

$$b = \frac{1}{2A}\left[\emptyset_i(y_j - y_k) + \emptyset_j(y_k - y_i) + \emptyset_k(y_i - y_j)\right] \tag{2.4}$$

$$c = \frac{1}{2A}\left[\emptyset_i(x_k - x_j) + \emptyset_j(x_i - x_k) + \emptyset_k(x_j - x_i)\right] \tag{2.5}$$

where, A is the area of the triangular element. Defining,

$$\begin{aligned} a_i &= (x_j y_k - x_k y_j)\frac{1}{2A},\quad a_j = (x_k y_i - x_i y_k)\frac{1}{2A},\quad a_k = (x_i y_j - x_j y_i)\frac{1}{2A} \\ b_i &= (y_j - y_k)\frac{1}{2A},\quad b_j = (y_k - y_i)\frac{1}{2A},\quad b_k = (y_i - y_j)\frac{1}{2A} \\ c_i &= (x_k - x_j)\frac{1}{2A},\quad c_j = (x_i - x_k)\frac{1}{2A},\quad c_k = (x_j - x_i)\frac{1}{2A} \end{aligned} \tag{2.6}$$

Then,

$$a = \emptyset_i a_i + \emptyset_j a_j + \emptyset_k a_k \tag{2.7}$$

$$b = \emptyset_i b_i + \emptyset_j b_j + \emptyset_k b_k \tag{2.8}$$

$$c = \emptyset_i c_i + \emptyset_j c_j + \emptyset_k c_k \tag{2.9}$$

Substituting Equations (2.7), (2.8), (2.9) into Equation (2.1), $\emptyset$ can now be expressed in terms of its nodal values in the triangular element considered. Thus,

$$\emptyset = (\emptyset_i a_i + \emptyset_j a_j + \emptyset_k a_k) + (\emptyset_i b_i + \emptyset_j b_j + \emptyset_k b_k)x + (\emptyset_i c_i + \emptyset_j c_j + \emptyset_k c_k)y \tag{2.10}$$

or

$$\emptyset = (a_i + b_i x + c_i y)\emptyset_i + (a_j + b_j x + c_j y)\emptyset_j + (a_k + b_k x + c_k y)\emptyset_k \tag{2.11}$$

Defining,

$$N_i = a_i + b_i x + c_i y \tag{2.12}$$

$$N_j = a_j + b_j x + c_j y \tag{2.13}$$

$$N_k = a_k + b_k x + c_k y \tag{2.14}$$

and

$$\{\phi^e\} = \begin{Bmatrix} \phi_j \\ \phi_j \\ \phi_k \end{Bmatrix} \qquad (2.15)$$

Then, the matrix form of Equation (2.11) will be,

$$[\phi] = [N_i \ N_j \ N_k] \ \{\phi^e\} \qquad (2.16)$$

This final form of Equation (2.16) is in agreement with the statement made earlier: "The value of the dependent variable at any point within the element is determined uniquely by the values of the variable at the nodes of the element and the position of the point under consideration in the element." More refined models can also be studied in finite element analysis [49], [3], merely by starting with a non-linear variation of the dependent variable in the element instead of a linear one as in Equation (2.1).

Different shapes (triangles, rectangles, etc.) and sizes can be used together and/or separately in the analysis; the only requirement is that the assumed function be continuous throughout the element. Mixed use of different shapes is admissible but complicates the computer program. In addition to the plane triangular elements, shown in Figure 4, there are several different shapes of elements which have been used to represent various continua. Some of these are summarized in Figure 5.

To the writer's knowledge, the use of all different possible element shapes has not been completely investigated. However, most of the element shapes seen in Figure 4 have been used in stress analysis of

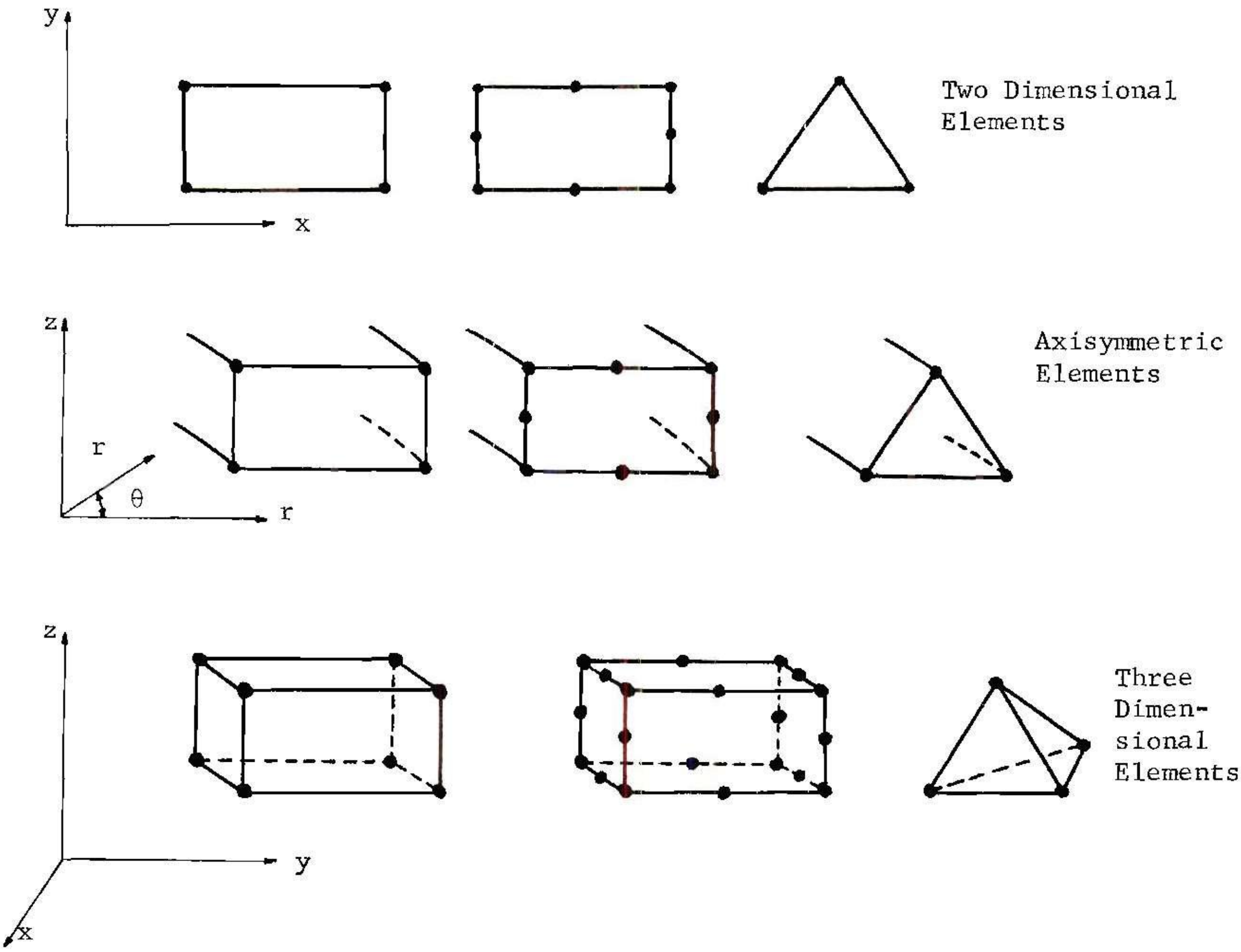

Figure 5. Some Typical Element Types

solids. Results obtained at Massachusetts Institute of Technology [6] and University of California at Berkeley [4] indicate that, in general, rectangular and more refined element shapes appear to yield slightly better approximations of stresses and deflections for a given nodal pattern than do triangular elements, since the former employ a more refined deformation approximation. However, because of their greater adaptability in fitting arbitrary boundary geometries and because they furnish sufficient accuracy, triangular elements have been used more widely in solution of field problems, even though some other shapes of elements may offer better approximations. The procedure in the formulation of these different shapes would remain basically unchanged, provided that equations of the type shown in Equation (2.16) can be derived. Later in this chapter matrix equations for a triangular ring element will be developed. Sizes of the elements can also be varied throughout the continuum. Thus, smaller elements can be used in the regions which show rapid variations in the properties of the continuum or in the values of the dependent variable.

Theoretically, the choice of element shape is critical since the analysis is actually performed on the substitute system, and the results can only be valid to the extent that the behavior of the actual continuum is simulated by the substitute system. In practice, however, idealization of the continuum does not present much difficulty. Previous work in this area indicates that reliable results have been frequently obtained by the use of rather coarse elements. However, the use of finer mesh idealization and, in the case of triangular elements, the use of equal sided triangles, improves the accuracy as would be expected.

Because of the well-known advantages of variational principles in treating arbitrary boundary conditions, the customary approach in deriving finite element equations, in application to field problems is to develop appropriate variational principles for the problem at hand. Thus, in this section, mathematical equations are developed in which elliptic and parabolic differential equations and their respective variational forms are employed. For problems in which exact variational forms cannot be found, the Galerkin method is utilized to arrive at the integral equations of the specific problem. It is clear that application of these formulations to different field problems is merely a matter of changing appropriate constants and variables. Solutions to some specific problems of this type are presented in Chapter IV.

The variational methods, or so-called direct methods, which have been mentioned previously, are used in many cases for solving boundary value problems. These methods, at least before the introduction of finite element analysis, were considered impractical by field engineers in the solution of field problems in fluid mechanics. Since variational analysis is central to the application of the finite element method considered in this study, the basic principles involved in this mathematical approach will be briefly reviewed.

The term functional is frequently used in variational calculus to describe functions defined by integrals whose arguments themselves are functions. Thus, given the functional below,

$$I = \int_{x_1}^{x_2} F(x,\ y(x),\ \frac{\partial y(x)}{\partial x})dx \qquad (2.17)$$

where F is a known function of the arguments x, y(x), $\frac{\partial y(x)}{\partial x}$, and y(x) is some function of x. The value of this integral depends not only on the points x_1 and x_2, the limits of integration, but also on the choice of function y(x). Then the function y = y(x) is sought (assuming its existence) which makes I stationary where y(x) is prescribed at $x = x_1$

and $x = x_2$. The basic theorem of calculus of variations states that for $y(x)$ to make Equation (2.17) stationary, it should satisfy Euler's equation below.

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\left(\frac{\partial y}{\partial x}\right)}\right) = 0 \qquad (2.18)$$

For the case of a function of several variables the problem is formulated in a similar way. For example, in a given region V bounded by a surface S there exists some function $\emptyset$ of three variables x, y, and z. Suppose on the surface S the values of this function are given, $\emptyset(s) = \psi(s)$, where $\psi(s)$ is known. It is required to choose from all the possible functions $\emptyset(x,y,z)$ one which fulfills the following conditions:

a) On the surface S the function $\emptyset(x,y,z)$ takes the given values, and

b) $\emptyset(x,y,z)$ gives an extremum of the integral

$$I = \iiint_V F\left(x,y,z,\emptyset,\frac{\partial\emptyset}{\partial x},\frac{\partial\emptyset}{\partial y},\frac{\partial\emptyset}{\partial z}\right)dxdydz \qquad (2.19)$$

It can be shown that the required function should be the solution of Euler's equation, namely

$$\frac{\partial F}{\partial\emptyset} - \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial p}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial q}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial F}{\partial r}\right\} = 0 \qquad (2.20)$$

where

$$p = \frac{\partial\emptyset}{\partial x},\quad q = \frac{\partial\emptyset}{\partial y},\quad r = \frac{\partial\emptyset}{\partial z}$$

Consider the integral below,

$$I = \iiint_V \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 + 2\phi f(x,y,z) \right] dxdydz \qquad (2.21)$$

where f(x,y,z) is a known function. Here,

$$F = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 + 2\phi f(x,y,z) \qquad (2.22)$$

$$\frac{\partial F}{\partial \phi} = 2f(x,y,z)$$

$$\frac{\partial F}{\partial p} = 2\frac{\partial \phi}{\partial x}, \quad \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial p}\right\} = 2\frac{\partial^2 \phi}{\partial x^2},$$

$$\frac{\partial F}{\partial q} = 2\frac{\partial \phi}{\partial y}, \quad \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial q}\right\} = 2\frac{\partial^2 \phi}{\partial y^2},$$

$$\frac{\partial F}{\partial r} = 2\frac{\partial \phi}{\partial z}, \quad \frac{\partial}{\partial z}\left\{\frac{\partial F}{\partial r}\right\} = 2\frac{\partial^2 \phi}{\partial z^2},$$

Substituting these values in Euler's equation one can obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = f(x,y,z) \qquad (2.23)$$

Thus, the function $\phi(x,y,z)$, which gives the extremum of the integral Equation (2.21) should satisfy a Poisson equation with the given boundary conditions.

This problem may also be inverted. In order to solve a Poisson equation with specified boundary conditions, one can find a function $\phi(x,y,z)$ which, when substituted, makes the integral Equation (2.21) stationary, and it will be the required solution. Thus, the present study will employ finite element analysis in conjunction with the above mentioned inverted process in the calculus of variations for the solution of elliptic and parabolic differential equations encountered in fluid mechanics.

Elliptic Partial Differential Equations

Elliptic partial differential equations arise usually from equilibrium or steady-state problems, and their solutions in relation to the calculus of variations frequently maximize or minimize an integral representing the energy of the system. The most familiar elliptic equations are Poisson's equation

$$\nabla^2 \phi = g(x,y) \tag{2.24}$$

and Laplace's equation

$$\nabla^2 \phi = 0 \tag{2.25}$$

Poisson's equation, for example, models the St. Venants theory of torsion, the slow motion of an incompressible fluid, and electricity and magnetism problems. Laplace's equation, for example, arises in field problems associated with the steady flow of heat in homogeneous conductors, with the irrotational flow of an incompressible fluid and with seepage problems.

The general form of a two-dimensional elliptic partial differential equation takes the form

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) + g(x,y) = 0 \tag{2.26}$$

where K_x, K_y, and $g(x,y)$ are known functions of x and y. Equation (2.26) together with appropriate boundary conditions define a field problem uniquely. However, as mentioned above, an alternate formulation is possible using the principles of calculus of variations. Applying the above stated Euler theorem of calculus of variations, it can be shown that the problem defined by Equation (2.26) is identical to that of finding a function $\emptyset(x,y)$ which makes the functional I stationary, where

$$I = \iint_A \left[\frac{K_x}{2}\left(\frac{\partial\emptyset}{\partial x}\right)^2 + \frac{K_y}{2}\left(\frac{\partial\emptyset}{\partial y}\right)^2 - g(x,y)\emptyset\right]dxdy \tag{2.27}$$

With no change in the boundary conditions, the integration is carried out over the whole region. Thus, it can be verified that the function $\emptyset$ of Equation (2.27) satisfies Equation (2.26) when I is stationary.

Since triangular elements are considered, Equation (2.16) as formulated earlier can be used.

$$\emptyset = [N_i \; N_j \; N_k]\{\emptyset^e\} \tag{2.16}$$

Equation (2.26) can now be solved by making the integral (2.27) stationary in an element. Thus,

$$\left\{\frac{\partial I}{\partial \emptyset^e}\right\} = \begin{Bmatrix} \frac{\partial I}{\partial \emptyset_i} \\ \frac{\partial I}{\partial \emptyset_j} \\ \frac{\partial I}{\partial \emptyset_k} \end{Bmatrix} = 0 \tag{2.28}$$

Differentiation with respect to the nodal values $\emptyset_i$, $\emptyset_j$, $\emptyset_k$ yields

$$\frac{\partial I}{\partial \emptyset_i} = \iint_A \left[K_x \frac{\partial\emptyset}{\partial x}\frac{\partial}{\partial \emptyset_i}\left(\frac{\partial\emptyset}{\partial x}\right) + K_y \frac{\partial\emptyset}{\partial y}\frac{\partial}{\partial \emptyset_i}\left(\frac{\partial\emptyset}{\partial y}\right) - g(x,y)\frac{\partial\emptyset}{\partial \emptyset_i}\right]dxdy = 0 \tag{2.29}$$

$$\frac{\partial I}{\partial \phi_j} = \iint_A [K_x \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi_j}(\frac{\partial \phi}{\partial x}) + K_y \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi_j}(\frac{\partial \phi}{\partial y}) - g(x,y)\frac{\partial \phi}{\partial \phi_j}]dxdy = 0 \quad (2.30)$$

$$\frac{\partial I}{\partial \phi_k} = \iint_A [K_x \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi_k}(\frac{\partial \phi}{\partial x}) + K_y \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi_k}(\frac{\partial \phi}{\partial y}) - g(x,y)\frac{\partial \phi}{\partial \phi_k}]dxdy = 0 \quad (2.31)$$

Since ϕ has been defined linearly in terms of x and y in an element, analysis of each term of Equations (2.29), (2.30) and (2.31) yields a set of constants and values of ϕ_i, ϕ_j, ϕ_k. In accordance with Equation (2.16) the following derivatives can now be taken,

$$\frac{\partial \phi}{\partial x} = \frac{\partial N_i}{\partial x}\phi_i + \frac{\partial N_j}{\partial x}\phi_j + \frac{\partial N_k}{\partial x}\phi_k \quad (2.32)$$

$$\frac{\partial}{\partial \phi_i}(\frac{\partial \phi}{\partial x}) = \frac{\partial N_i}{\partial x} \quad (2.33)$$

$$\frac{\partial N_i}{\partial x} = b_i, \frac{\partial N_j}{\partial x} = b_j, \frac{\partial N_k}{\partial x} = b_k \quad (2.34)$$

Thus,

$$\frac{\partial \phi}{\partial x} = b_i\phi_i + b_j\phi_j + b_k\phi_k \quad (2.35)$$

$$\frac{\partial}{\partial \phi_i}(\frac{\partial \phi}{\partial x}) = b_i \quad (2.36)$$

Similarly,

$$\frac{\partial \phi}{\partial y} = \frac{\partial N_i}{\partial y}\phi_i + \frac{\partial N_j}{\partial y}\phi_j + \frac{\partial N_k}{\partial y}\phi_k \quad (2.37)$$

$$\frac{\partial}{\partial \phi_i}(\frac{\partial \phi}{\partial y}) = \frac{\partial N_i}{\partial y} \quad (2.38)$$

$$\frac{\partial N_i}{\partial y} = c_i, \frac{\partial N_j}{\partial y} = c_j, \frac{\partial N_k}{\partial y} = c_k \tag{2.39}$$

Also,

$$\frac{\partial \emptyset}{\partial y} = c_i \emptyset_i + c_j \emptyset_j + c_k \emptyset_k \tag{2.40}$$

$$\frac{\partial}{\partial \emptyset_i}\left(\frac{\partial \emptyset}{\partial y}\right) = c_i \tag{2.41}$$

and

$$\frac{\partial}{\partial \emptyset_j}\left(\frac{\partial \emptyset}{\partial x}\right) = b_j \tag{2.42}$$

$$\frac{\partial}{\partial \emptyset_k}\left(\frac{\partial \emptyset}{\partial x}\right) = b_k \tag{2.43}$$

$$\frac{\partial}{\partial \emptyset_j}\left(\frac{\partial \emptyset}{\partial y}\right) = c_j \tag{2.44}$$

$$\frac{\partial}{\partial \emptyset_k}\left(\frac{\partial \emptyset}{\partial y}\right) = c_k \tag{2.45}$$

Assuming g(x,y), K_x and K_y to be constant within the element, one can substitute the results obtained above into Equations (2.29), (2.30) and (2.31), to form the matrix equation below.

$$\frac{\partial I}{\partial \emptyset^e} = [S^e]\begin{Bmatrix} \emptyset_i \\ \emptyset_j \\ \emptyset_k \end{Bmatrix} - \{F^e\} = 0 \tag{2.46}$$

or,

$$[S^e]\ \{\emptyset^e\} = \{F^e\} \tag{2.47}$$

where $[S^e]$ is known as element stiffness matrix and $\{F^e\}$ is known as element load vector which have the forms shown in Equations (2.48) and (2.49), respectively.

$$[S^e_{1,m}] = A[K_x b_1 b_m + K_y c_1 c_m] \tag{2.48}$$

$$\{F^e\} = g\frac{A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{2.49}$$

where (1,m) corresponds to the columns and rows of matrix $[S^e]$. The vector $\{F^e\}$ is known from structural anology as the element load vector. In this study it will be referred to as load vector to have correspondence with other available literature. Also, the stiffness matrix might be given various names in relation to its different properties in different field problems. The various names used in the identification of the matrix reflect the desire on the part of engineers to explain mathematical concepts in terms of their physical counterparts. Originally, the matrix has been called "Stiffness Matrix" because the finite element method was primarily a method devised for structural analysis, and stiffness of a plate or a beam is a common concept for structural engineers. However, when the method is applied to field problems, such as seepage, "Stiffness Matrix" of a porous medium does not have any physical meaning related to the problem being analyzed. In field problems, the stiffness matrix for an element consists of the properties of the material in the field (permeability, conductivity, etc.) and the nodal coordinates expressed in a local coordinate system. Thus, it could actually be called an "Element Characteristic Matrix," "characteristic" in the sense that it contains shape characteristics (coordinates) and material properties. However, this matrix will be referred to as the stiffness matrix throughout this thesis to assure correspondence with available literature.

Referring to Equation (2.47), for an element, the elliptic partial differential equation is expressed as an algebraic equation through the variational approach. The same procedure is applied to all elements. Then, the total stiffness matrix is arrived at by combining the element stiffness matrices in a manner similar to the method used to obtain a structural stiffness matrix. For a field of elements, a system of banded equations results. Thus,

$$[S]\ \{\phi\} = \{F\} \tag{2.50}$$

Here, [S] is the global matrix of coefficients (stiffness matrix) which incorporates the properties of the materials in the field and the geometry of the elements. $\{\phi\}$ is the vector of unknown ϕ's at the nodes and {F} is the load vector. This system of equations can now be solved by a number of computational procedures for the unknown nodal values of ϕ.

It is clear that the same type of formulation can be carried out for the partial differential equations where g(x,y) = 0. Thus, for the partial differential equation,

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) = 0 \tag{2.51}$$

the resulting equations will be in the form

$$[S]\{\phi\} = 0 \qquad (2.52)$$

where the stiffness matrix [S] has the same form as in Equation (2.50). Again, from this point on the same solution process follows which yields the unknown nodal values.

Axisymmetric Elements

In cylindrical coordinates the elliptic partial differential equation, analyzed before, can be written as

$$\frac{\partial}{\partial r}(K_r \frac{\partial \phi}{\partial r}) + \frac{K_r}{r}\frac{\partial \phi}{\partial r} + \frac{\partial}{\partial \Theta}(\frac{K_\Theta}{r^2}\frac{\partial \phi}{\partial \Theta}) + \frac{\partial}{\partial z}(K_z \frac{\partial \phi}{\partial z}) = 0 \qquad (2.53)$$

where K_r, K_Θ and K_z are known functions of r, Θ, z. If axial symmetry exists for the problem being analyzed then the partial derivatives with respect to Θ can be neglected. Thus, Equation (2.53) becomes,

$$\frac{\partial}{\partial r}(K_r \frac{\partial \phi}{\partial r}) + \frac{K_r}{r}\frac{\partial \phi}{\partial r} + \frac{\partial}{\partial z}(K_z \frac{\partial \phi}{\partial z}) = 0 \qquad (2.54)$$

This equation can be expressed in variational form as seen below,

$$I = 2\pi \iint_A [K_r \frac{r}{2}(\frac{\partial \phi}{\partial r})^2 + K_z \frac{r}{2}(\frac{\partial \phi}{\partial z})^2] dr dz \qquad (2.55)$$

A typical axisymmetric triangular element used in the solution of axisymmetric problems, is shown in Figure 6.

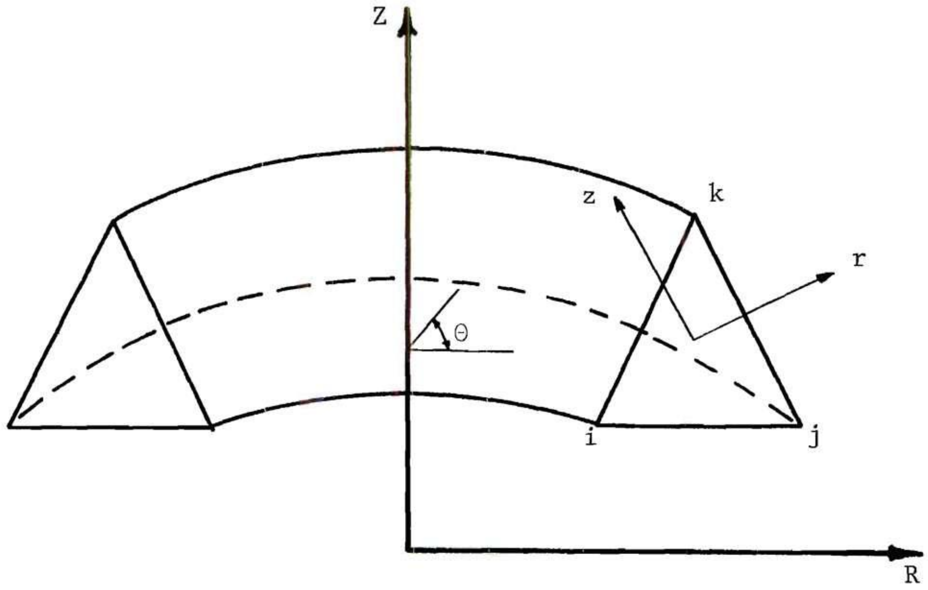

Figure 6. An Axisymmetric Element

If $\emptyset$ is assumed to be linear within the element, once again an equation similar to Equation (2.16) can be obtained. Thus,

$$\emptyset = [N_i \; N_j \; N_k] \{\emptyset^e\} \tag{2.56}$$

For axisymmetric elements, N_i, N_j, N_k are given as

$$N_i = a_i + b_i r + c_i z \tag{2.57}$$

$$N_j = a_j + b_j r + c_j z \tag{2.58}$$

$$N_k = a_k + b_k r + c_k z \tag{2.59}$$

where

$$a_i = \frac{1}{2A}(r_j z_k - r_k z_j),\ a_j = \frac{1}{2A}(r_k z_i - r_i z_k),\ a_k = \frac{1}{2A}(r_i z_j - r_j z_i), \quad (2.60)$$

$$b_i = \frac{1}{2A}(z_j - z_k),\ b_j = \frac{1}{2A}(z_k - z_i),\ b_k = \frac{1}{2A}(z_i - z_j),\ c_i = \frac{1}{2A}(r_k - r_j),$$

$$c_j = \frac{1}{2A}(r_i - r_k),\ c_k = \frac{1}{2A}(r_j - r_i)$$

From this point, the procedure is the same as that for plane triangular elements. Thus, Equation (2.54) can be solved by making the functional in Equation (2.55) stationary. This leads to an equation similar to Equation (2.43), namely

$$[S^e]\{\emptyset^e\} = 0 \quad (2.61)$$

and $[S^e]$ has the form,

$$[S^e_{l,m}] = V[K_r\, b_l\, b_m + K_z\, c_l\, c_m] \quad (2.62)$$

where (l,m) correspond to rows and columns of matrix $[S^e]$, and V is the volume of an element. After this point, the same procedure is applied. First the individual elements are assembled, then the system of linear algebraic equations is solved for unknown $\emptyset$'s.

Three dimensional elements can also be used in the idealization process. Although the computer programs developed in this study utilize two-dimensional elements, they can be easily extended to three dimensional studies.

Parabolic Partial Differential Equations

Many problems in physics and engineering require numerical solution of the linear parabolic partial differential equations. In two-dimensional cartesian coordinates, this equation can be written as

$$\frac{\partial}{\partial x}(K_x \frac{\partial \emptyset}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial \emptyset}{\partial y}) + g(x,y) = K_t \frac{\partial \emptyset}{\partial t} \qquad (2.63)$$

where K_x, K_y, K_t and g(x,y) are known functions of x and y.

Derivation of finite element equations of Equation (2.63) can be based on the utilization of restricted variational forms or on Galerkin's method. If one assumes that the conditions at a particular instant are analyzed, the time variation of $\emptyset$ in Equation (2.63) can be considered as a prescribed function of position only. Then, the associated restricted variational form [17], [18] which has to be made stationary becomes

$$I = \iint_A [\frac{K_x}{2}(\frac{\partial \emptyset}{\partial x})^2 + \frac{K_y}{2}(\frac{\partial \emptyset}{\partial y})^2 - g\emptyset + K_t \frac{\partial \emptyset}{\partial t}\emptyset] dxdy \qquad (2.64)$$

When Equation (2.64) is made stationary, the resulting equation has the form below.

$$\iint_A (\frac{\partial N_i}{\partial x} \sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial N_i}{\partial y} \sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y}) \emptyset_l \; dxdy$$

$$+ \iint_A N_i K_t \sum_{l=i}^{j,k} N_l \frac{\partial \emptyset_l}{\partial t} dxdy = \iint_A N_i g \; dxdy \qquad (2.65)$$

However, as mentioned earlier, this is not a solution to an exact variational form. Thus, instead of working on restricted variational forms and trying to fit a variational functional to the problem at hand, one can utilize the Galerkin method which is a more direct approach to arrive at the integral equations of finite element analysis [17], [11], [53].

The Galerkin method of approximation can be described as follows [8]. Suppose that a solution to a differential equation

$$L\{\emptyset\} = g(x,y) \tag{2.66}$$

is required, where $\emptyset$ is a function satisfying boundary conditions, and L is a linear differential operator. If the function $\emptyset$ is approximated by a trial function having the form

$$\emptyset = \sum_{i=1}^{n} N_i(x,y)\emptyset_i \tag{2.67}$$

then the Galerkin method, which is sometimes referred to as "The Method of Weighting Functions," consists of requiring the weighted integral residual of the differential equation to be equal to zero. Accordingly,

$$\iint_A [L\{\emptyset\} - g(x,y)]\, M_i \, dxdy = 0, \; i = 1,2,\ldots,n \tag{2.68}$$

where M_i is an arbitrary weighting function. Similar to previous formulations, Equations (2.67) and (2.68) result in n simultaneous equations to be solved.

Thus, by equating the weighted integral residual of the parabolic partial differential equation to zero one can form the i^{th} integral finite element equation of Equation (2.63). Thus,

$$\iint_A N_i \{[\frac{\partial}{\partial x}(K_x\frac{\partial}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial}{\partial y})] \sum_{l=i}^{j,k} N_l\emptyset_l + g$$

$$- K_t\frac{\partial}{\partial t}\sum_{l=i}^{j,k} N_l\emptyset_l \} dxdy = 0 \qquad (2.69)$$

The second order differential terms appearing in the integral equation above necessitate the continuity of the first derivativatives of the function $\emptyset$. This can be taken care of by including additional boundary terms in the integral equation above. This process suggested by C. V. Smith, Jr. is explained in detail in reference [38]. The weighing function, M_i, is chosen as N_i which is defined by Equation (2.12) and $\emptyset$ is defined by Equation (2.16), as before. Second derivatives appearing in the first two terms of the Equation (2.69) can be modified by using Green's theorem. Thus,

$$\iint_A u\nabla^2 v dA = - \iint_A (\nabla u) \cdot (\nabla v)dA + \int_S u\nabla v \cdot \bar{n}ds \qquad (2.70)$$

where ∇u is gradient of u and $\bar{n}$ is unit outward normal, such that

$$\nabla v \cdot \bar{n} \ ds = \frac{\partial v}{\partial n} ds \qquad (2.71)$$

Thus Equation (2.69) takes the form

$$- \iint_A (\frac{\partial N_i}{\partial x} \sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial N_i}{\partial y} \sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y})\emptyset_l dxdy + \iint_A N_i \ g \ dxdy$$

$$- \iint_A N_i K_t \sum_{l=i}^{j,k} N_l \frac{\partial \emptyset_l}{\partial t} dxdy + \int_S N_i \sum_{l=i}^{j,k} (K_x\frac{\partial N_l}{\partial x} \cdot n_x$$

$$+ K_y \frac{\partial N_l}{\partial y} \cdot n_y)\emptyset_l ds = 0 \qquad (2.72)$$

where n_x, n_y are the direction cosines of the outward normal to the boundary surface. The surface integral in the above equation exists only for the boundary elements [38]. Thus, Equation (2.72) reduces to Equation (2.65) when boundary conditions are of the Dirichlet type. If the boundary conditions are of Neumann type, the additional line integration term will also exist in Equation (2.65), which is characteristic of natural boundary conditions of variational forms.

The same procedure can be carried out for other nodes, and the matrix equation takes the form

$$[S^e] \{\emptyset\} + [P^e] \{\frac{\partial \emptyset}{\partial t}\} = \{F^e\} \tag{2.73}$$

where $[S^e]$ has the form given in Equation (2.48), and $\{F^e\}$ has the form given in Equation (2.49), and $[P^e]$ has the form given below,

$$[P^e_{l,m}] = \frac{K_t A}{12} [12a_l a_m + b_l b_m(X) + (b_l c_m + c_m b_l)(XY) + c_l c_m(Y)] \tag{2.74}$$

where (l,m) corresponds to columns and rows of matrix $[P^e]$, and

$$X = \sum_{l=i}^{j,k} x_l^2 \tag{2.75}$$

$$Y = \sum_{l=i}^{j,k} y_l^2 \tag{2.76}$$

$$XY = \sum_{l=i}^{j,k} x_l y_l \tag{2.77}$$

At this point, the assembly of elements results in total [S] and [P] matrices and a total { F } vector for the continuum considered. Thus, the final form of the finite element equations is

$$[S]\{\emptyset\} + [P]\frac{\partial \emptyset}{\partial t} = \{F\} \tag{2.78}$$

To proceed with the transient part of the solution, it will be assumed that $\frac{\partial \emptyset}{\partial t}$ associated with each degree of freedom of the discrete system varies linearly within a time increment (Δt), as first suggested by Clough and Wilson [46]. Thus, from a direct integration over the time interval, Δt for all nodal points, the following equation for $\emptyset$ at the end of a time interval can be obtained.

$$\{\emptyset\}_t = \{\emptyset\}_{t-\Delta t} + \frac{\Delta t}{2}\left(\frac{\partial \emptyset}{\partial t}\right)_{t-\Delta t} + \frac{\Delta t}{2}\left(\frac{\partial \emptyset}{\partial t}\right)_t \tag{2.79}$$

or,

$$\left(\frac{\partial \emptyset}{\partial t}\right)_t = -\left(\frac{\partial \emptyset}{\partial t}\right)_{t-\Delta t} + [\{\emptyset\}_t - \{\emptyset\}_{t-\Delta t}]\frac{2}{\Delta t} \tag{2.80}$$

Thus, if the initial values of $\emptyset$ are known, Equations (2.78) and (2.80) can be solved simultaneously to obtain the values of $\emptyset$ at the time ($t + \Delta t$). Details of this numerical process will be given in Chapter III, and at this point, it is clear that these simultaneous solutions are carried out on a simultaneous set of equations.

Axisymmetric Elements

The procedure explained above can also be used to solve transient axisymmetric problems. In order to avoid repetition, the mathematical formulation will not be given again. If the solution to an axisymmetric partial differential equation

$$\frac{\partial}{\partial r}(K_r \frac{\partial \emptyset}{\partial r}) + \frac{K_r}{r}\frac{\partial \emptyset}{\partial r} + \frac{\partial}{\partial z}(K_z \frac{\partial \emptyset}{\partial z}) = K_t \frac{\partial \emptyset}{\partial t} \qquad (2.81)$$

is required, then, for an element, the finite element equation will be,

$$[S^e]\{\emptyset\} + [P^e]\{\frac{\partial \emptyset}{\partial t}\} = 0 \qquad (2.82)$$

where $[S^e]$ and $[P^e]$ have the respective forms shown below.

$$[S^e_{l,m}] = V[K_r b_l b_m + K_z c_l c_m] \qquad (2.83)$$

$$[P^e_{l,m}] = \frac{K_t V}{12}[12a_l a_m + b_l b_m(R) + (b_l c_m + b_m c_l)(RZ) + c_l c_m(Z)] \qquad (2.84)$$

and

$$R = \sum_{l=i}^{j,k} r_l^2 \qquad (2.85)$$

$$Z = \sum_{l=i}^{j,k} z_l^2 \qquad (2.86)$$

$$RZ = \sum_{l=i}^{j,k} r_l z_l \qquad (2.87)$$

Thus, after an assembly process, using the procedure explained before, it is possible to solve for the values of $\emptyset$ at $(t + \Delta t)$ if the initial conditions are known.

Convective Diffusion Equation

If the transport of some quantity (tracer) in a flowing fluid is analyzed, then the differential equation describing the phenomena in two-dimensional rectangular coordinates takes the form

$$\frac{\partial}{\partial x}(K_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial \phi}{\partial y}) - u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y} = K_t \frac{\partial \phi}{\partial t} \quad (2.88)$$

where ϕ represents the concentration of the tracer in question, $u\frac{\partial \phi}{\partial x}$ and $v\frac{\partial \phi}{\partial y}$ represent the convection or advection of the tracer in x and y directions, respectively, and the first two terms on the left side of Equation (2.88) represent the transport of material through diffusion, following the convention of Bird et al. [5]. In Equation (2.88) K_x and K_y are diffusion coefficients and u and v are velocity components in x and y directions, respectively, all of which are assumed to be known functions of x and y. Thus, their values can be assumed to be constant within an element. Also K_t is the time constant which is usually equal to unity in diffusion problems.

Once again, finite element equations can be formulated either through the use of restricted variational forms or through Galerkin's method. If one assumes that conditions at a particular instant are analyzed, then the associated restricted variational form which has to be made stationary becomes (see Appendix B)

$$I = \iint_A [\frac{K_x}{2}(\frac{\partial \phi}{\partial x})^2 + \frac{K_y}{2}(\frac{\partial \phi}{\partial y})^2 + K_t \frac{\partial \phi}{\partial t}\phi] e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} dxdy \quad (2.89)$$

where K_x, K_y, u and v are uniform constants of the problem and ϕ is the unknown function. When Equation (2.89) is made stationary the resulting i^{th} equation has the form below.

$$\iint_A e^{-\frac{xu}{K_x}-\frac{yv}{K_y}}\left[\frac{\partial N_i}{\partial x}\sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial N_i}{\partial y}\sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y}\right]\emptyset_l \, dxdy$$

$$+ \iint_A e^{-\frac{xu}{K_x}-\frac{yv}{K_y}} N_i K_t \sum_{l=i}^{j,k} N_l \frac{\partial \emptyset_l}{\partial t} dxdy = 0 \qquad (2.90)$$

Before working on a stationary principle, Equation (2.89) can be written in a simpler form using the transformation

$$\emptyset' = \emptyset\, e^{\left(-\frac{xu}{K_x} - \frac{yv}{K_y}\right)} \qquad (2.91)$$

Thus,

$$I = \iint_A \left\{\frac{K_x}{2}\left(\frac{\partial \emptyset'}{\partial x}\right)^2 + \frac{K_y}{2}\left(\frac{\partial \emptyset'}{\partial y}\right)^2 + \left(\frac{u}{2}\frac{\partial \emptyset'}{\partial x} + \frac{v}{2}\frac{\partial \emptyset'}{\partial y}\right)\emptyset'\right.$$

$$\left. + \left(\frac{u^2}{8K_x} + \frac{v^2}{8K_y}\right)\emptyset'^2 + \emptyset' \frac{\partial \emptyset'}{\partial t}\right\} dx\, dy \qquad (2.92)$$

As before, a stationary principle at this point will result in matrix equation

$$[S^e]\{\emptyset'\} + [P^e]\left\{\frac{\partial \emptyset'}{\partial t}\right\} = 0 \qquad (2.93)$$

where,

$$[S^e] = A[K_x b_l b_m + K_y c_l c_m + \frac{u}{2}(a_l b_m + a_m b_l) + \frac{v}{2}(a_l c_m + a_m c_l) + \left(\frac{u^2}{4K_x} + \frac{v^2}{4K_y}\right)\cdot$$

$$(a_l c_m + b_l b_m \frac{X}{12} + (b_l c_m + b_m c_l)\frac{XY}{12} + c_l c_m \frac{Y}{12})] \qquad (2.94)$$

where l and m again correspond to rows and columns of the matrix $[S^e]$, X, Y and XY are as given by Equations (2.75), (2.76) and (2.77), and $[P^e]$ is given by Equation (2.74). From here on assembly of elements result in total [S] and [P] matrices which can be solved for the unknown nodal values by using the procedure explained previously. The only addition being the transformation of the results, again using Equation (2.91).

However, Equation (2.90) can also be derived if the Galerkin method is utilized to arrive at the integral equations of the finite element method. However, this time either through physical reasoning or through observing the exact variational form of steady convective diffusion equation, which is

$$I = \iint_A \{\frac{K_x}{2} (\frac{\partial \phi}{\partial x})^2 + \frac{K_y}{2} (\frac{\partial \phi}{\partial x})^2 \} e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} dxdy, \qquad (2.95)$$

the weighing function for the i^{th} equation is chosen as,

$$M_i = N_i \, e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \qquad (2.96)$$

Thus, forming the weighted integral residual of Equation (2.88) and setting it equal to zero one can write,

$$\iint_A N_i \, e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} [\frac{\partial}{\partial x} (K_x \frac{\partial}{\partial x}) + \frac{\partial}{\partial y} (K_y \frac{\partial}{\partial y})$$

$$- u \frac{\partial}{\partial x} - v \frac{\partial}{\partial y} - K_t \frac{\partial}{\partial t}] \sum_{l=i}^{j,k} N_l \phi_l \, dxdy = 0 \qquad (2.97)$$

Again applying Green's theorem to Equation (2.97),

$$- \iint_A \{ \frac{\partial}{\partial x} [N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}}] \sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial}{\partial y} [N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}}] \sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y} \} \phi_l \, dxdy$$

$$- \iint_A N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} u \sum_{l=i}^{j,k} \frac{\partial N_l}{\partial x} \phi_l \, dxdy - \iint_A N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} v \sum_{l=i}^{j,k} \frac{\partial N_l}{\partial y} \phi_l \, dxdy$$

$$- \iint_A N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} K_t \sum_{l=i}^{j,k} N_l \frac{\partial \phi_l}{\partial t} \, dxdy + \int_S N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \sum_{l=i}^{j,k} K_l \frac{\partial N_l}{\partial n} \phi_l \, ds = 0 \qquad (2.98)$$

Again the surface integral appears only for boundary elements. The first integral of Equation (2.98) can be written as

$$I_1 = - \iint_A \{ e^{-\frac{xu}{K_y} - \frac{yv}{K_y}} [\frac{\partial N_i}{\partial x} \sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial N_i}{\partial y} \sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y}] \} \phi_l \, dxdy$$

$$+ \iint_A N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} u \sum_{l=i}^{j,k} \frac{\partial N_l}{\partial x} \phi_l \, dxdy + \iint_A N_i e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} v \sum_{l=i}^{j,k} \frac{\partial N_l}{\partial y} \phi_l \, dxdy \qquad (2.99)$$

When substituted back into Equation (2.98), the last two integrals of Equation (2.99) will cancel the second and third integrals of Equation (2.98) leaving the i^{th} integral finite element equation as

$$\iint_A e^{-\frac{xu}{K_x}-\frac{yv}{K_y}} \left[\frac{\partial N_i}{\partial x}\sum_{l=i}^{j,k} K_x \frac{\partial N_l}{\partial x} + \frac{\partial N_i}{\partial y}\sum_{l=i}^{j,k} K_y \frac{\partial N_l}{\partial y}\right]\emptyset_l \, dxdy$$

$$+ \iint_A N_i \, e^{-\frac{xu}{K_x}-\frac{yv}{K_y}} K_t \sum_{l=i}^{j,k} N_l \frac{\partial \emptyset_l}{\partial t} \, dxdy = \int_S N_i \, e^{-\frac{xu}{K_x}-\frac{yv}{K_y}} \sum_{l=i}^{j,k} K_l \frac{\partial N_l}{\partial n}\emptyset_l ds \qquad (2.100)$$

Equation (2.100) reduces to Equation (2.90) when boundary conditions are of Dirichlet type, as explained before. From here on, the same equations can be developed for stiffness and mass matrices.

Boundary Conditions

The general form of the boundary conditions frequently encountered in the solution of field problems can be written as

$$\alpha\emptyset + \sigma \frac{\partial \emptyset}{\partial n} + \lambda = 0 \qquad (2.101)$$

where $\emptyset$ is the unknown function being searched for and α, σ and λ are some known constants, and $\frac{\partial \emptyset}{\partial n}$ is the derivative of $\emptyset$ normal to the boundary. On the basis of Equation (2.101), three types of problems can be identified, depending on the type of the boundary condition.

a) The Dirichlet problem; when in Equation (2.101) $\sigma = 0$. That is, $\emptyset$ is specified on the boundary.

b) The Neumann problem; when in Equation (2.101) $\alpha = 0$. That is, normal derivatives of $\emptyset$ are specified on the boundary.

c) The Cauchy problem; when the boundary conditions have the general form given by Equation (2.101).

For example, problems of flow through porous media are usually of mixed boundary value type, where Dirichlet conditions apply over a part of the boundary, and Neumann or Cauchy conditions apply over the rest of the boundary. If a case with a free surface is considered, the problem becomes a special and interesting one. The location of the free surface is not known a priori and has to be found as part of the solution. The free surface is both a stream line (where $\frac{\partial \phi}{\partial n} = 0$ and $\phi = y + \frac{p}{\gamma}$) and a line of constant pressure (where $\phi = y$). Thus, in a way, an additional boundary condition is given instead of the location of the boundary. This makes the free surface problem theoretically solvable. In fact, such a solution has been recently provided by Finn [25].

Treatment of complex boundary conditions in finite element analysis is relatively simple compared to other numerical procedures. This characteristic of the finite element analysis is even more clear when the derivation of finite element equations are based on the methods of variational calculus, Berg [4], Hildebrand [23].

The importance of the natural boundary conditions in variational analysis lies in the fact that, by adding suitable boundary terms to a functional, it is possible to alter the natural boundary conditions without changing the Euler equation. For example, for the functional

$$I = \iint_A \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] dxdy \tag{2.102}$$

the Euler equation of which is Laplace's equation, the natural boundary condition is

$$\frac{\partial \phi}{\partial n} = 0 \tag{2.103}$$

If some boundary terms are added to the functional (2.107),

$$I = \iint_A [(\frac{\partial \phi}{\partial x})^2) + (\frac{\partial \phi}{\partial y})^2]dxdy + \int_S f(s)\phi^2 ds \qquad (2.104)$$

where f(s) is an arbitrary known function, then the functional (2.104) has again the Laplace's equation as its Euler equation, but the natural boundary condition becomes

$$\frac{\partial \phi}{\partial n} + f(s)\phi = 0 \qquad (2.105)$$

which means that a combination of the normal derivative and the function itself is specified for the given boundary region. If the boundary terms added are of the form shown in Equation (2.106)

$$I = \iint_A [(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2]dxdy + \int_S f(s)\phi ds \qquad (2.106)$$

then the functional again has the Laplace's equation as its Euler equation, but the associated natural boundary condition becomes

$$\frac{\partial \phi}{\partial n} + f(s) = 0 \qquad (2.107)$$

which means that normal derivative is specified for the given boundary region. For such boundary conditions, the finite element equations will not be changed totally but there will be some additional terms resulting from the stationary value of the last integral in Equations (2.104) and (2.106) for those elements that are on the boundary.

Once the final form of the matrix equations is obtained, boundary conditions must be inserted which permit the solution of the simultaneous equations. When the value of $\emptyset$ is given as a boundary condition of a specific problem, these values are simply inserted in all the appropriate equations as known nodal values. If Neumann boundary conditions are specified for an element on the boundary, then Equation (2.106) should be made stationary for that boundary element which will give additional terms for the nodes on the boundary.

CHAPTER III

NUMERICAL ANALYSIS

General

It should be pointed out here that numerical solutions obtained for a problem are no better than the mathematical models from which they are derived. It is also well known that, for complicated problems, engineering mathematical analysis is never precisely complete because the mathematical model is usually a somewhat simplified one, owing in part to the incomplete understanding of the physical system in question. Less essential aspects may be neglected, certain linearizations are presumed, or particular aspects are assumed constant in the range under consideration so as to yield mathematical models tractable under the application of the available analytical tools. Nonetheless, if a model serves to describe the phenomena under consideration with sufficient accuracy, it must be regarded as a useful one. Thus, it remains for the engineer to decide what approximations to make and when and under what circumstances the approximate model describes the real system sufficiently accurate.

In some detailed studies, even if the mathematical models, i.e. the differential equations, are formulated accurately their exact analytical solution may prove to be beyond the reach of purely mathematical analysis. One basic way of dealing with such cases involves the solution of the governing equations of the problem by using the numerical methods. This mode of analysis is proving to be useful for an increasing number of previously unsolved and unsolvable problems.

In developing a general computational scheme in this thesis, the usual finite difference techniques, which have been widely used in solving differential equations numerically, have been avoided. The main emphasis has been given to the relatively recent "Finite Element Method". The main rationale in this attempt is the advantages which the finite element analysis possesses over other numerical schemes in the solution of field problems.

In Chapters I and II of this thesis an outline of the finite element method and detailed formulations of mathematical equations for solving elliptic and parabolic differential equations are given. In this chapter, the main consideration will be given to the numeric and computer phase of finite element analysis. In doing so, some comparisons with the other widely used numerical procedure, "The Finite Difference Method," will be presented.

Both finite element and finite difference methods seek to replace the differential equations by simultaneous algebraic expressions which give relations between values of the dependent variable and the values of the independent variable or variables. The numerical solution then consists in solving a series of simultaneous algebraic equations for the values of the dependent variable at a number of discrete points throughout the solution field. Differences between the various methods lie only in the procedures used to arrive at the node point equations. In finite difference methods, the domain of the independent variables is replaced by a finite set of points, usually referred to as mesh points, and one seeks to determine the approximate values of the desired solution at these mesh points. The values at the mesh points are required to satisfy the difference equations obtained either by replacing partial derivatives by difference quotients, or by certain other more sophisticated techniques which in turn lead to a series of simultaneous algebraic equations rather than differential equations. Detailed treatment of the finite difference methods, which will not be presented here, can be found in Collatz [9], Hildebrand [22] and Varga [41]. In the finite element technique, the solution field is divided into finite elements. Then, the governing differential equations of the problem are solved over each element in terms of the solution values at certain points, which result in an algebraic expression for the element. When all such elements are assembled, a set of simultaneous algebraic equations are obtained.

In particular, if a uniform rectangular finite element mesh is chosen to approximate the field, a direct comparison between finite element and finite difference equations can be made. To be able to make this comparison, resultant algebraic equations for a grid shown in Figure 7 will be developed for the finite element analysis and compared with known equations of the finite difference analysis.

Assembly of Elements

In previous chapters certain matrix equations have been developed on the basis of the type of element and the type of the problem being analyzed. These governing matrix equations are formulated for a single typical element in the field. The form of these matrix equations mainly depends on the assumption of the mathematical form for the primary unknown of the problem and on the type of solution technique used. The interesting point is that structural or non-structural elements can be identical in shape. Furthermore, the triangle in elasticity problems can have the same form for displacements as a similar fluid flow element has for velocity potential or stream function. The major difference between the elasticity and fluid flow problems lies in the boundary conditions to be satisfied and in the differential equations to be solved. In other words, the numerical solution and the assembly procedure will be identical in both cases.

In practice, the assembly of elements to find a global stiffness matrix for the continuum to be analyzed, is performed automatically by a digital computer. However, this operation will be performed algebraically here in order to provide a comparison with a typical finite difference formulation [52], [50].

In this example, the assembly procedure for a typical triangular element will be presented which will utilize the matrix equations of the previously developed elliptic equation.

Referring back to Equation (2.47), the general matrix equation for an element can be written as

$$[S^e]\{\emptyset^e\} = \{F^e\} \tag{3.1}$$

A typical element of the stiffness matrix has the form (assuming isotropic material in the element to avoid complications in the algebra)

$$S^e_{lm} = KA(b_l b_m + c_l c_m) \tag{3.2}$$

where l and m correspond to rows and columns of the matrix $[S^e]$. Writing b and c in terms of nodal coordinates,

$$S^e_{ijk} = \frac{K}{4A} [(y_i - y_k)(y_k - y_i) + (x_k - x_j)(x_i - x_k)] \quad (3.3)$$

with the total matrix being in the form

$$[S^e_{ijk}] = \begin{bmatrix} S_{ii} & S_{ij} & S_{ik} \\ S_{ji} & S_{jj} & S_{jk} \\ S_{ki} & S_{kj} & S_{kk} \end{bmatrix} \quad (3.4)$$

The load matrix has the form

$$\{F^e_{ijk}\} = \frac{Ag}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad (3.5)$$

where g is constant within the element. In Figure 7, four elements with nodes spaced along a rectangular square grid are shown to meet at point 0. This point is situated at an interface of regions (A) and (B) with different values of K. For element II, defined by the nodes (0,1, 2) one can form the stiffness matrix as

$$[S_{012}] = \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{bmatrix} = \frac{K_A}{2h^2} \begin{bmatrix} 2h^2 & -h^2 & -h^2 \\ -h^2 & h^2 & 0 \\ -h^2 & 0 & h^2 \end{bmatrix} \quad (3.6)$$

Similarly for other elements,

$$[S_{023}] = \frac{K_A}{2h^2} \begin{bmatrix} 2h^2 & -h^2 & -h^2 \\ -h^2 & h^2 & 0 \\ -h^2 & 0 & h^2 \end{bmatrix} \quad (3.7)$$

$$[S_{034}] = [S_{041}] = \frac{K_B}{2h^2}\begin{bmatrix} 2h^2 & -h^2 & -h^2 \\ -h^2 & h^2 & 0 \\ -h^2 & 0 & h^2 \end{bmatrix} \qquad (3.8)$$

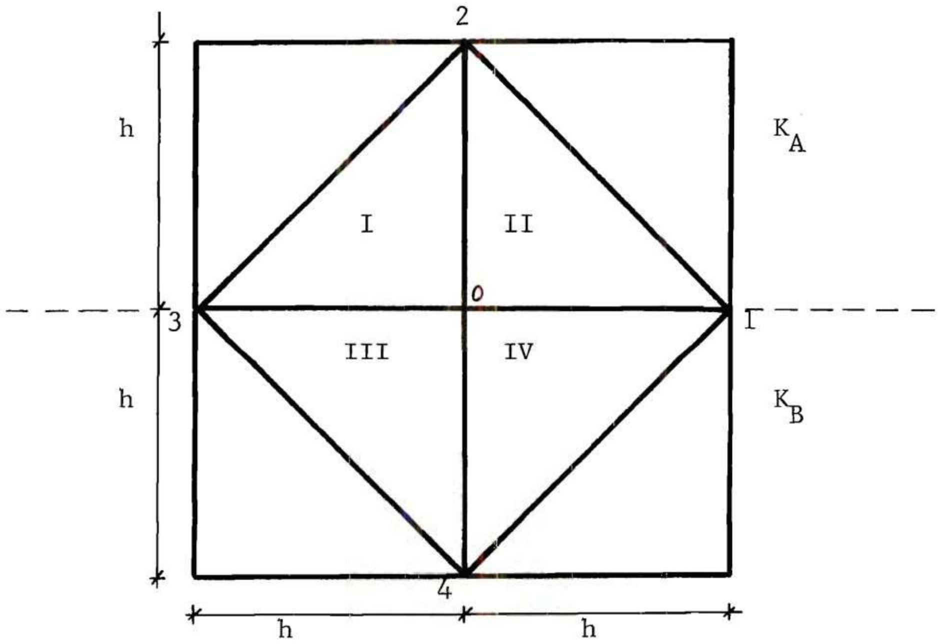

Figure 7. Triangular Element Assembly

Adding appropriate coefficients by using Equation (3.9),

$$\sum \emptyset_n \sum S^e_{in} = \sum F_i \qquad (3.9)$$

one can write

$$\emptyset_0 \{(2h^2 + 2h^2)\frac{K_A}{2h^2} + (2h^2 + 2h^2)\frac{K_B}{2h^2}\} - \emptyset_1 \{(h^2)\frac{K_A}{2h^2} + (h^2)\frac{K_B}{2h^2}\} - \emptyset_2 \{(h^2)\frac{K_A}{2h^2}$$

$$+ (h^2)\frac{K_A}{2h^2}\} - \emptyset_3 \{(h^2)\frac{K_A}{2h^2} + (h^2)\frac{K_B}{2h^2}\} - \emptyset_4 \{(h^2)\frac{K_B}{2h^2} + (h^2)\frac{K_B}{2h^2}\} = \frac{2}{3}gh^2 \qquad (3.10)$$

or simplifying,

$$\emptyset_0\{2K_A + 2K_B\} - \emptyset_1\{K_A + K_B\}/2 - \emptyset_2 K_A - \emptyset_3\{K_A + K_B\}/2 - \emptyset_4 K_B = \frac{2}{3}gh^2 \quad (3.11)$$

The above equation is almost identical to the finite difference approximation to the same problem. However, such an equation requires not only the consideration of the governing equations, but also the establishment of the special interface conditions. In the finite element approximation no special treatment is necessary whether the values of K vary between elements or remain constant.

The only difference between the above equation and that provided by the usual finite difference approximation is in the "loads" contributed to each node which would be simply (gh^2). This difference, however, automatically balances out since the total "load" is still the same over the whole region. Note that if K_a and K_b are identical, the well known finite difference approximation to the Laplace operator is obtained. Thus,

$$4K\emptyset_0 - K\emptyset_1 - K\emptyset_2 - K\emptyset_3 - K\emptyset_4 = \frac{2}{3}gh^2 \quad (3.12)$$

The use of this alternate method which appears to add little in terms of increased accuracy to the results obtained by a long-in-use, well known method, is justified by several advantages. These can be summarized as;

a) It is simple to deal with non-homogeneous and anisotropic problems.

b) The elements can be in any shape and size to follow arbitrary boundaries and to allow for regions of rapid variations of the function sought.

c) From the automation point of view, the finite element method produces symmetric, positive definite matrices after the introduction of boundary conditions of the problem. The matrices can be placed in a band form and can be solved with a minimum computer storage and time.

d) In finite element analysis, there is no need to make approximations in the mathematical analysis of the problem. The exact form of the governing differential equations are solved in the process. The approximation made in finite element analysis is in the idealization of the continuum as formed of small elements and in the assumed distribution of the unknown variable over the domain which itself is not restricted to a certain shape.

Solution of Simultaneous Algebraic Equations

Having achieved some familiarity with finite element formulations and assembly processes, attention can now be turned to the question of how to obtain solutions to a system of linear algebraic equations, thus derived. Procedures for solving a given set of simultaneous algebraic equations are usually divided into two categories, so-called direct and indirect methods.

Direct methods, such as the Gauss elimination scheme, Cholesky scheme and matrix methods, are those that yield the required solution in a finite number of steps. The computational algorithm for a direct method is usually somewhat complicated but nonrepetitious. Indirect or iterative methods yield the required solution as the limit of a sequence of steps. The computational algorithm is usually quite simple but requires repeated applications.

In this thesis, the main attention will be given to Cholesky's scheme of direct methods which is particularly efficient in the solution of simultaneous algebraic equations that have symmetric and positive definite coefficient matrices.

A system of n linear algebraic equations in n unknowns can be written as a single matrix equation

$$\begin{bmatrix} a_{11} & a_{11} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ c_n \end{Bmatrix} \qquad (3.13)$$

or simply as

$$[A]\ \{X\} = \{C\} \qquad (3.14)$$

An alternative scheme is to write the system of n equations as

$$\sum_{j=1}^{n} a_{ij}\, x_j = c_i \quad i = 1,2,\ldots.n \qquad (3.15)$$

If [A] represents a symmetric positive definite matrix, which is actually a characteristic of matrices obtained in finite element algorithms, then such a matrix can be decomposed into the product of a lower triangular matrix and an upper triangular matrix, each of which is the transpose of the other.

$$[A] = [U]^T[U] \tag{3.16}$$

where

$$[U] = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ 0 & 0 & \dots & u_{nn} \end{bmatrix} \tag{3.17}$$

From Equation (3.16) one can see that

$$a_{ii} = u_{1i}^2 + u_{2i}^2 + u_{3i}^2 + \dots + u_{ii}^2 \tag{3.18}$$

or

$$a_{ii} = \sum_{k=1}^{i} u_{ki}^2 \quad \text{For } (i = j) \tag{3.19}$$

and

$$a_{ij} = u_{1i}\, u_{1j} + u_{2i}\, u_{2j} + \dots + u_{ii} u_{ij} \tag{3.20}$$

or

$$a_{ij} = \sum_{k=1}^{i} u_{ki}\, u_{kj} \quad \text{For } (i < j) \tag{3.21}$$

Since elements of [A] are known, one can determine the elements of [U] by reversing the equations given above.

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \quad \text{For } (1 \leq i = j) \tag{3.22}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki}\, u_{kj}}{u_{ii}} \quad \text{For } (1 \leq i < j) \tag{3.23}$$

$$u_{ij} = 0 \quad \text{For } (i>j) \tag{3.24}$$

Thus, the solution of a system of equations follows below as shown. If [A] can be written in terms of [U] as given in Equation (3.16), then one can write Equation (3.14) as

$$[U]^T[U]\{X\} = \{C\} \tag{3.25}$$

Let

$$[U]\{X\} = \{X\}^* \tag{3.26}$$

Then Equation (3.25) can be written as

$$[U]^T\{X\}^* = \{C\} \tag{3.27}$$

A direct solution for the unknown values of the vector $\{X\}^*$ can be obtained by the use of Equation (3.28) and (3.29)

$$x_1^* = \frac{c_1}{u_{11}} \tag{3.28}$$

and

$$x_i^* = \frac{c_i - \sum_{k=1}^{i-1} u_{ki} x_k^*}{u_{ii}} \quad \text{For } (1<i) \tag{3.29}$$

In a similar manner, the unknowns of the vector $\{X\}$ of Equation (3.26) can be determined, since

$$x_n = \frac{x_n^*}{u_{nn}} \tag{3.30}$$

and

$$x_i = \frac{x_i^* - \sum_{k=i+1}^{n} u_{ik} x_k}{u_{ii}} \quad \text{For } (i<n) \tag{3.31}$$

Note that Equations (3.30) and (3.31) give the desired solution of Equation (3.14).

Cholesky's method, described above, has proven to be quite convenient and efficient from the computarization point of view. A recent discussion in a paper by Tezcan and Kostro [39] indicates this clearly. In their paper, they compare the Gauss algorithm with the Cholesky algorithm and conclude that the Cholesky solution is much faster than Gauss', and that the solution time by Gauss' method increases at a relatively higher rate than that by Cholesky when the band width increases.

Another problem with all such computerized algorithms is that they are limited by the storage locations available on the computer system. That is, there is a limit for the number of unknowns that can be handled in the simultaneous linear equations. For example, the Univac 1108 has a capacity of 60k. In most finite element problems the number of nodes to be considered or the number of unknowns to be solved for in a typical problem can range from two hundred to three hundred, and more. If a minimum of two hundred unknowns is considered, the stiffness matrix itself will require some 80k locations if the computations are carried out in double precision, which is above the 60k capacity of the Univac 1108. Thus, one must either utilize drum and/or tape units of the computer for auxillary storage with a corresponding decrease in efficiency of the whole process, or one can formulate some short cuts in the storage procedures using the characteristics of the matrices formed in the assembly processes.

The above mentioned method of decomposition is particularly efficient in this respect when applied to symmetric band matrices which is as indicated earlier, a characteristic of the matrices formed in the finite element analysis. Maximum band width is related to the maximum difference between the nodal numbers of any element in the field. For such matrices, fewer calculations are required due to the fact that elements outside the band are all equal to zero. If the general form of such a symmetric banded matrix is considered, only the upper portion of the band including the diagonal elements has to be stored to perform the steps in the algorithm shown earlier in this section. Furthermore, if those elements are stored as a rectangular array with the diagonal elements occupying the first column of the array then the limitations of storage of the stiffness and mass matrices would be reduced considerably. In finite element problems with two hundred unknowns, the maximum band width can be easily kept at twentyfive or thirty, thus lowering the storage requirements to 12k in double precision calculations. The previously explained decomposition algorithm can be performed on this rectangular array. This process is efficiently used in the computer program employed in this thesis. A printout of the program is given in Appendix C.

Time Dependent Problems

The procedure for developing matrix equations for time dependent field problems is explained in detail in Chapter II. The resulting matrix equation has the form

$$[S]\{\emptyset\} + [P]\{\frac{\partial \emptyset}{\partial t}\} = \{F\} \qquad (3.32)$$

where [S] is the stiffness matrix and [P] is the matrix generated from the time dependent terms in the differential equations. From here on this [P] matrix will be referred to as the mass matrix.

The solution to Equation (3.32) can be obtained by a step-by-step technique suggested by Wilson and Clough [46]. In this algorithm the variation of $\{\frac{\partial \emptyset}{\partial t}\}$ is assumed to be linear in each time interval Δt. Thus, from a direct integration over the time interval for all nodal points, the following equation can be obtained which reflects the solution at time t in terms of the values at time $(t - \Delta t)$.

$$\{\emptyset\}_t = \{\emptyset\}_{t-\Delta t} + (\{\frac{\partial \emptyset}{\partial t}\}_{t-\Delta t} + \{\frac{\partial \emptyset}{\partial t}\}_t)\frac{\Delta t}{2} \qquad (3.33)$$

Thus, Equation (3.33) together with Equation (3.32) can now be solved simultaneously for the values of $\emptyset$ at time t if the initial conditions at time $t = 0$ are given.

This simultaneous time and space solution process can be formulated neatly for computer applications, thus decreasing the algebra, computer storage and computer time. At time t substituting Equation (3.33) into Equation (3.32), one can write

$$([P]\frac{2}{\Delta t} + [S])\{\emptyset\}_t = [P](\{\frac{\partial \emptyset}{\partial t}\}_{t-\Delta t} + \frac{2}{\Delta t}\{\emptyset\}_{t-\Delta t}) + \{F\} \qquad (3.34)$$

Again substituting Equation (3.32) into Equation (3.34), this time at time $(t - \Delta t)$, one can write

$$([P]\frac{2}{\Delta t} + [S])\{\emptyset\}_t = ([P]\frac{2}{\Delta t} - [S])\{\emptyset\}_{t-\Delta t} + 2\{F\} \quad (3.35)$$

If one defines

$$[S]^* = [P]\frac{2}{\Delta t} + [S] \quad (3.36)$$

and

$$\{F\}^* = \{F\} + \frac{2}{\Delta t}[P]\{\emptyset\}_{t-\Delta t} \quad (3.37)$$

then from Equation (3.38) one can solve for the unknowns $\{\emptyset\}^*$

$$[S]^*\{\emptyset\}^* = \{F\}^* \quad (3.38)$$

Once the $\{\emptyset\}^*$'s are determined, the problem reduces to solving Equation (3.39) for the values of $\emptyset_t$ which are the nodal values of the function sought at time t.

$$\{\emptyset\}_t = 2\{\emptyset\}^* - \{\emptyset\}_{t-\Delta t} \quad (3.39)$$

Incrementing by Δt and repeating the same process, continuous solutions can be obtained in time and space coordinates for unsteady problems.

Description of the Program

The program (presented in Appendix C) is written in Fortran IV computer language. The whole program is divided into seven subprograms and a main program. As a rule, unique registers are chosen for the more important variables or quantities. To avoid making the present program too complicated, some limiting features are built into it. The more important ones include: the restriction to cylindrical and cartesian coordinates only; the restriction to triangular elements; the restriction of the variation of the unknown in an element to be linear; the restriction of the coefficients which are assumed to be the constant within each element and for time variation; the restriction of the linear variation of $\frac{\partial \phi}{\partial t}$. The present state of the program limits the element number at 400 and the unknown number at 250, with the band width in stiffness and mass matrices limited to 30. These limits can easily be extended to 600 in the case of elements and 450 in the case of unknowns by reducing the maximum band width to 20.

The various parts of the program and their specific functions are described as follows. A summary of flow of operations can be seen in Figure 8.

The "MAIN" Program. This is of course the most important part of the whole computational procedure. It controls the flow of operations in the whole program. It reads the necessary data for the specific problem, performs the automatic assembly of elements forming global stiffness and mass matrices, performs the time space calculations of unsteady problems and finally prints out the results obtained.

Subroutine SET. Performs the change of axis for each element, i.e. it forms the local coordinates of each node of the element. With this information, control goes back to the Main program.

Subroutine VEC. Performs the multiplication of a symmetric banded matrix, stored as a rectangular array, with a vector. The resultant vector is stored in the memory, and the control goes back to the Main program.

Subroutine CINTEG. It computes the exponential terms appearing in convective problems. This subroutine is only used in problems where convective terms exist. Results are stored in a COMMON location to be used in the main program later in the process.

Subroutine ELEM. This subroutine forms the stiffness and mass matrices for each element which are then assembled by the Main program to form the global matrices. The type of the stiffness or mass matrix to be formed is determined by some constants which are fed into the Main program as input data.

Subroutine BOUND. This subroutine introduces the boundary conditions to the final stiffness matrix assembled by the Main program. These boundary conditions have the forms given in Chapter II.

Subroutine DCB. It performs the decomposition of a symmetric banded matrix, stored as a rectangular array, into lower and upper triangular matrices, and stores the upper triangular part as a rectangular array.

Subroutine SBAND. This is the simultaneous equation solver which uses the previously decomposed matrices and with the results stored as a vector, control goes back to the Main program.

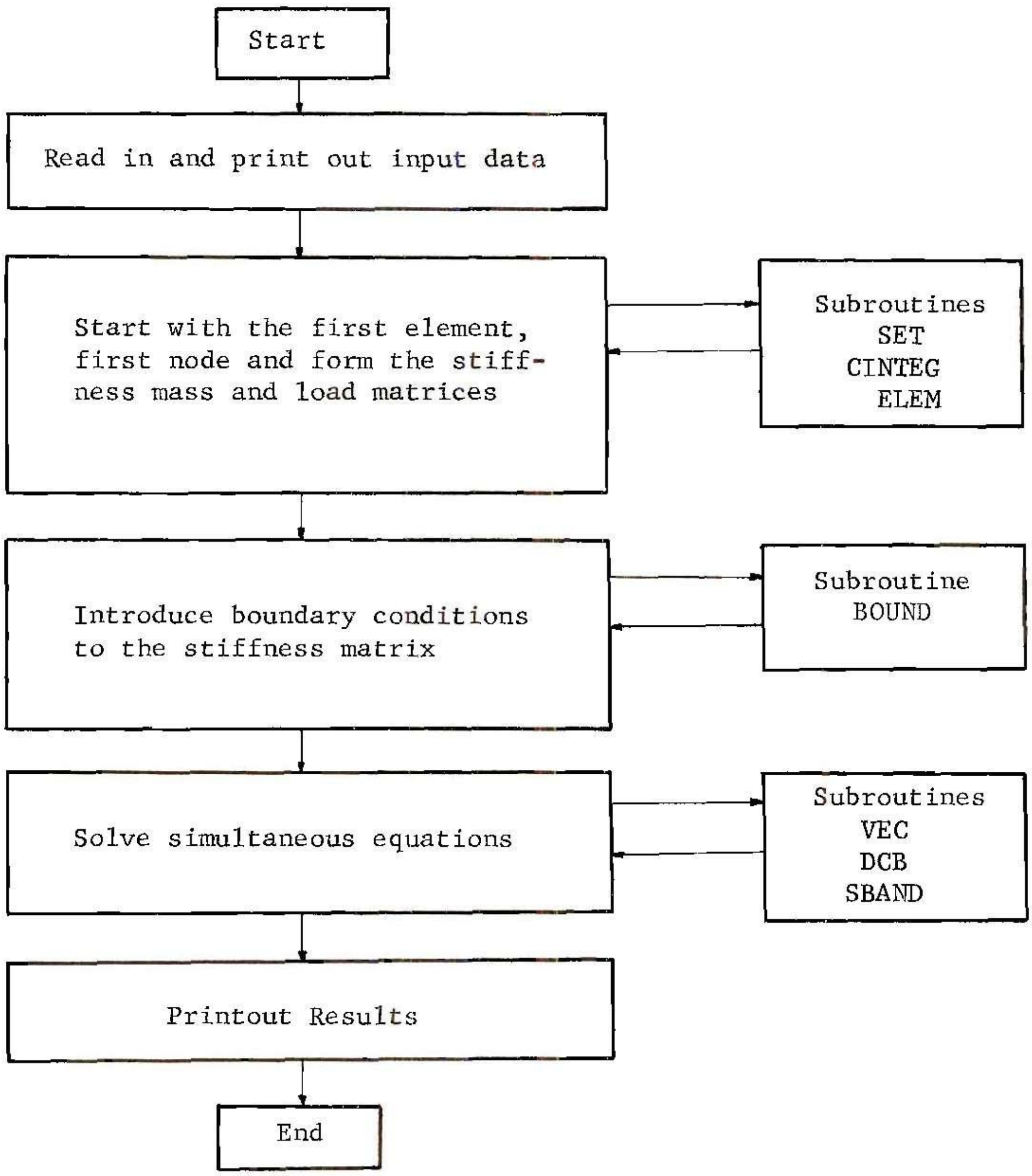

Figure 8. Flow of Operations

CHAPTER IV

NUMERICAL SOLUTIONS

In this chapter, numerical solutions to the differential equations, treated in detail in earlier chapters will be given. These examples are chosen in the field of fluid mechanics, specifically in heat conduction, irrotational and rotational incompressible flow, and ground water seepage. In some of these problems, comparisons with known exact solutions will be given to indicate the accuracy of the numerical solutions. As will become apparent, for these problems the solutions obtained through finite element analysis have considerable accuracy. Some examples will be given of problems for which no exact solution is available.

The variety of possibilities for application of the computer program is obvious. Also obviously, the appropriate employment of the mathematical forms is by no means limited to the problem types specifically treated in this dissertation. Indeed, any physical phenomena properly modeled by elliptic and parabolic differential equations, accompanied by appropriate boundary conditions can be investigated using the techniques developed here. Some restrictions imposed on these models, as explained in Chapters II and III, must be taken into consideration.

Heat Conduction

Heat transfer is an important field in present-day engineering sciences. It is of great significance in power engineering, chemical technology, civil engineering, aerospace engineering, and others. For example, the design of thermal apparatus, the design of building walls undergoing thermal effects, the heat insulation of buildings, furnaces, pipelines, the heating of machines, thermal stresses in bridges and many other problems involve unsteady or steady heat conduction.

The present work includes numerical solutions of some steady and unsteady heat conduction problems in cartesian and in cylindrical coordinates. In each of these examples, the material is assumed to have temperature-independent thermal conductivity. In all problems, the continuum described by the problem is idealized using triangular elements. The type of mesh used for some of these problems can be seen in Figure 9.

Example One. In this example a very long cylindrical chimney is heated at the inside by some hot gasses. The temperature outside the chimney is usually the ambient air temperature. A mathematical model which approximately describes the steady temperature distribution in the chimney wall is developed below. Figure 10 shows the physical aspects of the example.

Consider an infinite hollow cylinder (cylindrical tube) with $a \leq r \leq b$, $-\infty \leq z \leq +\infty$. If the heat transfer between the inner and outer cylinder surfaces and the surroundings occurs uniformly over the whole surface then the steady temperature distribution within the cylinder walls will depend only on the radius (axisymmetric problem).

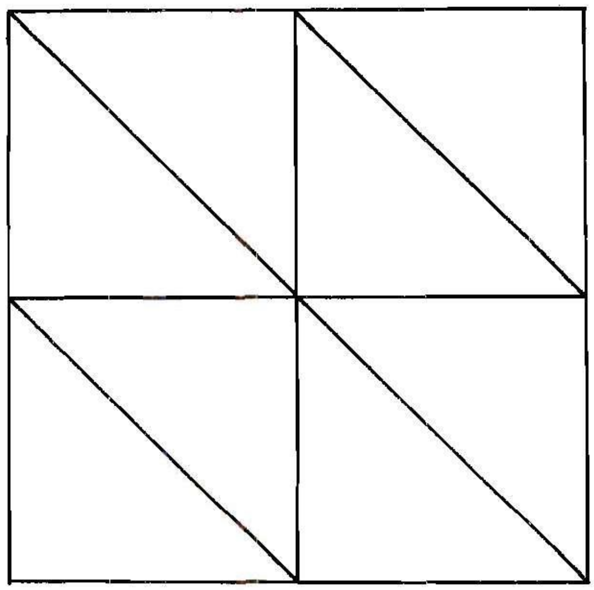

Figure 9. Type of Mesh Used in Heat Transfer Problems

The boundaries at $r = a$ and $r = b$ are kept at a constant temperature T_a and T_b. For this example, $a = 10$ inches, $b = 20$ inches, $T_a = 10°F$, and $T_b = 0°F$. The problem is to determine the steady state temperature distribution through the chimney wall.

If the cylindrical tube is isotropic and homogeneous, then the steady state temperature described above satisfies the following mathematical equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0 \tag{4.1}$$

The boundary conditions are

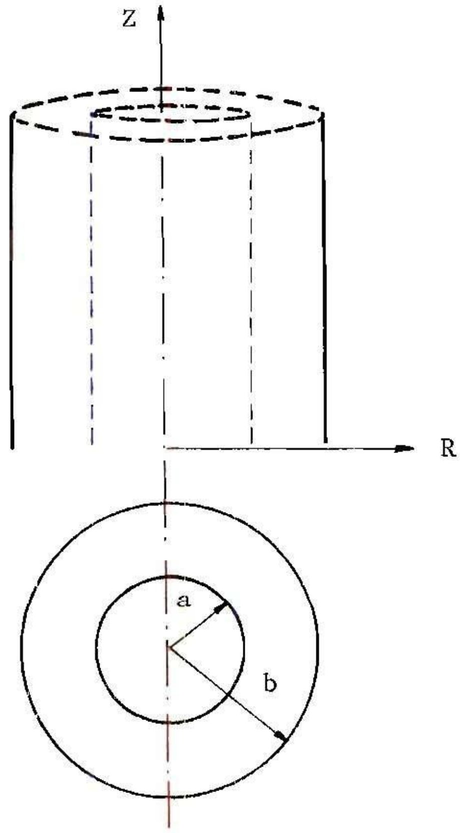

Figure 10. Schematic View of the Cylinder

$$\text{B. C. 1 :} \quad \text{at} \quad r = 10 \text{ in.} \quad T = 10°F \tag{4.2}$$

$$\text{B. C. 2 :} \quad \text{at} \quad r = 20 \text{ in.} \quad T = 0°F \tag{4.3}$$

Although the problem described above is one-dimensional, the mathematical model should be two-dimensional because the program developed is for two-dimensional problems. Thus, the mathematical model can take the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{4.4}$$

with an additional boundary condition

$$\text{B. C. 3 :} \quad \text{at } z = \pm 20 \text{ in.} \quad \frac{\partial T}{\partial z} = 0°\text{F/in} \tag{4.5}$$

Solution to Equation (4.1) can be written as T = T(r). It can be observed that T(r) also satisfies Equation (4.4) with the given boundary conditions (4.2), (4.3) and (4.5). Since there is a unique solution, the solution to Equation (4.4) can be given as T(r). This in turn implies a one-dimensional solution although the analysis is two-dimensional.

For this example, the continuum described above is idealized by

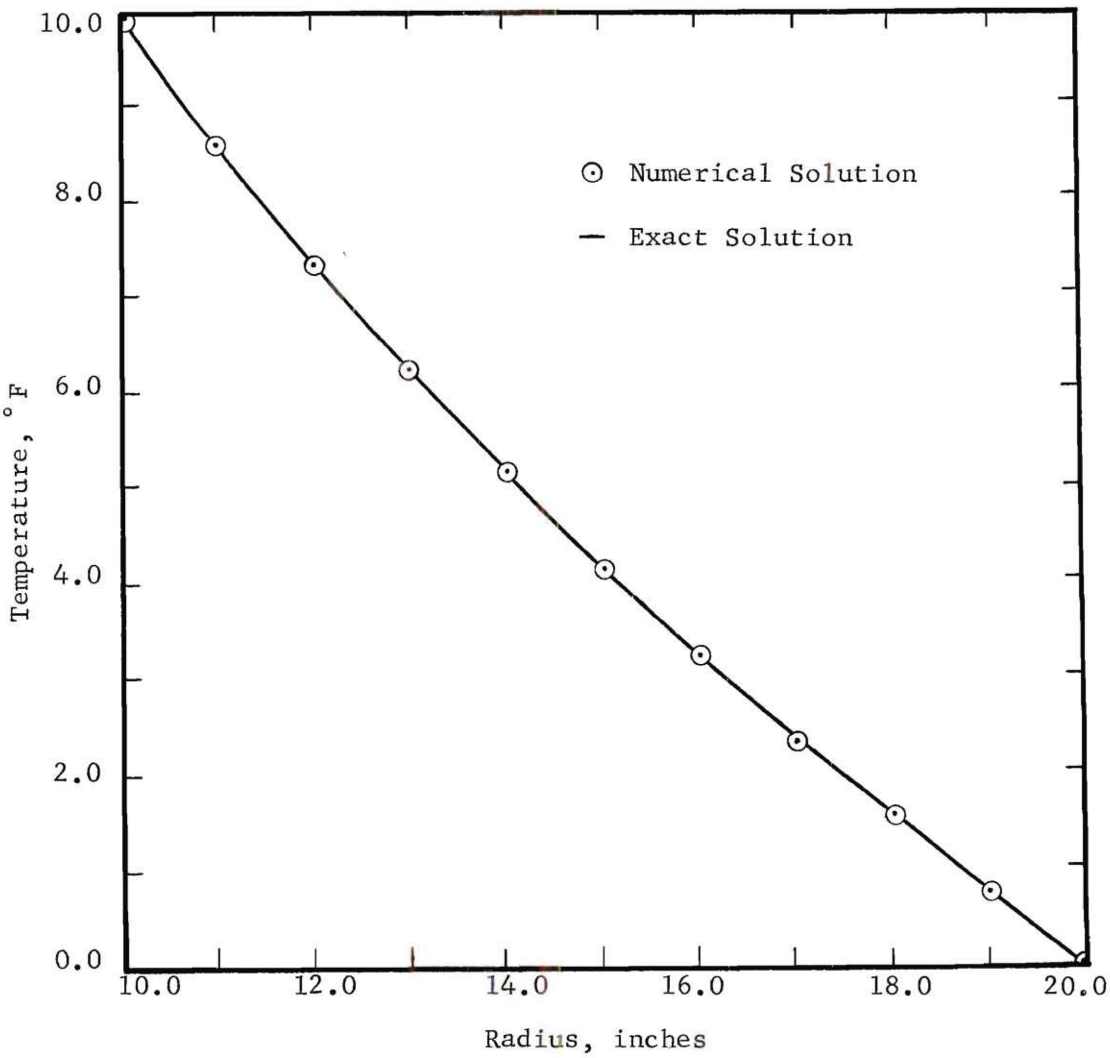

Figure 11. Steady Heat Transfer in a Cylinder Wall

160 triangular elements which resulted in 99 nodal points. The computer execution time for this problem was thirteen seconds. The finite element

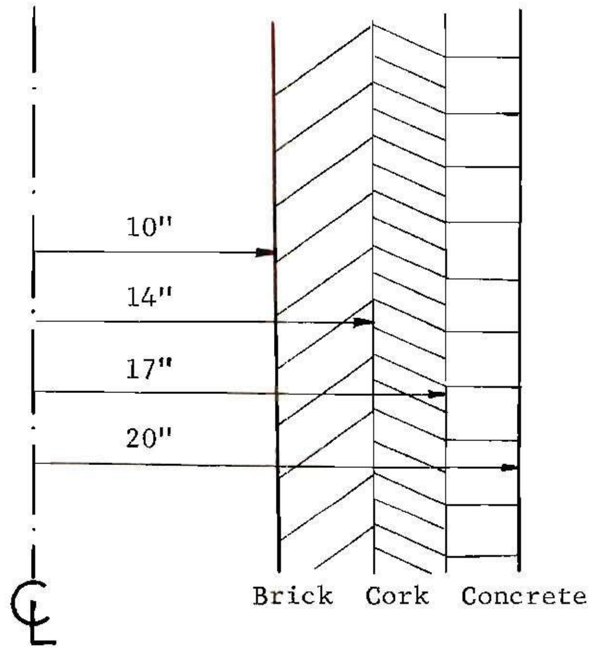

Figure 12. Non-Homogeneous Cylinder Wall

results and a comparison with the exact solution [5], [32] can be seen in Figure 11. The calculated error for this solution is around 0.5 percent.*

<u>Example Two</u>. In this example, the problem described in example one is extended to an unsteady non-homogeneous problem. It is assumed that the wall of the chimney is composed of three layers of material which have different thermal diffusivities, as shown on Figure 12. The first layer which is between r = 10 inches and r = 14 inches is a brick layer with a thermal diffusivity of 0.02 ft^2/hr. The second layer

*The average error was calculated using the formula

$$\text{Percent error} = \frac{\text{exact result - numerical result}}{\text{exact result}} \times 100$$

which is between r = 14 inches and r = 17 inches is a cork layer (insulation material) with a thermal diffusivity of 0.006 ft^2/hr. The third layer which is between r = 17 inches and r = 20 inches is a concrete layer with a thermal diffusivity of 0.008 ft^2/hr. It is further assumed that the wall is initially at 0°F. and at time t = 0 the inside temperature increases to 10°F and outside temperature remains at 0°F.

If the cylindrical tube is isotropic then the unsteady temperature distribution described above satisfies the following mathematical model.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{D}\frac{\partial T}{\partial t} \qquad (4.6)$$

where D is the thermal diffusivity. The boundary conditions are given as,

I. C. : at $t < 0$ hr. $T = 0°F$ (4.7)

B. C. 1 : at $r = 10$ in. $T = 10°F$ (4.8)

B. C. 2 : at $r = 20$ in. $T = 0°F$ (4.9)

Finite element results for this problem can be seen in Figure 13. Idealization of the continuum for this problem is identical to that explained in example one.

Example Three. In some industrial buildings, homes or even side walks, floor slabs are heated by radiant heating. If a radiant heating unit and the floor slab are assumed to be a single unit, then the steady temperature distribution in a floor with a square shape may

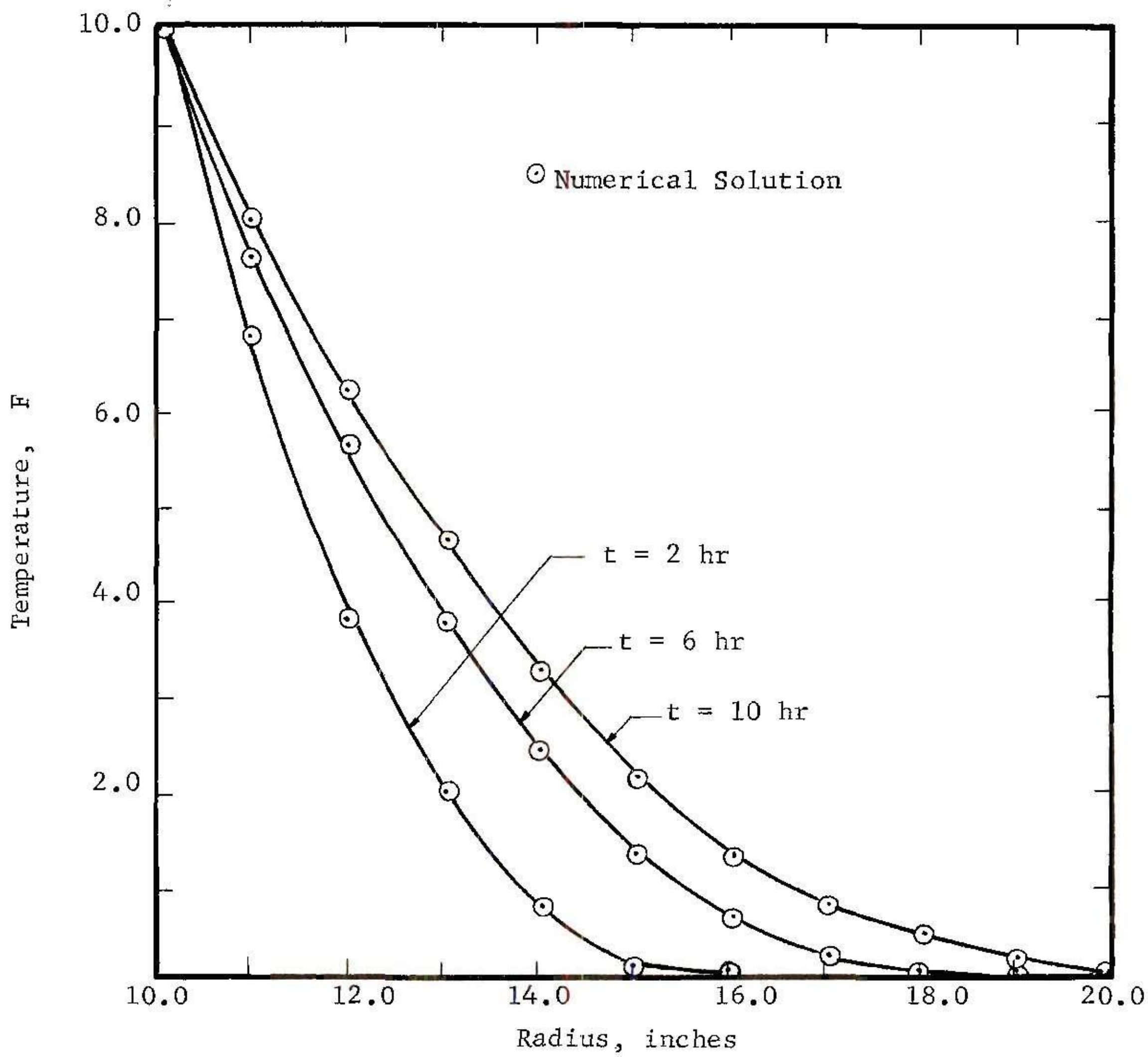

Figure 13. Unsteady Heat Transfer in a Non-Homogeneous Cylinder Wall

be approximated by the solution of the equation for steady heat flow across a square continuum with internal generation of heat, see Figure 14. If the thermal conductivity K parallel to x-axis differs from its value K parallel to y-axis, then the steady temperature distribution equation takes the form

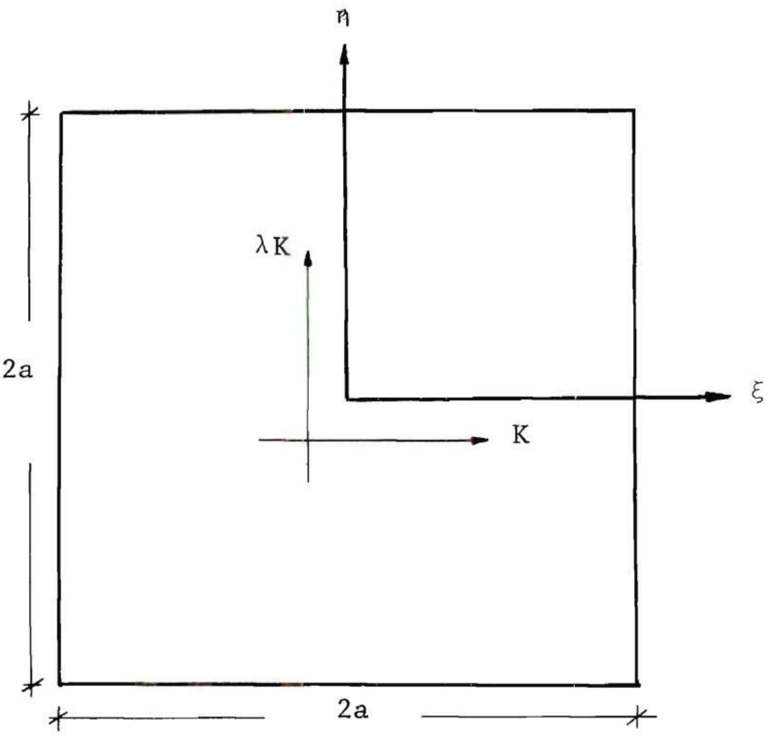

Figure 14. Radiant Floor Heating

$$\frac{\partial^2 \phi}{\partial \xi^2} + \lambda \frac{\partial^2 \phi}{\partial \eta^2} = - \frac{qa^2}{K} \tag{4.10}$$

where q is the number of units of heat generated per unit area per second, (2a) is the length of side of the square plate and ξ and η are the dimensionless coordinates.

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad \phi = \frac{T}{T_o} \tag{4.11}$$

where T_o is the ambient air temperature. Purely for illustrative purposes, let ϕ be the solution of equation

$$\frac{\partial^2 \phi}{\partial \xi^2} + 3 \frac{\partial^2 \phi}{\partial \eta^2} = - 16 \tag{4.12}$$

where $-1 = \xi = +1$, $-1 = \eta = +1$ and,

$$\text{B. C. 1 :} \quad \text{at } \xi = \pm 1 \quad \phi = 0 \tag{4.13}$$

$$\text{B. C. 2 :} \quad \text{at } \eta = \pm 1 \quad \phi = 0 \tag{4.14}$$

Finite element results for this example are given in Figure 15. This figure clearly indicates the anisotropic condition modeled in Equation (4.12). The temperature distribution in ξ direction is not symmetric to the temperature distribution in η direction. As expected the highest value of ϕ is observed at the center of the floor which corresponds to the point defined as $\xi = 0$ and $\eta = 0$. In this example, computer execution time was again thirteen seconds.

Example Four. In this example unsteady temperature distribution in a wall is studied. The wall can be considered as an infinite plate. Initially, the wall is assumed to be at a constant temperature and at time $t = 0$, it is subjected to differential heat sources from each side. These conditions give rise to unsteady temperature variation in the wall. The mathematical model for this problem is explained below.

An infinite plate (wall) is usually understood to be one such that the width and the length are very large compared to its thickness. Thus, an infinite plate represents a body restricted by two parallel planes. A change in temperature occurs only along its thickness.

It is assumed that the initial temperature distribution over the plate thickness is given as a constant value T_o. At the initial time, the bounding surface temperatures are instantaneously changed to some temperature T_a which is then maintained constant. The problem is to

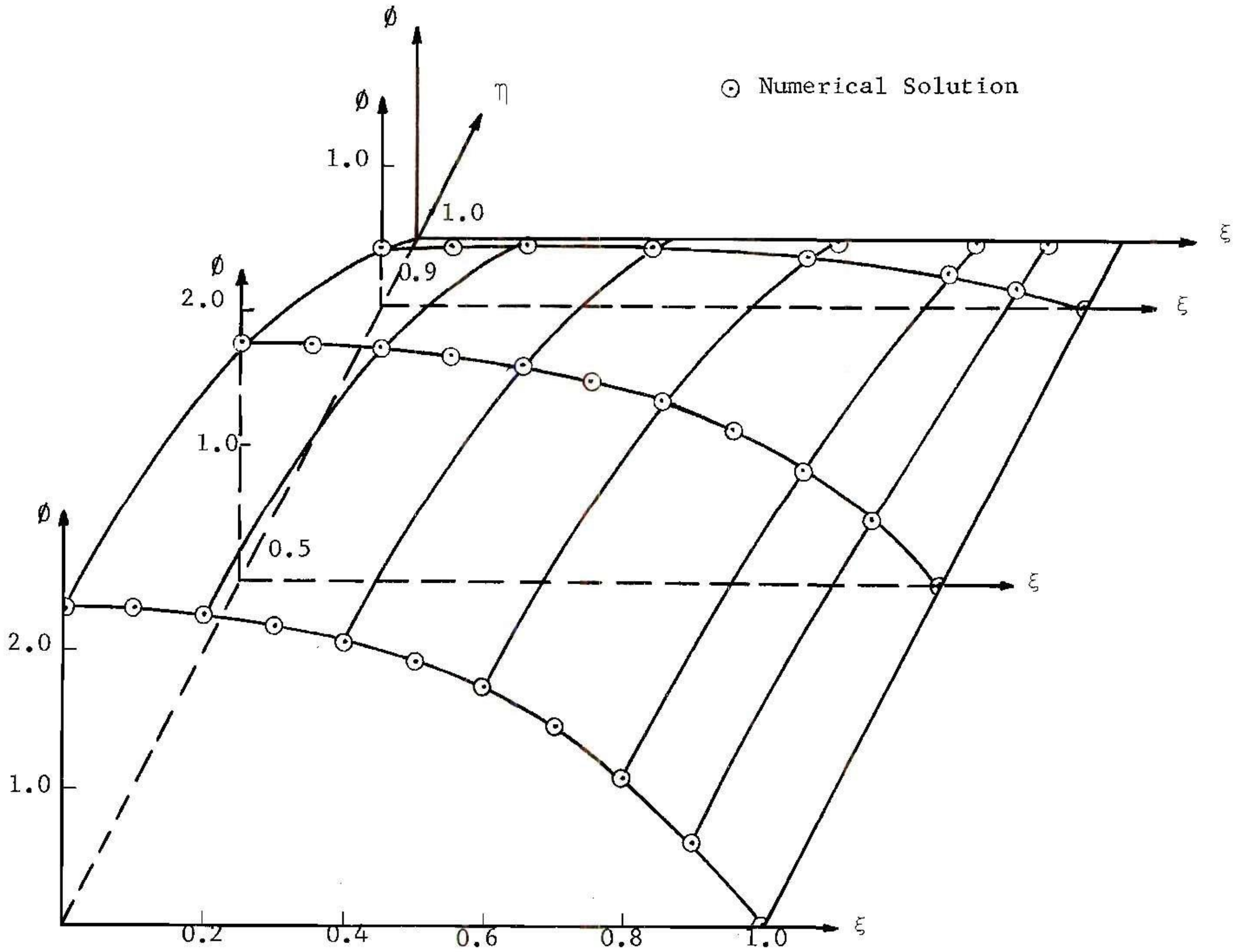

Figure 15. Steady Heat Transfer in a Homogeneous Anisotropic Floor With Internal Heat Generation (Radiant Floor Heating)

determine the temperature distribution over the plate thickness at any instant.

If the origin of coordinates is placed at the center of the plate, and if the plate is homogeneous and isotropic, then the problem stated above takes the dimensionless mathematical form

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \xi^2} \tag{4.15}$$

where,

$$\phi = \frac{T - T_a}{T_o - T_a} \tag{4.16}$$

$$\xi = \frac{x}{b}, \quad \tau = \frac{D t}{b^2} \tag{4.17}$$

where (b) is the half the plate thickness, and D is the ratio of specific heat to conductivity. The transformed boundary conditions take the form

$$\text{I. C. :} \quad \text{at } \tau = 0 \quad \phi = 1 \tag{4.18}$$

$$\text{B. C. :} \quad \text{at } \xi = \pm 1 \quad \phi = 0 \tag{4.19}$$

For this problem, the computer execution time was 0.14 seconds per time step. The finite element results and a comparison with the exact solution [5], [7] of the cooling process described above can be seen in Figure 16. Computed error for this solution is 1.0 percent.

Example Five. In this example, the homogeneous problem described in Example Four is extended to an unsteady non-homogeneous problem. It is again assumed that the wall shown in Figure 17 has a thickness (2b) and composed of five layers of material which have different thermal diffusivities. The coordinate axis is located at the center of the wall as seen in Figure 17, and the wall is assumed to be symmetric with respect to center. The second layer which is between $\xi = 0.3$ and $\xi = 0.7$ has a thermal diffusivity 5 times greater than the first

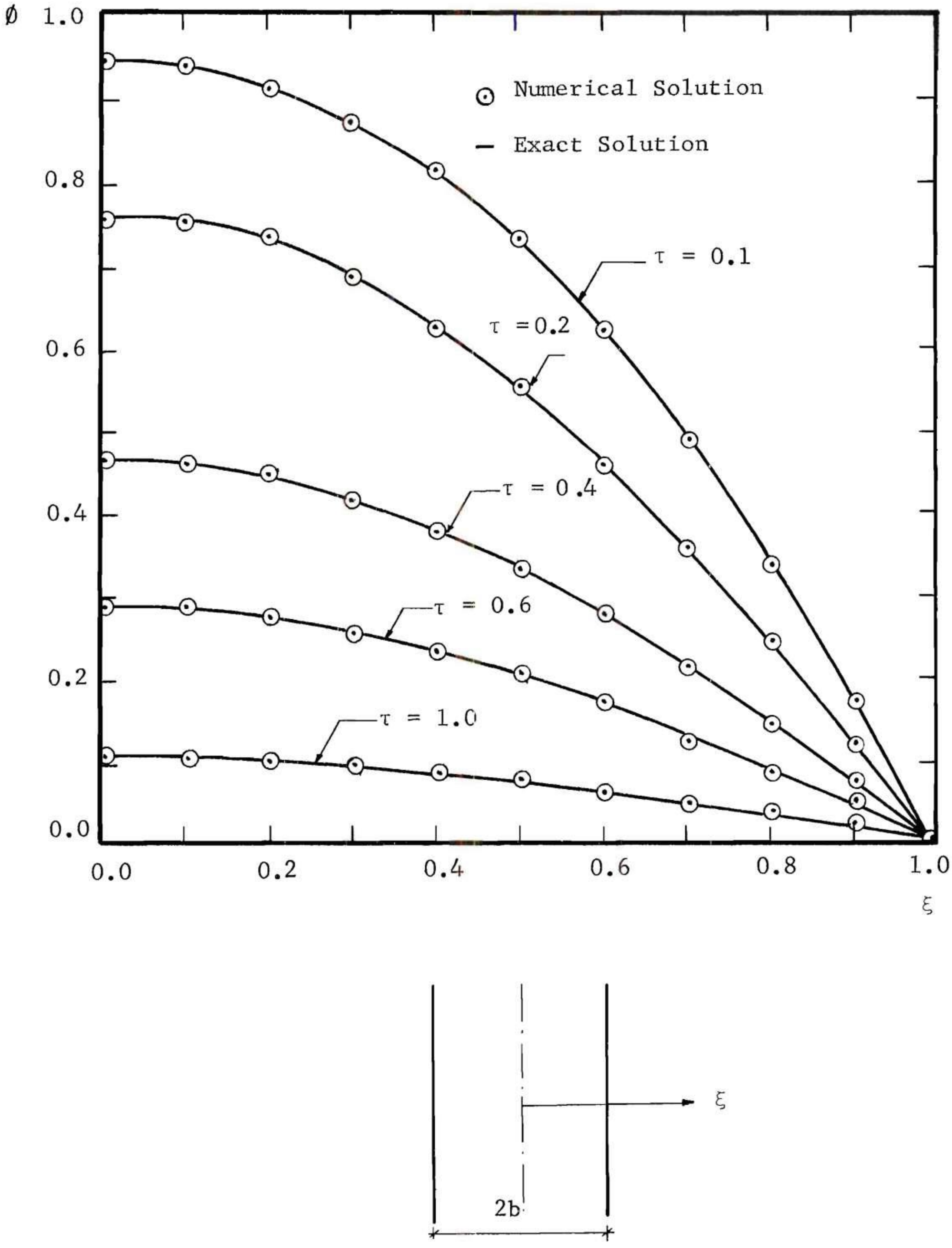

Figure 16. Unsteady Heat Transfer in a Wall

layer which is between $\xi = 0.0$ and $\xi = 0.3$. The third layer which is between $\xi = 0.7$ and $\xi = 1.0$ has a thermal diffusivity 2 times greater than the first layer. If the same cooling process as given in Example Four is studied for this case, then the mathematical model and the boundary conditions will be the same as given in Example Four. The only difference will be the variation of the thermal diffusivities of layers and the interface conditions between these layers which are taken care in the mathematical analysis. The finite element results for this example can be seen in Figure 18. In this figure the non-homogeneous regions of the wall can be easily detected from the abrubt variation of the temperature distributions in the wall. As expected, a final steady state condition is reached at the end of the cooling process

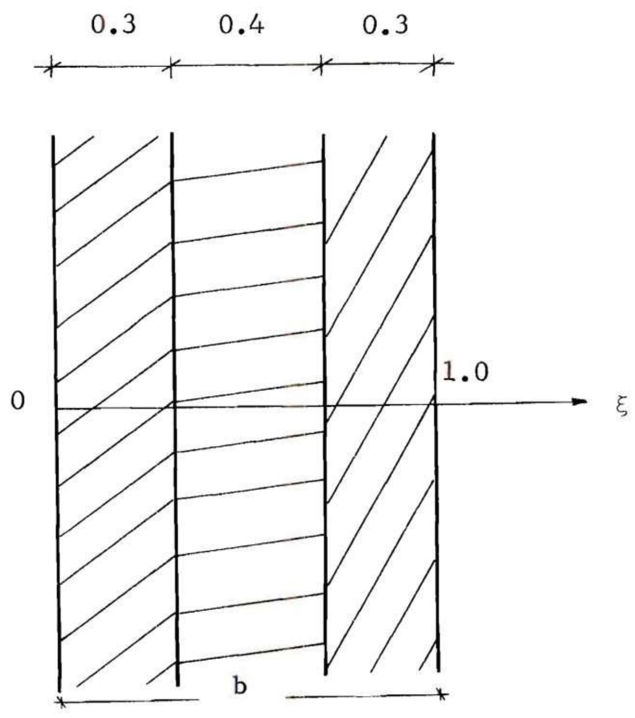

Figure 17. Composite Wall

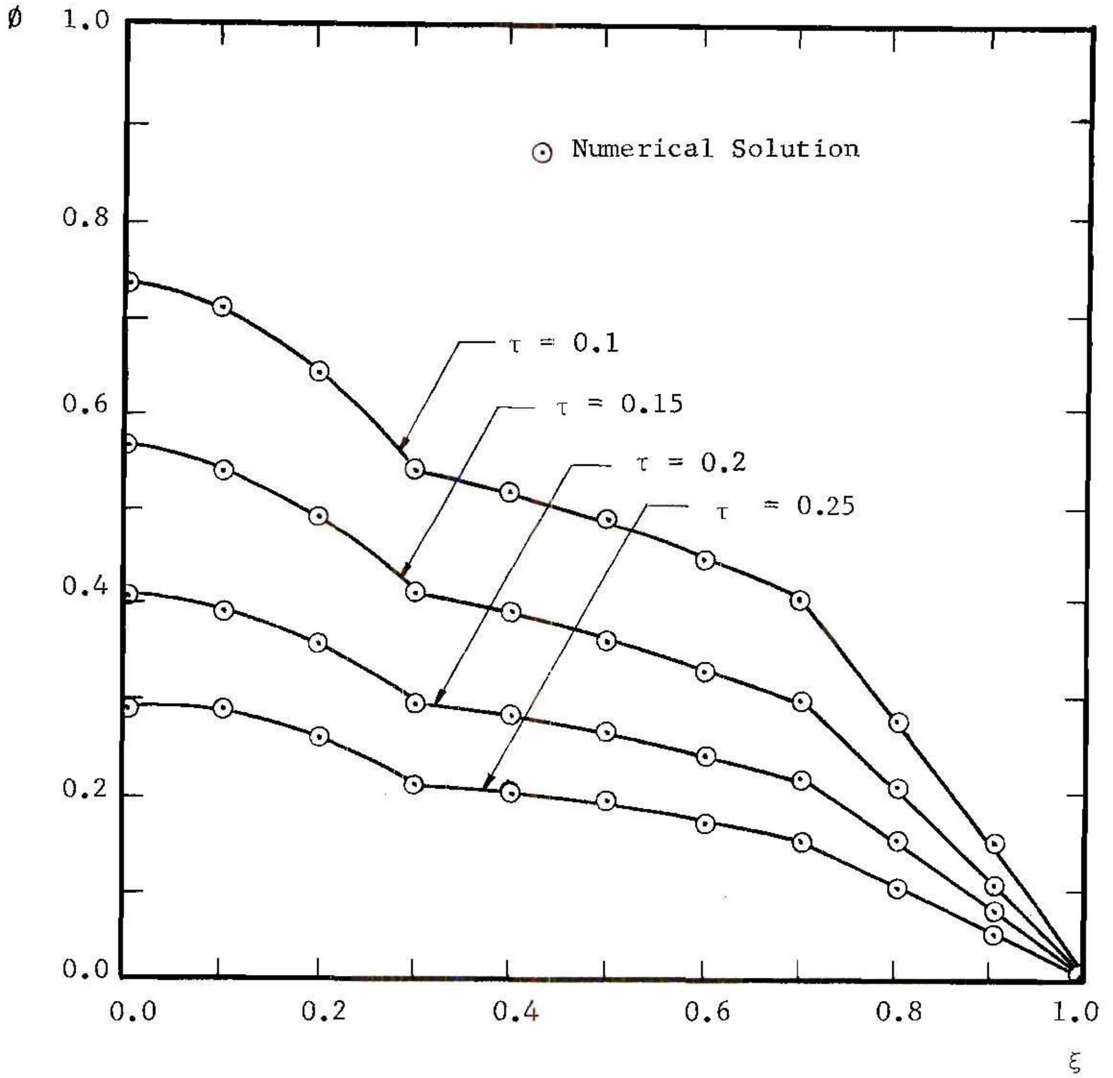

Figure 18. Unsteady Heat Transfer in a Non-Homogeneous Composite Wall

which corresponds to $\phi = 0$. For this problem computer execution time was 0.14 seconds per time step.

Example Six. During production in the factory a cylindrical reinforcement rod is subjected to a cooling process. Initially, the temperature of the rod is greater than the ambient air temperature surrounding the rod. Thus, this difference in initial rod and air temperature gives rise to unsteady temperature variation in the rod. This problem takes the mathematical form below.

Consider a cylinder whose length is infinitely large compared to its diameter. If the heat transfer between the cylinder surface and the surroundings occurs uniformly over the whole surface then its temperature will depend only on time and radius (axisymmetric problem).

The cylinder has a prescribed radial temperature distribution, T(r), which is taken as a constant value, T_o, to simplify the problem in this example. At an initial time instant the cylinder surface is instantaneously cooled to some temperature T_a which is assumed constant during the entire cooling process. The problem is to determine the temperature distribution in the cylinder as a function of time.

If the origin of the coordinates is placed at the symmetry axis of the cylinder, then the above problem takes the dimensionless mathematical form

$$\frac{\partial \emptyset}{\partial \tau} = \frac{\partial^2 \emptyset}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \emptyset}{\partial \xi} \tag{4.20}$$

where,

$$\emptyset = \frac{T - T_a}{T_o - T_a} \tag{4.21}$$

$$\xi = \frac{r}{R} \tag{4.22}$$

$$\tau = \frac{Dt}{R} \tag{4.23}$$

where R is the radius of the cylinder and D is the ratio defined in Example four. The transformed boundary conditions take the form

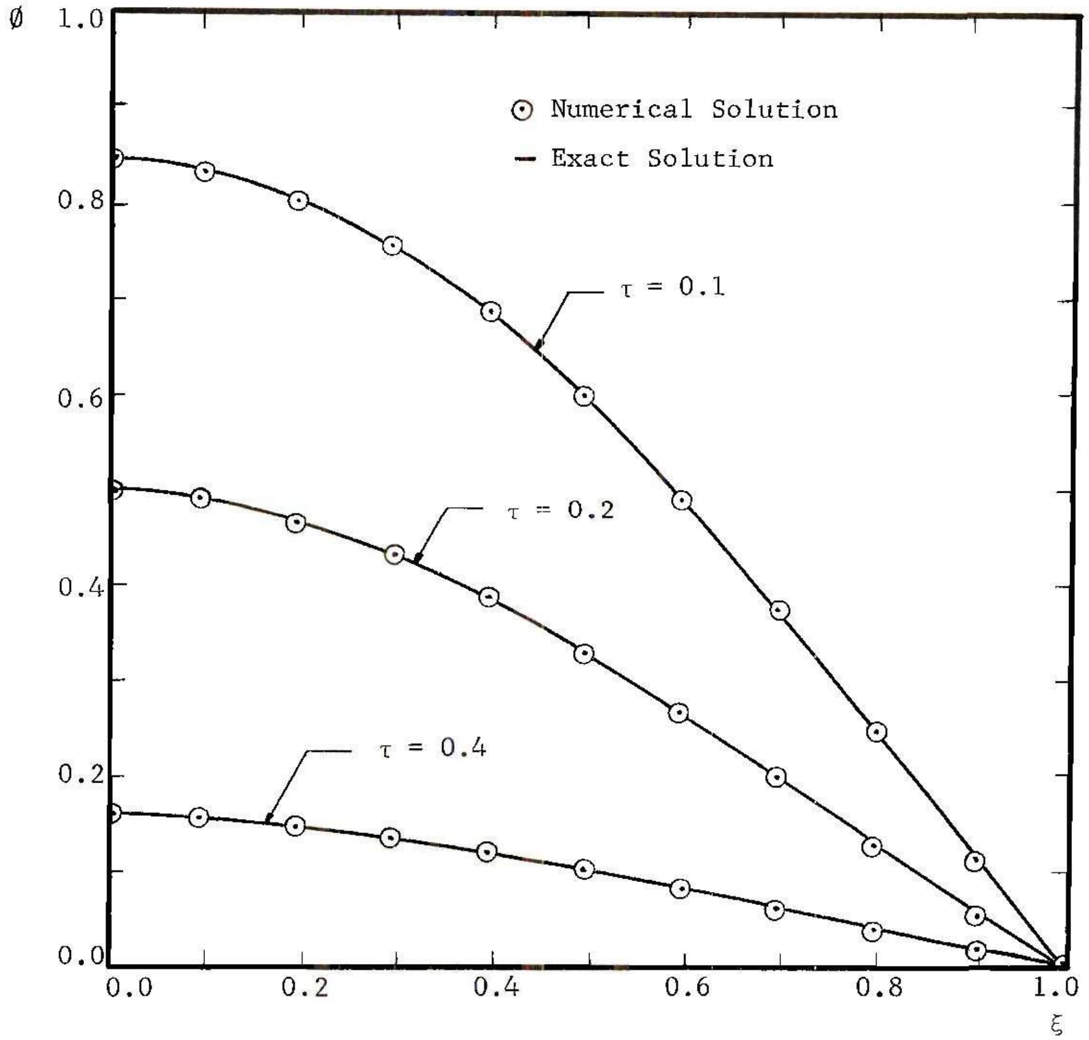

Figure 19. Unsteady Heat Transfer in a Cylindrical Rod

$$\text{I. C. : at } \tau = 0 \quad \emptyset = 1 \tag{4.24}$$

$$\text{B. C. : at } \xi = 1 \quad \emptyset = 0 \tag{4.25}$$

Finite element results and a comparison with the exact solution [5], [7] can be seen in Figure 19. Computer execution time and percent error for this solution were the same as in Example four.

Fluid Flow

The broad field of fluid mechanics can be considered as a field parallel to solid mechanics, since it is built upon the same fundamental laws of motion. The subject branches out into various specialities such as aerodynamics, hydraulic engineering, gas dynamics, and the like. It deals with the statics, kinematics and dynamics of fluids. Available methods of analysis stem from the application of Newton's laws of motion, the first and second laws of thermodynamics, the principle of conservation of mass, Newtons law of viscosity, and others.

In this section numerical solutions will be given for some examples in the above field. Just as in heat transfer problems the numerical solutions obtained through finite element analysis in fluid flow problems are in very good agreement with exact results, and for problems where no exact solutions exist the writer relies on the accuracy of the method as demonstrated in previous examples.

Example Seven. Irrotational incompressible potential flows are flows for which the vorticity vector ω is equal to zero

$$\omega = \nabla \times \bar{v} = 0 \tag{4.26}$$

where $\bar{v}$ is the velocity vector. A function, the curl of which is equal to zero, can always be represented by the gradient of a scalar function as shown by the vector identity curl (grad $\emptyset$) = 0. Thus, the velocity vector can be written as a gradient of a scalar function $\emptyset(x,y)$.

$$\bar{v} = \nabla\emptyset \tag{4.27}$$

In this context, the function ϕ is called the velocity potential and because of the existence of this potential function, irrotational flow is often called potential flow. If the fluid is incompressible, the continuity equation is

$$\nabla \cdot \bar{v} = 0 \tag{4.28}$$

Thus, ϕ must also satisfy the well known relation

$$\nabla \cdot (\nabla \phi) = 0 \tag{4.29}$$

which is Laplace's equation. It is also known that a stream function is related to the velocity potentials by

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \phi}{\partial x} \tag{4.30}$$

$$v = - \frac{\partial \Psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{4.31}$$

Thus, the stream function also satisfies Laplace's equation and in two dimensional cartesian coordinates

$$\nabla \cdot (\nabla \Psi) = 0 \tag{4.32}$$

In this example a cylinder of radius (a) is considered with uniform flow velocity V_o at infinity. A schematic view is given in Figure 20. If the flow around this cylinder is irrotational and incompressible potential lines and stream lines should satisfy Equations (4.29) and (4.32), respectively. The problem is to determine the velocity distribution around the cylinder surface where $v \approx \frac{\Delta \phi}{\Delta s} \approx \frac{\Delta \psi}{\Delta n}$ and Δs and Δn denote the distances between a pair of adjacent equipotential and stream-lines respectively. In this problem the distribution of ϕ and ψ values were obtained using the finite element method. Given a = 4.6 m, V_o = 4.64 m/sec for illustrative purposes. The results and a comparison with exact solution is shown in Figure 21. The computer execution time for this problem was 10 seconds.

Example Eight. Equation (4.29) also models the viscous flow of fluids at low Reynolds numbers through homogeneous and isotropic porous media. In this example, the flow is around many cylindrical piles which support a foundation. Since the problem can be considered as symmetrical both in flow direction and in the transverse direction, two rows of cylinders are studied as the physical model.

Consider two cylinders of radius (a), the centers of which are separated by (3a) in the flow direction and by (3a/2) in the transverse direction. If the flow passing around these cylinders is incompressible then the potential lines describing this flow should satisfy Equation

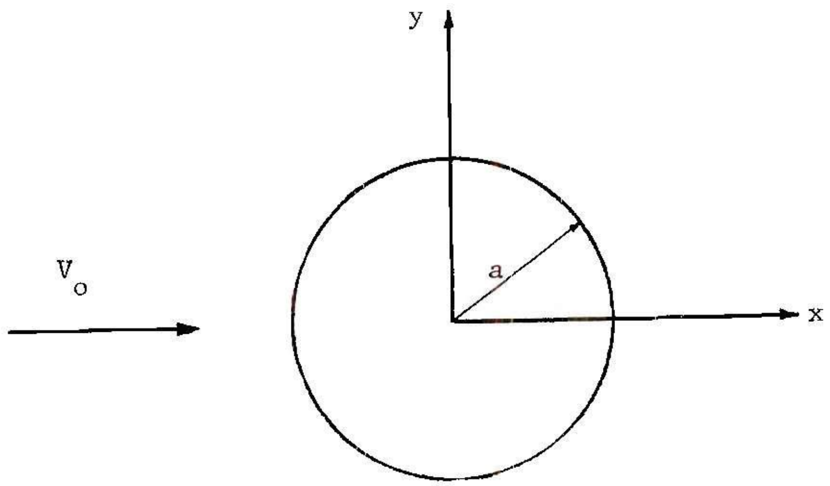

Figure 20. Schematic View of Cylinder

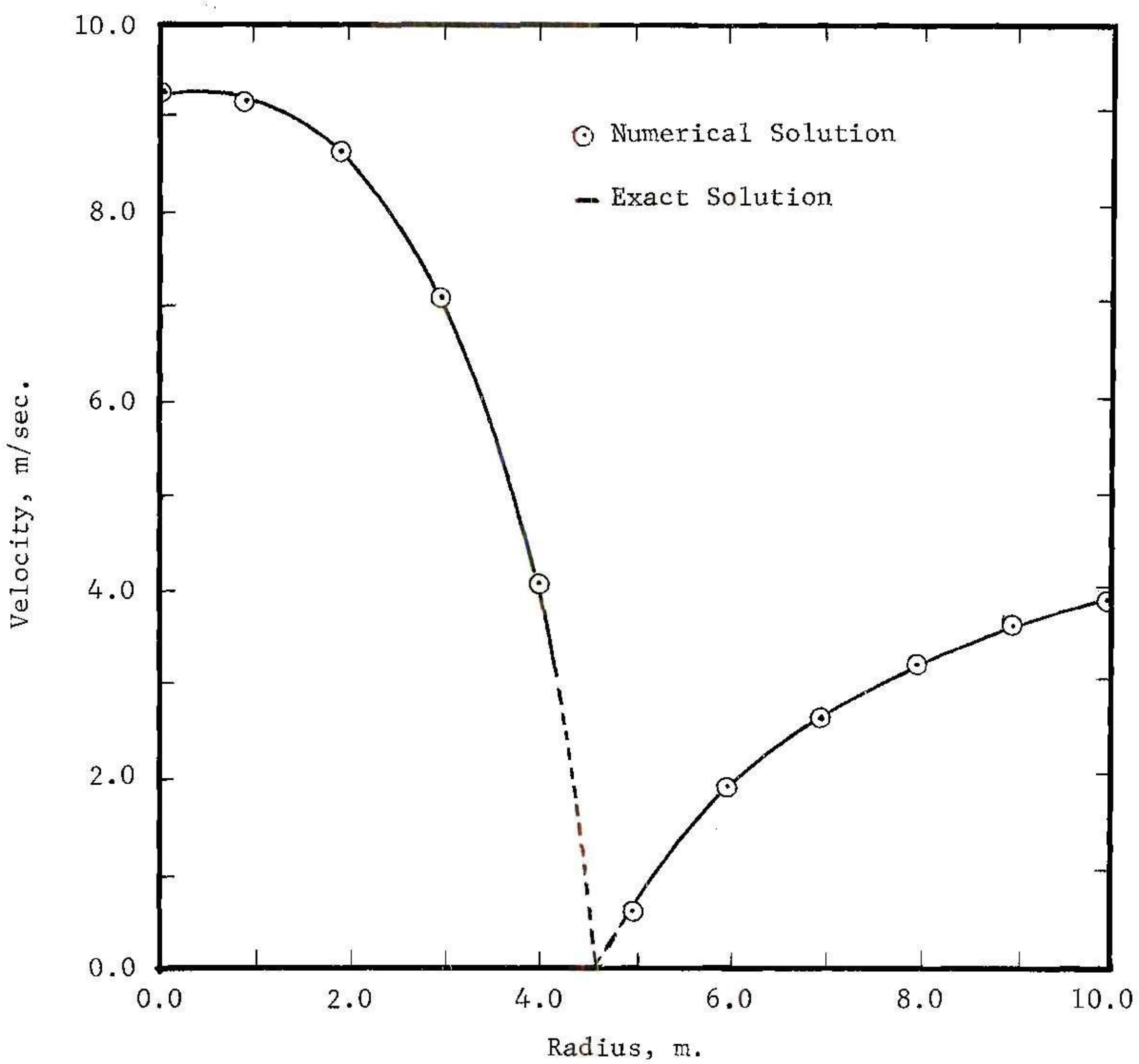

Figure 21. Velocity Distribution Around a Cylinder

(4.29). If the problem is to determine the potential line configuration between the rows of cylinders, the finite element results can be seen in Figure 22. This solution can be extended to flows where many such rows exist. Idealization of this problem was obtained using 212 elements which resulted in 129 nodal points. Computer execution time for this problem was 20 seconds.

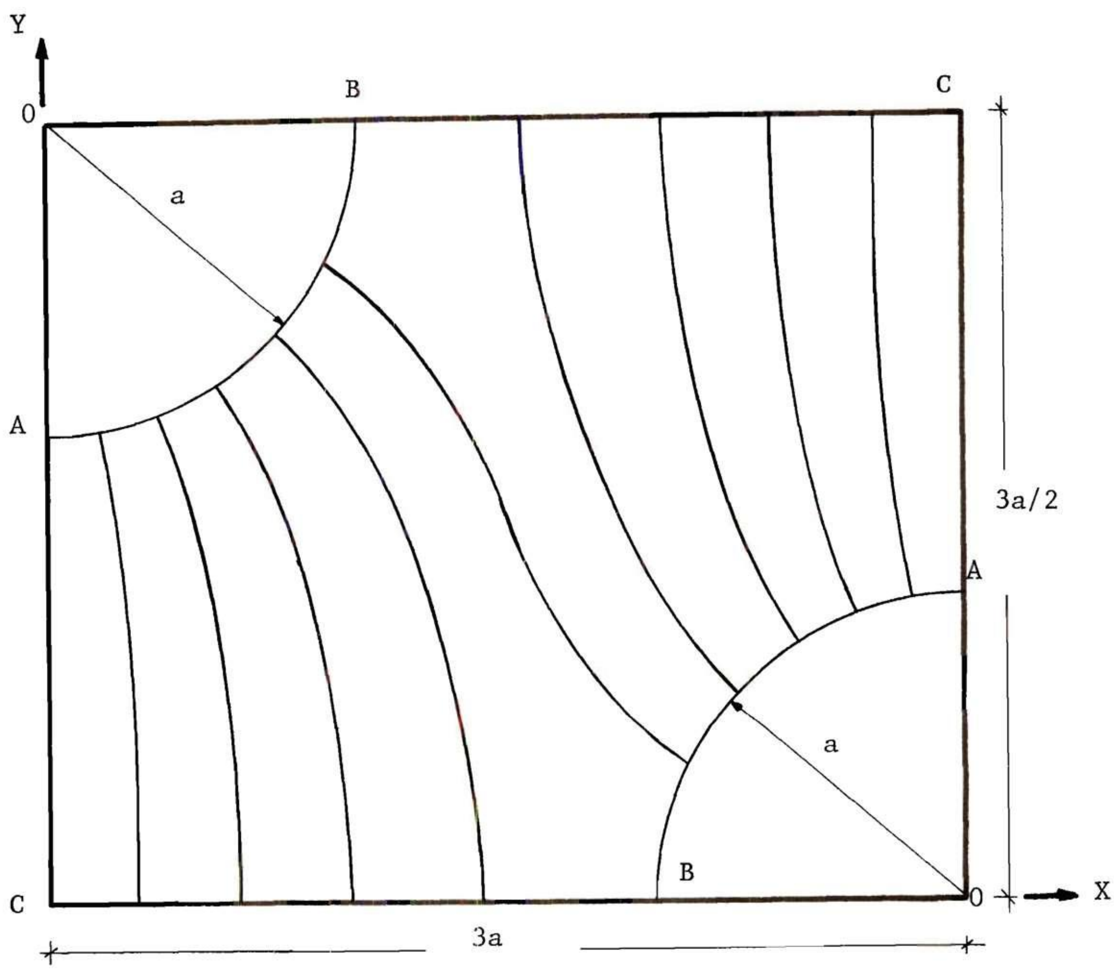

Figure 22. Potential Lines Between Two Rows of Cylinders

The potential line configuration in Figure 22 represents the relative magnitudes of velocities at various locations between the cylinders. If the flow is in the x-direction (Figure 22), the distance between the potential lines increases with decreasing distance from the cylinder along line $\overline{CB}$. This implies a reduction in the velocity with decreasing distance from the approached cylinder. Thus, stagnation points at locations B and maximum velocities at locations A are indicated.

Example Nine. Hydrodynamic lubrication is an important problem area in many engineering fields. The basic problem can be reduced to the study of flow characteristics of a liquid separating two walls which are moving relative to each other. The mathematical model describing this model can be summarized as follows.

A semi-infinite body of liquid with constant density ρ and constant viscosity μ is bounded by two walls parallel to x-axis. The distance between the walls is a constant, h. Initially, the fluid and the solid surfaces are at rest; but at time t = 0 the lower solid surface is set in motion in the positive x-direction with a velocity U_O. Consequently, the velocity distribution between these two walls is a function of time. Assuming that there is no pressure gradient or gravity force in the x-direction and that the flow is laminar, then this problem one of flow formation in Couette flow. The solution to this problem can be obtained as an exact solution of the Navier-Stokes equations. A schematic view of this problem can be seen in Figure 23.

The problem described above takes the dimensionless mathematical

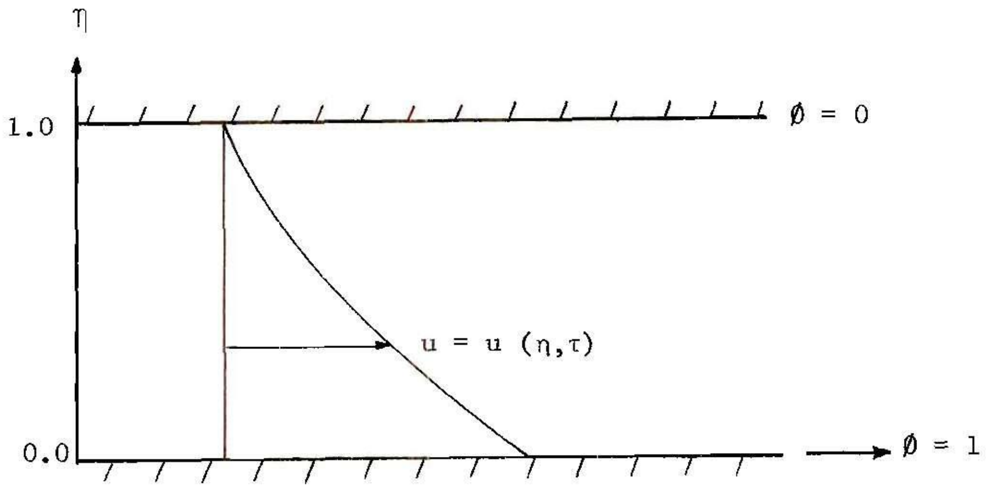

Figure 23. Schematic View of Unsteady Couette Flow

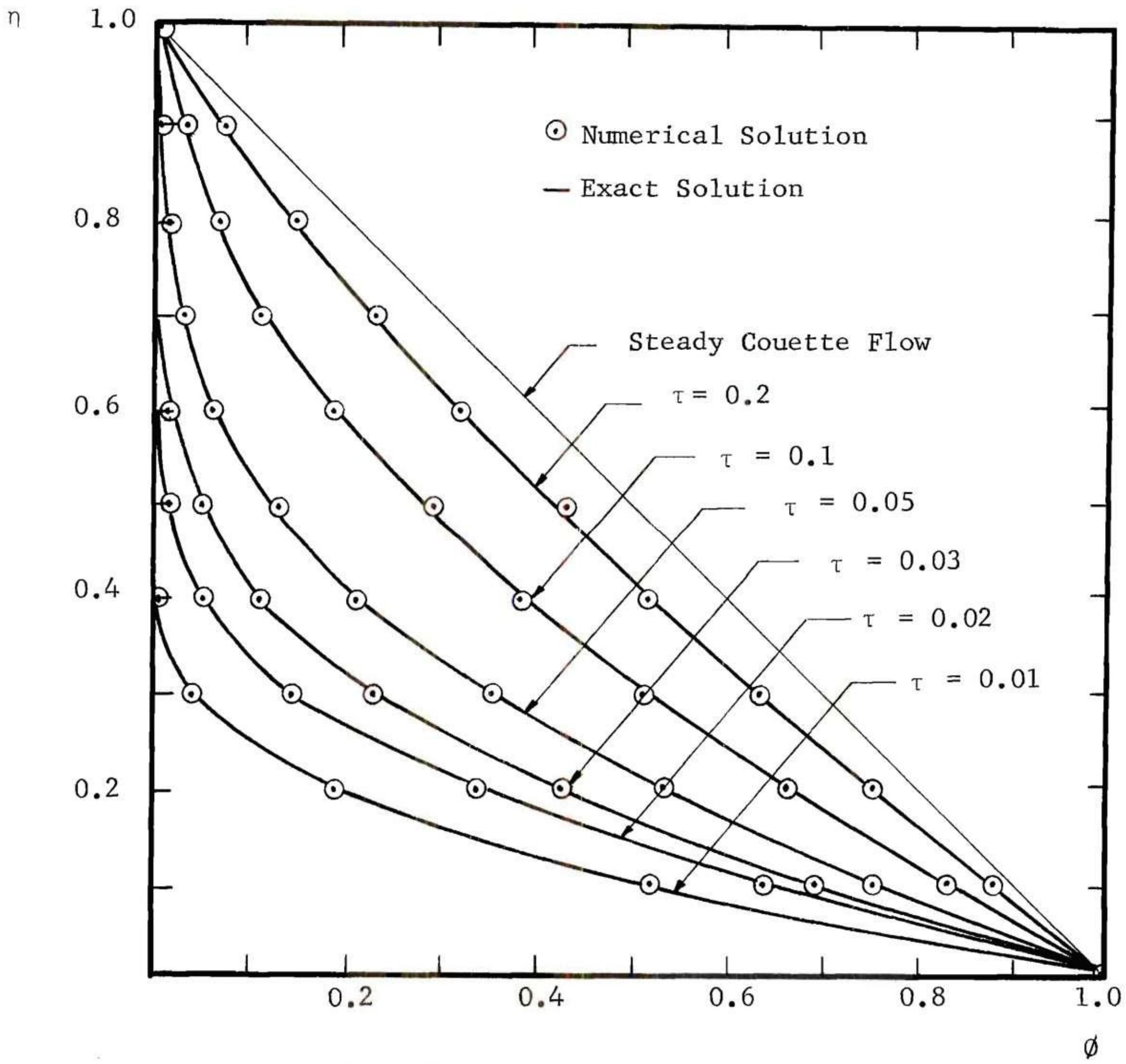

Figure 24. Flow Formation in Couette Flow

form below.

$$\frac{\partial \emptyset}{\partial \tau} = \frac{\partial^2 \emptyset}{\partial \eta^2} \tag{4.33}$$

The dimensionless parameters are

$$\phi = \frac{u}{U_o}, \quad \eta = \frac{y}{h}, \quad \tau = \frac{\nu t}{h^2} \tag{4.34}$$

where ν is the kinematic viscosity, u is the velocity in x-direction. The transformed boundary conditions take the form,

$$\text{I. C. :} \quad \text{at } \tau = 0 \quad \phi = 0 \tag{4.35}$$

$$\text{B. C. 1 :} \quad \text{at } \eta = 0 \quad \phi = 1 \tag{4.36}$$

$$\text{B. C. 2 :} \quad \text{at } \eta = 1 \quad \phi = 0 \tag{4.37}$$

Once again, the above described continuum was idealized by using 200 elements with 121 nodal points. Finite element results and a comparison with the exact solution [37] can be seen in Figure 24. Computer execution time and percentage error for this problem were 0.14 seconds per time step and about one percent, respectively.

Example Ten. Another extension of the finite element method can be the study of dispersion in a porous medium. This phenomenon can be modeled by the convective dispersion equation presented in Chapter III. The one-dimensional form of this equation in cartesian coordinates is

$$D \frac{\partial^2 \phi}{\partial x^2} - u \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t} \tag{4.38}$$

where ϕ is the concentration of the dispersing mass, D is the dispersion coefficient and u is the seepage velocity in the x-direction. Shamir and Harleman [35] studied steady and unsteady problems of this type in detail and presented a numerical scheme for solving such problems. In this example, a typical unsteady problem studied in their

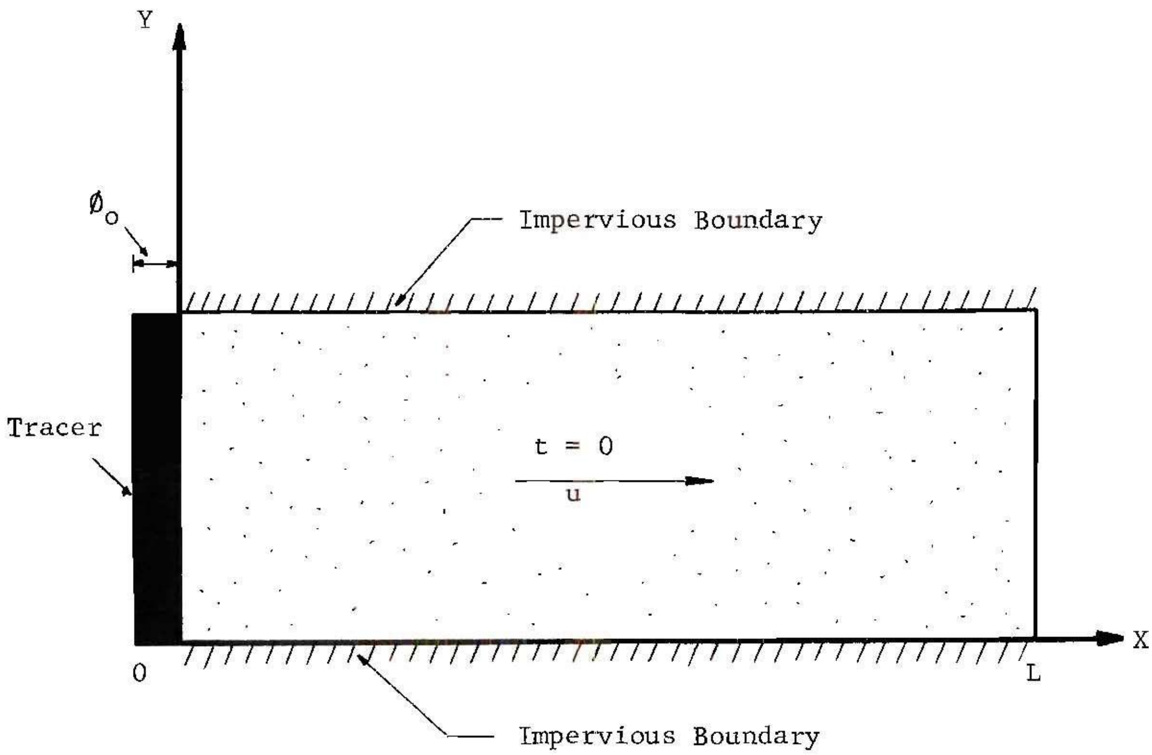

Figure 25. Longitudinal Dispersion Model

report was solved by the finite element method.

The problem is to determine the one-dimensional dispersion of a tracer (concentration) introduced to the porous medium at a constant rate at x = 0 cm. There is a constant rate of seepage in the x-direction. The velocity distribution is assumed to be uniform in y-direction. Initially, the distribution of this tracer in the porous medium is zero. A schematic description of this problem is given in Figure 25.

The dimensionless form of the unsteady convective diffusion equation in one-dimensional studies is

$$\frac{\partial^2 \phi}{\partial \xi^2} - \lambda \frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \tau} \tag{4.39}$$

where

$$\phi = \frac{\phi}{\phi_o}, \quad \xi = \frac{X}{L}, \quad \lambda = \frac{uL}{D}, \quad \tau = \frac{Dt}{L^2} \tag{4.40}$$

In applying the numerical solution the following data, which are taken from the above report, are used.

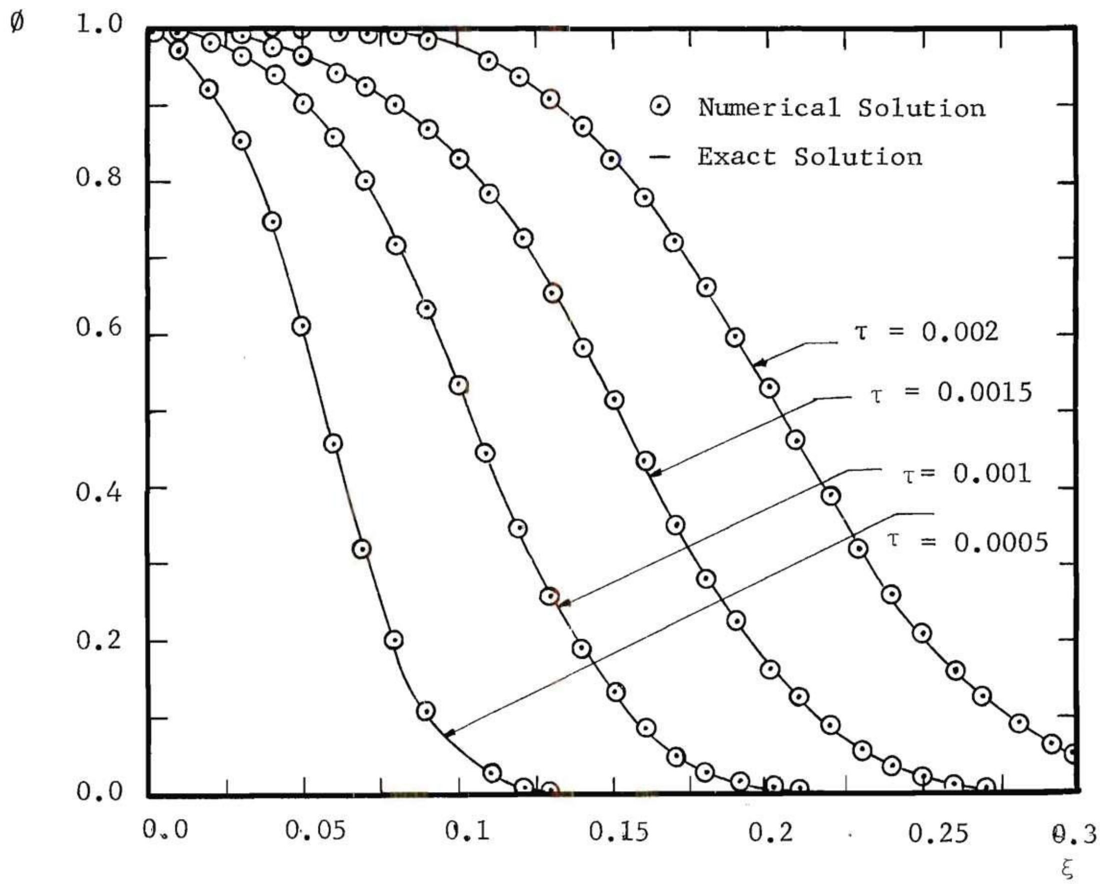

Figure 26. Convective Diffusion in Porous Media

$$0 \leq x \leq 10 \text{ cm.}$$

$$u = 0.1 \text{ cm/sec}$$

$$D = 0.01 \text{ cm}^2/\text{sec} \qquad (4.41)$$

Thus the convective diffusion equation to be solved and the transformed boundary conditions take the form

$$\frac{\partial^2 \phi}{\partial \xi^2} - 100 \frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \tau} \qquad (4.42)$$

$$\text{I. C. :} \quad \text{at } \tau = 0 \quad \phi = 0 \qquad (4.43)$$

$$\text{B. C. 1 :} \quad \text{at } \xi = 0 \quad \phi = 1 \qquad (4.44)$$

$$\text{B. C. 2 :} \quad \text{at } \xi = 1 \quad \frac{\partial \phi}{\partial \xi} = 0 \qquad (4.45)$$

Numerical results and a comparison with the exact solution [35] are given in Figure 26.

Although the finite element results are very accurate for this particular example, extreme care should be given to the coefficients appearing in the exponential terms in the solution of this type of problem. Very large numbers appearing in the exponent create truncation and roundoff errors in computer computations which may be significant and may result in unstable solutions. Thus, special care should be given to problems in this category before finalizing the input values for a specific problem.

In this example, 200 elements were used in idealizing the continuum given in Figure 25. The computer execution time was 0.16 seconds per time step. The computed error for this solution is around 2.0 percent.

This example was actually a laboratory model of flow of a certain concentration in a confined aquifer. A practical application of this example could be the study of the flow of some concentration in a confined aquifer between two rivers. Of course, the appropriate constants of the problem may have to be changed for a specific application.

Example Eleven. Another engineering application of the finite element method can be the study of ground water seepage under a dam. If the earth materials are layered and anisotropic, it becomes very difficult for an engineer to estimate the quantity of seepage passing underneath this dam.

In this problem an arbitrarily conceptualized dam is located on a layered and anisotropic media. The problem may be the determination of the equipotential lines under a constant head. A schematic description of this problem is given in Figure 27.

The mathematical model describing this problem is

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4.46}$$

where K_x and K_y are permeabilities of the medium in the principal local seepage directions, respectively. ϕ in this model represents the equipotential function. As the boundary conditions of the problem, ϕ is assumed to vary from 10 on the upstream side of the dam to 0 on the downstream side of the dam.

Finite element results for this problem can be seen on Figure 28. In this problem, the continuum analyzed was idealized by 297 elements and the computer execution time was 15 seconds.

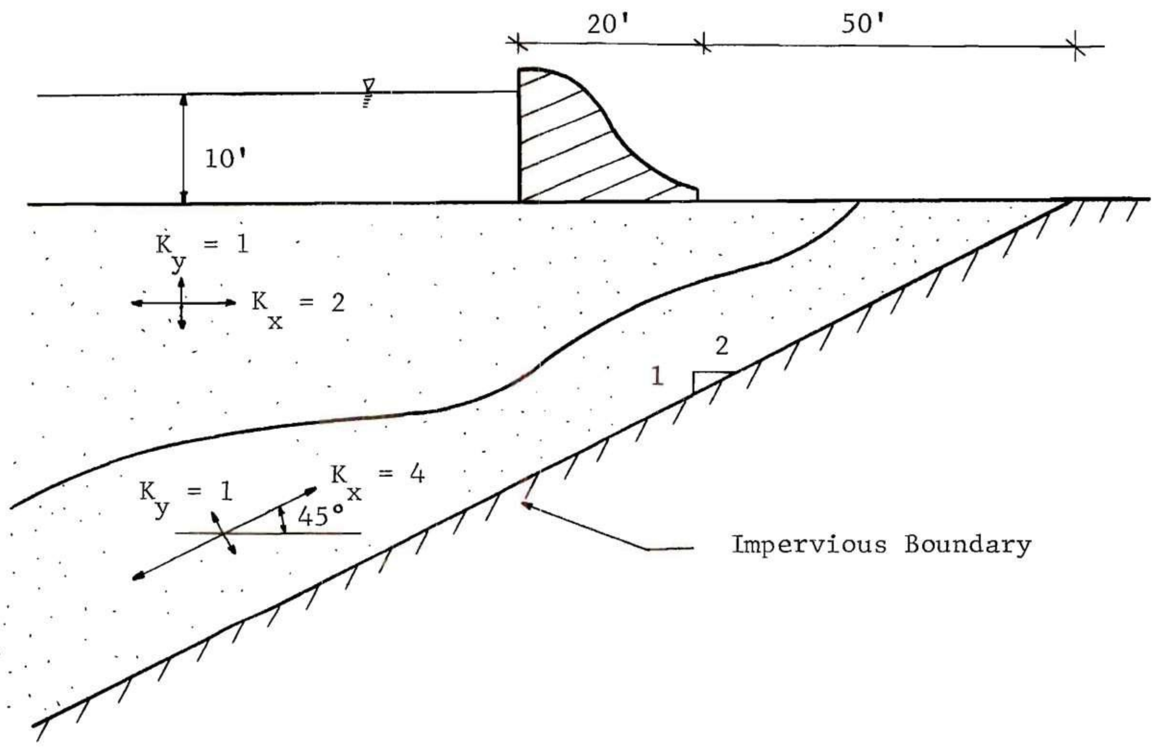

Figure 27. Concrete Dam on Non-Homogeneous and Anisotropic Media

Example Twelve. Parabolic equations analyzed in Chapter II can also be utilized in the study of unsteady groundwater seepage problems in confined aquifers. In this example, the non-homogeneous problem described in Figure 29 is analyzed. At time $t < 0$ the water surface elevation at the boundary on the left is 60 ft. At time $t = 0$ this water surface is dropped to 40 ft. The boundary on the right is considered to be a constant water surface elevation which is kept at 60 ft all the time.

The physical problem described above takes the mathematical form

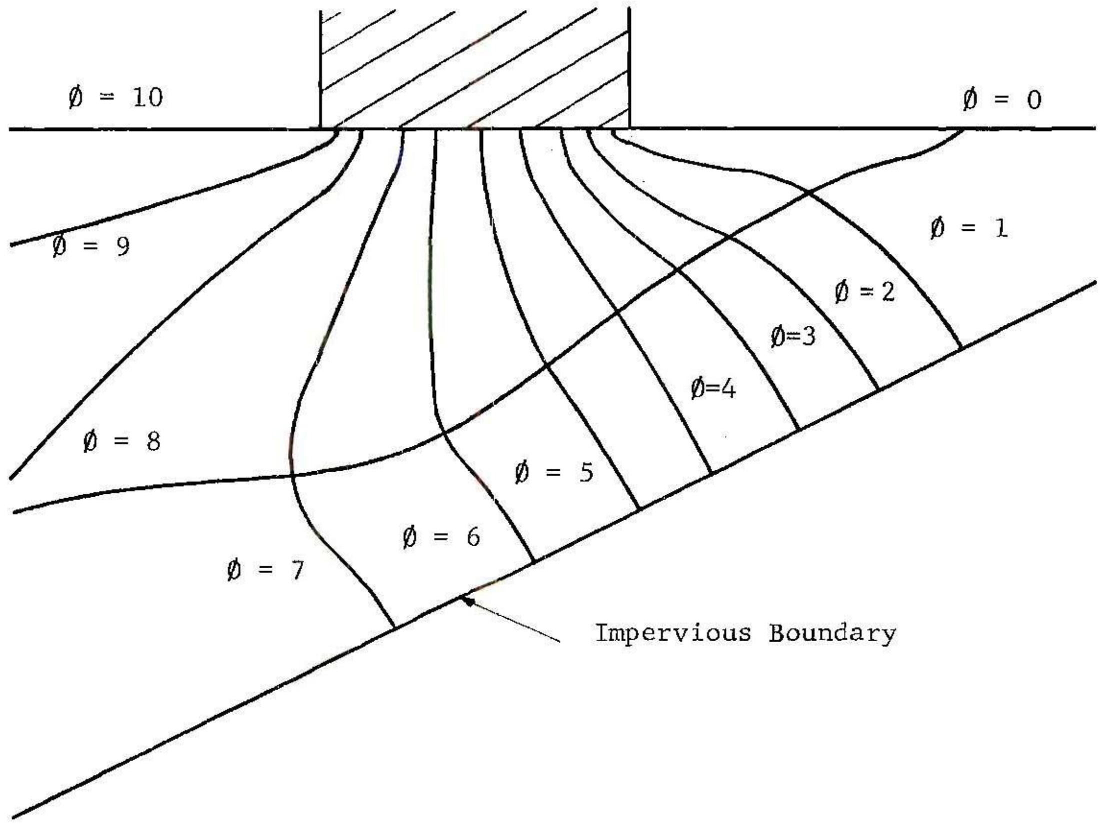

Figure 28. Equipotential Line Configuration Under a Dam on Non-Homogeneous and Anisotropic Media

$$\frac{\partial^2 h}{\partial \xi^2} = \frac{\partial h}{\partial \tau} \tag{4.47}$$

and

$$\xi = \frac{x}{L}, \quad \tau = \frac{Tt}{L^2} \tag{4.48}$$

where T is the transmissibility of the confined medium and L is the total length of the medium. The transformed boundary conditions are

I. C. : at $\tau = 0$ $h = 60$ ft. (4.49)

B. C. 1 : at $\xi = 0$ $h = 40$ ft. (4.50)

B. C. 2 : at $\xi = 1$ $h = 60$ ft. (4.51)

If the non-homogeneous media in the confined aquifer is characterized by the transmissibilities identical to those described in Figure 29, then the unsteady drop of head in this aquifer takes the form shown in Figure 30. Computer execution time for this problem was 0.14 seconds per time step.

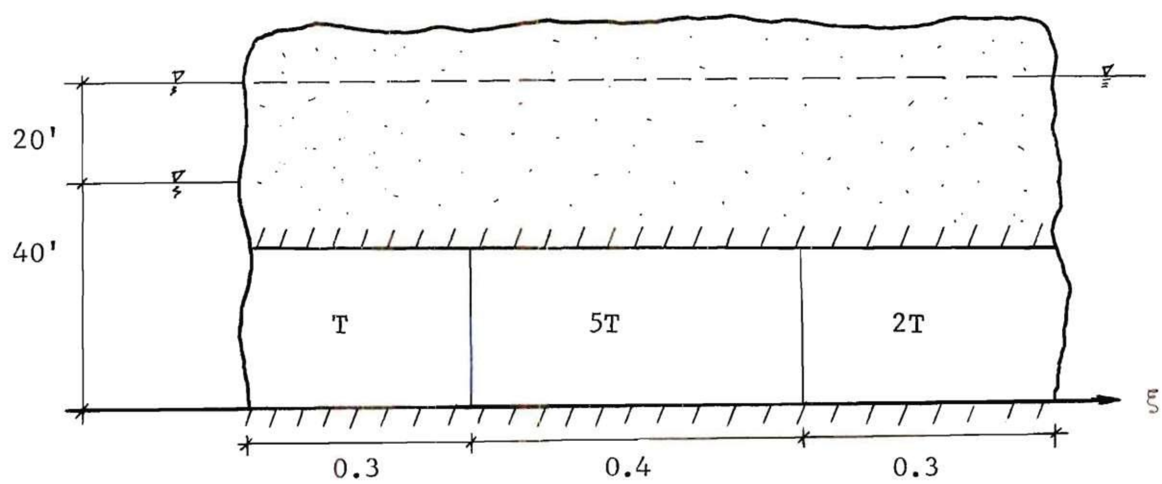

Figure 29. Confined Groundwater Seepage

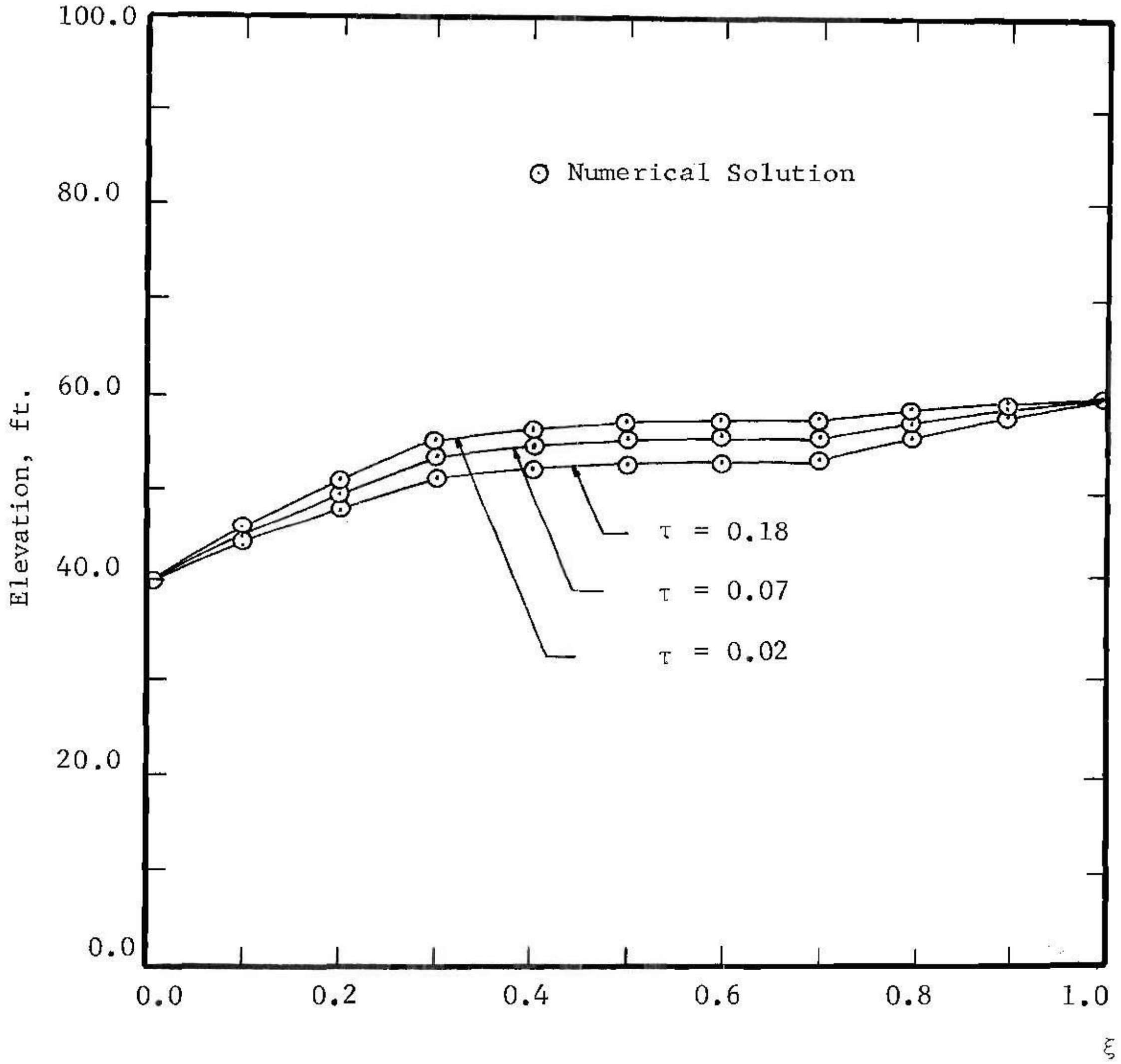

Figure 30. Unsteady Groundwater Seepage in a Confined Aquifer

CHAPTER V

CONCLUSION

In the present dissertation, a detailed analysis and formulation of the finite element method in application to continuum problems is presented. The use of variational calculus to obtain the matrix equations of the finite element analysis is illustrated for problems which have exact variational forms. For problems where an exact variational form cannot be found, for example parabolic partial differential equations, the use of Galerkin's method to obtain the finite element equations is described. The Galerkin method extends the development of matrix equations of the finite element analysis to many other problems which do not have exact variational forms.

The engineering problems which have been investigated are; steady and unsteady heat transfer problems, steady and unsteady seepage problems and steady and unsteady incompressible fluid flow problems. Although some of the above mentioned problems have been treated by solid mechanics and fluid mechanics analysts, an intensive treatment of the mathematical formulations of such problems does not exist in the presently available literature. Thus, it is hoped that the analysis presented in this dissertation represents a major contribution to this area of research. The investigation of the convective diffusion phenomenon by the finite element method is an additional significant extension to fluid mechanics literature. For this problem the use of Galerkin's method and the development of the specific restricted variational form utilized to derive matrix finite element equations constitute two separate contributions to the field.

The finite element method and the program developed for this dissertation are not restricted at all by the irregular geometry, nonhomogeneity and anisotropy of physical problems. The program can handle Dirichlet and/or Neumann boundary conditions. The author's experience has been that the finite element method is much easier to apply and to program than is the case with the finite difference analyses. Mathematical formulations presented in earlier chapters and the above mentioned generalities clearly indicate this. It is known that finite difference analysis presents difficulties in formulation and in programming if the solution of a problem with above mentioned generality is attempted.

Results given in Chapter IV also indicate that finite element analysis is both accurate and efficient for problems of the type analyzed here. The comparisons of exact solutions with numerical solutions yield the following error percentages. For steady problems the percent error was in the range of 0.4 percent to 1.0 percent. For unsteady problems the upper limit increased to 2.0 percent. These percentages could possibly be reduced if smaller elements and smaller time increments are used, although the cost of solution is increased. The results also show the efficiency of the Cholesky algorithm which is utilized in the solution of simultaneous algebraic equations. For steady problems with approximately 200 to 300 elements, the execution time was 13 to 21 seconds. For unsteady problems with the same number of elements the computer execution time was approximately 0.14 seconds per time step.

Needless to say, much care should be given in selecting a global indexing scheme for node points since only a few numbering schemes will result in the minimum possible band width of the system matrices which in turn affects the computer execution time. The specific characteristics of these best numbering schemes will depend on the nature of the continuum being considered. The general rule to be applied is to keep the differences between the nodal numbers within each element as small as possible.

Although the program is written for two dimensional studies, it can be extended to three dimensional studies; however, computer storage and execution times will be increased considerably. The applications presented here are restricted to the solution of the selected elliptic and parabolic partial differential equations. It is the belief of the author that the finite element method can be equally efficiently utilized in the solution of hyperbolic partial differential equations and is recommended for further study.

APPENDIX A

The following has been adopted to represent the primary variables:

A = Area;

a, b, c = Coefficients defined in Equation (2.1);

D = Constant, thermal diffusivity, diffusion coefficient;

{f} = Displacements due to distributed loads;

{F} = Load matrix;

I = Integral expression;

K = Constant, permeability, conductivity;

K_x, K_y, K_z = (x, y, z) components of K;

N_i, N_j, N_k = Defined by Equations (2.12), (2.13) and (2.14);

L = Linear differential operator;

p = Pressure;

r, z = Local cylindrical coordinates;

R, Z = Global cylindrical coordinates;

M_i = Weighing function;

{P} = Distributed loads (per unit volume);

[P] = Mass matrix;

{R} = Concentrated loads;

[S] = Stiffness matrix;

t = Time coordinate;

T = Temperature, transmissibility;

u = Velocity in (x) direction;

U = Strain energy;

x,y	=	Local cartesian coordinates;
X, Y	=	Global cartesian coordinates;
v	=	Velocity in (y) direction;
V	=	Volume;
w	=	Potential energy of load system;
α, σ, λ	=	Constants defined by Equation (2.106);
$\{\beta\}$	=	Displacements due to concentrated loads;
δ	=	Variation, infinitesimal change;
∇	=	Laplacian;
Δ	=	Increment;
ξ	=	Dimensionless length;
η	=	Dimensionless length;
γ	=	Specific weight;
$\emptyset$	=	Dependent variable defined by Equation (2.1);
$\emptyset_i$, $\emptyset_j$, $\emptyset_k$	=	Nodal values of dependent variable;
Θ	=	Angle, measured from positive x-axis;
ρ	=	Density of fluid;
μ	=	Dynamic viscosity of fluid;
ν	=	Kinematic viscosity of fluid;
τ	=	Dimensionless time;
ω	=	Vorticity vector;
Ψ	=	Stream function;

APPENDIX B

VARIATIONAL PRINCIPLE FOR STEADY CONVECTIVE-DIFFUSION EQUATION

The basic problem in the calculus of variations, as mentioned before, is to determine a function ϕ such that a certain definite integral involving that function and certain of its derivatives takes on a stationary value. The well known Euler theorem of calculus of variations states that for ϕ to make the definite integral stationary it should satisfy the Eulers' Equation (B.2). In this appendix the existence of a variational form for steady convective-diffusion equation will be shown using the Euler's theorem of calculus of variations.

Assumed variational form for the convective diffusion equation has the form,

$$I = \iint_A \left[\frac{K_x}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{K_y}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right] e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} dxdy \qquad (B.1)$$

One can verify this variational form with the application of the Euler theorem,

$$\frac{\partial F}{\partial\phi} - \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial\phi_x}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial\phi_y}\right\} = 0 \qquad (B.2)$$

where F is the functional inside the brackets in Equation (B.1) including the exponential term and subscript (x) and (y) indicates differentiation with respect to (x) and (y). Each term of Equation (B.2) can be written as,

$$\frac{\partial F}{\partial \emptyset} = 0 \qquad (B.3)$$

$$\frac{\partial F}{\partial \emptyset_x} = K_x \frac{\partial \emptyset}{\partial x} e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \qquad (B.4)$$

$$\frac{\partial}{\partial x}\{\frac{\partial F}{\partial \emptyset_x}\} = [K_x \frac{\partial^2 \emptyset}{\partial x^2} - u\frac{\partial \emptyset}{\partial x}] e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \qquad (B.5)$$

$$\frac{\partial F}{\partial \emptyset_y} = K_y \frac{\partial \emptyset}{\partial y} e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \qquad (B.6)$$

$$\frac{\partial}{\partial y} \frac{\partial F}{\partial \emptyset_y} = [K_y \frac{\partial^2 \emptyset}{\partial y^2} - v \frac{\partial \emptyset}{\partial y}] e^{-\frac{xu}{K_x} - \frac{yv}{K_y}} \qquad (B.7)$$

Summing these terms according to Equation (B.2) and dividing by ($e^{-\frac{xu}{K_x} - \frac{yv}{K_y}}$) one can write,

$$K_x \frac{\partial^2 \emptyset}{\partial x^2} + K_y \frac{\partial^2 \emptyset}{\partial y^2} - u \frac{\partial \emptyset}{\partial x} - v \frac{\partial \emptyset}{\partial y} = 0 \qquad (B.8)$$

with Neuman boundary conditions given as the natural boundary conditions. This is the steady convective diffusion equation. Thus it is shown that if one can find a function $\emptyset$ which makes Equation (B.1) stationary then $\emptyset$ should also satisfy the steady convective-diffusion equation given above. The author realizes that if a variational form which describes the problem in the total domain is considered then the above variational form is restricted to homogeneous domains and uniform flow problems.

APPENDIX C

THE COMPUTER PROGRAM

A computer program was written in Fortran IV for use with the Univac 1108 digital computer. This program consists of seven subroutines and a main program. Individual functions of these parts are explained in Chapter III. Capacity of this program is presently limited to 400 elements and 250 nodal points. In order not to complicate the program, only the main storage of the computer is utilized throughout the process.

The computer program developed for this study can find wide application in the field of engineering, as demonstrated in Chapter IV. The input data required for the program are the global coordinates of the nodal points, appropriate constants, boundary conditions of the problem, and a set of data which could be called "element connectivity data". The last set of data indicates the correspondence between elements and their respective nodal points which is necessary to form the stiffness and mass matrices properly. After this data is inserted, the program forms the appropriate matrices and solves the resulting set of equations for the unknown nodal values.

A partial listing of this program is given on the following pages.

```
-HDG,P FINITE ELEMENT ANALYSIS ARAL M. M.
-FOR,IS SEVOSH
      DOUBLE PRECISION XE(3),YE(3),SE(3,3),RE(3),PE(3,3),S(250,30)
     2,P(250,30),R(250),PHI(250),DEP(250),PHIF(250),    PHIIN(250),V(250)
     3,TR(250),RBN(250)
      DIMENSION  X(2,250),NODE(3,400),XK(400),YK(400),ALPHA(400),F(400),
     2UX(400),VY(400),TK(400),NODD(50),BCD(50),NODN(2,25),BCN(25),
     3NEWN(3),XKK(250),YKK(250),UXK(250),VYK(250),NODPLI(2(25),BCPHI(25)
C
C*******************************************************************ARAL
C                             FINITE ELEMENT ANALYSIS
C                                 MUSTAFA M. ARAL
C                          GIT SCHOOL OF CIVIL ENGINEERING
C
C*******************************************************************ARAL
      COMMON/BLOCK2/ DRX,DRY
C*******************************************************************ARAL
      READ(5,61) JDATA
   62 WRITE(6,60)
   60 FORMAT(26X,-FINITE ELEMENT ANALYSIS-/30X,-MUSTAFA M. ARAL-)
      READ(5,1) ITIME,ICON,IAXI
      READ(5,2) NELEM,NNODE,NBCN,NBCD,NBCPHI
      IF(ITIME.EQ.0) GO TO 4
      READ(5,3) TI,TF,TST
    4 CONTINUE
      WRITE(6,5)
      WRITE(6,6)
      WRITE(6,7) ITIME,ICON,IAXI
      WRITE(6,8)
      WRITE(6,9) NELEM,NNODE,NBCN,NBCD,NBCPHI
      IF(ITIME.EQ.0) GO TO 12
      WRITE(6,10)
      WRITE(6,11) TI,TF,TST
   12 CONTINUE
      WRITE(6,14)
      WRITE(6,15)
      DO 16 J=1,NNODE
```

```
      READ(5,17) I,X(1,I),X(2,I)
   16 WRITE(6,18) (J,X(1,J),X(2,J))
      WRITE(6,19)
      DO 20 L=1,NELEM
   20 READ(5,21) I,(NODE(J,I),J=1,3),XK(I),YK(I),ALPHA(I),F(I)
      WRITE(6,22)
      WRITE(6,23)((I,(NODE(J,I),J=1,3),XK(I),YK(I),ALPHA(I),F(I)),I=1
     2,NELEM)
      IF(ICON.EQ.0) GO TO 29
      DO 24 I=1,NELEM
   24 READ(5,25) UX(I),VY(I)
      WRITE(6,26)
      WRITE(6,27)
      WRITE(6,28) ((I,UX(I),VY(I)),I=1,NELEM)
      DO 51 L=1,NNODE
   51 READ(5,50) I,XKK(I),YKK(I),UXK(I),VYK(I)
      WRITE(6,52)
      WRITE(6,53)((I,XKK(I),YKK(I),UXK(I),VYK(I)),I=1,NNODE)
   29 CONTINUE
      IF(ITIME.EQ.0) GO TO 35
   30 READ(5,31) (TK(I),I=1,NELEM)
      WRITE(6,32)
      WRITE(6,33)
      WRITE(6,34)((I,TK(I)),I=1,NELEM)
   46 READ(5,81)(PHIIN(I),I=1,NNODE)
      WRITE(6,49)
      WRITE(6,998)
      WRITE(6,996) ((I,PHIIN(I)),I=1,NNODE)
   35 CONTINUE
      IF(NBCD.EQ.0) GO TO 40
      DO 36 I=1,NBCD
   36 READ(5,37) NODD(I),BCD(I)
      WRITE(6,38)
      WRITE(6,39)
      WRITE(6,37)(NODD(I),BCD(I),I=1,NBCD)
   40 IF(NBCN.EQ.0) GO TO 45
      DO 41 I=1,NBCN
```

```
41 READ(5,42) (NODN(J,I),J=1,2),BCN(I)
   WRITE(6,43)
   WRITE(6,44)
   WRITE(6,42) (((NODN(J,I),J=1,2),BCN(I)),I=1,NBCN)
45 IF(NBCPHI.EQ.0) GO TO 70
   DO 71 I=1.NBCPHI
71 READ(5,42) (NODPHI(J,I),J=1,2),BCPHI(I)
   WRITE(6,43)
   WRITE(6,44)
   WRITE(6,42) (((NODPHI(J,I),J=1,2),BCPHI(I)),I=1,NBCP)I)
70 CONTINUE
 1 FORMAT (3I2)
 2 FORMAT (5I10)
 3 FORMAT(3F10.6)
 5 FORMAT (1H0,10X,-INPUT DATA-)
 6 FORMAT (1H0,10X,-TIME-,6X,-CONVECTIVE-,6X,-AXISYMMETPY-)
 7 FORMAT (11X.I2,11X,I2,14X,I2)
 8 FORMAT (1H0,10X,-NELEM-,10X,-NNODE-,10X,-N5CN-,10X,-NBCD-,10X,-NBCPHI-)
  2PHI-)
 9 FORMAT (5X,I10,5X,I10,4X,I10,4X,I10,5X,I10)
10 FORMAT (1H0,10X,-TINIT-,10X,-TFINAL-,10X,-TSTEP-)
11 FORMAT (8X,F10.6,8X,F10.6,8X,F10.6)
14 FORMAT (1H0,10X,-NODAL COORDINATES REFERENCED TO GLOFAL SYSTEM-)
15 FORMAT (1H0,10X,-NODE-,10X,-X COORDINATE-,10X,-Y COORDINATE-)
17 FORMAT (I10,2F10.3)
18 FORMAT (4X,I10,10X,F10.2,12X,F10.2)
19 FORMAT (1H1,10X,-DATA FOR TRIANGULAR ELEMENTS. CONSTANTS OF THE PR
  2OBLEM AND ELEMENT CONNECTIVITY MATRIX-)
21 FORMAT (4I10,2F10.3,2F10.2)
22 FORMAT (1H0,9X,-I-,2X,-NODE(1,I)-,2X,-NODE(2,I)-,2X,-NODE(3,I)-,
  27X,-XK -,7X,-YK -,5X,-ALPHA-,9X,-F-)
23 FORMAT (4I11,4F10.2)
25 FORMAT (2F10.3)
26 FORMAT (1H0,-CONVECTIVE CONSTANTS-)
27 FORMAT (1H0,9X,-I-,8X,-UX-,8X,-VY-)
28 FORMAT (I11,2F10.3)
50 FORMAT (I10,4F10.3)
```

```
   52 FORMAT(1H0,9X,-I-,6X,-XKK(I)-,6X,-YKK(I)-,6X,-UXK(I)-,6X,-VYK(I)-)
   53 FORMAT(I11,4F12.4)
   31 FORMAT (8F10.2)
   32 FORMAT (1H0,-TIME CONSTANTS-)
   33 FORMAT (1H0,9X,-I-,8X,-TK-)
   34 FORMAT(I11,F10.2)
   37 FORMAT (I10,F10.2)
   38 FORMAT (1H1,-DIRICHLET BOUNDARY CONDITIONS-)
   39 FORMAT (1H0,8X,-I-,5X,-BC(I)-)
   42 FORMAT (2I10,F10.2)
   43 FORMAT (1H0,-NEUMAN BOUNDARY CONDITIONS-)
   44 FORMAT (1H0,4X,-NODE 1-,5X,-NODE 2-,5X,-BC(I)-)
   49 FORMAT (10X,- INITIAL VALUES OF PHI-)
   61 FORMAT(I10)
   81 FORMAT(8D10.2)
C*****************************************************************************ARAL
      IER=0.0
      DO 100 J=1,30
      DO 100 I=1,200
      S(I,J)=0.0D0
  100 P(I,J)=0.0D0
      DO 101 I=1,200
  101 R(I)=0.0D0
      IUBW=0
      NNN = 0
      DO 200 II=1,NELEM
      XKX=XK(II)
      YKY=YK(II)
      ANG=ALPHA(II)
      TKT=TK(II)
      FF=F(II)
      UXX=UX(II)
      VYY=VY(II)
      DO 201 I=1,3
  201 NEWN(I)=NODE(I,II)
      CALL SET (NEWN,XE,YE,X)
      CALL ELEM (XE,YE,FF,XKX,YKY,ANG,TKT,ICON,ITIME,SE,RE(PE,IAXI,UXX,V
```

```
     2YY)
      DO 300 JJ=1,3
      DO 300 J =1,3
      MM=NODE(J,II)
      NN=NODE(JJ,II)
      IF(NN.LT.MM) GO TO 300
      KK=NN-MM+1
      IF(NNN.GT.MM) GO TO 310
      NNN = MM
  310 CONTINUE
      IF(IUBW.GT.KK) GO TO 301
      IUBW=KK
      IF(IUBW.LT.30) GO TO 301
      WRITE(6,302)
      GO TO 1000
  301 CONTINUE
      S(MM,KK)= S(MM,KK)+SE(J,JJ)
      IF(ITIME.EQ.0) GO TO 300
      P(MM,KK)= P(MM,KK)+PE(J,JJ)
  300 CONTINUE
      DO 303 J=1,3
      LL=NODE(J,II)
  303 R(LL)=R(LL)+RE(J)
  200 CONTINUE
      IF(ICON.EQ.0) GO TO 199
      CALL CINTEG(XKK,YKK,UXK,VYK,NNODE,X,DEP)
  199 CONTINUE
      WRITE(6,311)
      WRITE(6,312) NNN,IUBW
      CALL BOUND (NBCN,NBCD,BCN,BCD,NODN,NODD,S,R,X,IUBW,NNODE,IAXI,NBCPHI
     2HI,NODPHI,BCPHI)
      IF(ITIME.EQ.I) GO TO 999
      CALL DCB(NNN,IUBW,S,IER)
      IF (IER.EQ.1) GO TO 1000
      CALL SBAND(NNN,IUBW,S,P,PHI)
      WRITE (6,990)
      WRITE(6,998)
```

```
      DO 997 I=1,NNN
  997 WRITE(6,996)  I,PHI(I)
      GO TO 1000
  999 CONTINUE
      TCONT = TI + TST
      DO 500 I=1,NNN
      DO 500 J=1,IUBW
  500 S(I,J)  = ((P(I,J)*2.0/TST)+S(I,J))
      CALL DCB(NNN,IUBW,S,IER)
      IF (IER.EQ.1) GO TO 1000
  501 CONTINUE
      DO 600 IT=1,50
      CALL VEC (NNN,IUBW,P,PHIIN,V)
      DO 502 I=1,NNN
  502 TR(I) = R(I) +(2.0*V(I)/TST)
      CALL SBAND (NNN,IUBW,S,TR,RBN)
      DO 503 I=1,NNN
  503 PHIIN(I) = 2.0*RBN(I)- PHIIN(I)
      IF(IT.EQ.50   ) GO TO 601
  600 TCONT = TCONT + TST
  601 CONTINUE
      WRITE(6,510) TCONT
      IF(ICON.EQ.0) GO TO 602
      DO 603 I=1,NNN
      PHIF(I)=PHIIN(I)/DEP(I)
  603 WRITE(6,511) I,PHIF(I)
      GO TO 599
  602 CONTINUE
      DO 512 I=1,NNN
  512 WRITE(6,511) I,PHIIN(I)
  599 CONTINUE
      TCONT = TCONT+TST
      IF(TCONT-TF) 501,501,1000
 1000 CONTINUE
      JDATA=JDATA-1
      IF(JDATA.EQ.0) GO TO 1001
      GO TO 62
 1001 CONTINUE
```

```
302 FORMAT(10X,-ERROR,IUBW GT 30, CHANGE ELEMENT LOCATIONS-)
311 FORMAT (1H0,10X,-NNN-,10X,-UBW-)
312 FORMAT(2I13)
510 FORMAT(10X,-VALUES OF PHI AT TIME =-,F8.4)
511 FORMAT(4(I6,D20.10))
990 FORMAT (1H1,-SOLUTION, VALUES OF PHI-)
998 FORMAT (1H0,18X,-I-,15X,-PHI(I)-)
996 FORMAT (I20,D20.10)
    STOP
    END
    SUBROUTINE CINTEG(XKK,YKK,UXK,VYK,NNODE,X,DEP)
    DOUBLE PRECISION DEP(250)
    DIMENSION XKK(250),YKK(250),UXK(250),VYK(250),X(2,250)
    DO 1 I=1,NNODE
    IF(YKK(I)) 3,2,3
  3 DEP(I)=DEXP((-X(1,I)*UXK(I))/(2*XKK(I))+(-X(2,I)*VYK(I))/(2*YKK(I))
   2))
    GO TO 1
  2 DEP(I)=DEXP((-X(1,I)*UXK(I))/(2*XKK(I)))
  1 CONTINUE
    RETURN
    END
    SUBROUTINE VEC(NNN,IUBW,D,E,G)
    DOUBLE PRECISION D(250,30),E(250),G(250)
    DO 1 I=1,NNN
  1 G(I) =0.0D0
    DO 2 I=1,NNN
    DO 2 J=1,IUBW
    K =I+J-1
    IF(K.GT.NNN) GO TO 2
    G(I) =G(I) +D(I,J)*E(K)
  2 CONTINUE
    DO 3 K=NNN,2,-1
    KMM =K-1
    DO 4 M=2,IUBW
    G(K) =G(K)+D(KMM,M)*E(KMM)
    KMM=KMM-1
```

```
      IF(KMM.LT.1) GO TO 3
    4 CONTINUE
    3 CONTINUE
      RETURN
      END
      SUBROUTINE SET (N,XE,YE,X)
      DOUBLE PRECISION XE(3),YE(3)
      DOUBLE PRECISION ORX,ORY
      DIMENSION N(3),X(2,250)
      COMMON/BLOCK2/ ORX,ORY
      ORX=0.0
      ORY=0.0
      DO 100 I=1,3
      J=N(I)
      ORX=ORX + X(1,J)
  100 ORY=ORY + X(2,J)
      ORX=ORX/3.0
      ORY=ORY/3.0
      DO 200 I=1,3
      J=N(I)
      XE(I) = X(1,J)-ORX
  200 YE(I) = X(2,J)-ORY
      RETURN
      END
      SUBROUTINE ELEM (XE,YE.FF,XKX,YKY,ANG.TKT,ICON,ITIME(SE,RE.PE,
     2IAXI,UXX,VYY)
      DOUBLE PRECISION SE(3,3),RE(3),PE(3,3),XE(3),YE(3),A(3),B(3),C(3),X(3),
     2X(3),Y(3)
      DOUBLE PRECISION AREA,SUMX2,SUMXY,SUMY2
      COMMON/BLOCK2/ ORX,ORY
      DO 100 I=1,3
      X(I)=XE(I)*DCOS(0.017453*ANG)+YE(I)*DSIN(0.017453*ANG)
  100 Y(I)=-XE(I)*DSIN(0.017453*ANG)+YE(I)*DCOS(0.017453*ANG)
      AREA=DABS(((X(2)*Y(3)-Y(2)*X(3))-(X(1)*Y(3)-X(3)*Y(1))+(X(1)*Y(2)-
     2X(2)*Y(1)))*0.5)
      A(1) = (X(2)*Y(3) - X(3)*Y(2))/(2.*AREA)
      A(2) = (X(3)*Y(1) - X(1)*Y(3))/(2.*AREA)
```

```
      A(3) = (X(1)*Y(2) - X(2)*Y(1))/(2.*AREA)
      B(1) = (Y(2) - Y(3))/(2.*AREA)
      B(2) = (Y(3) - Y(1))/(2.*AREA)
      B(3) = (Y(1) - Y(2))/(2.*AREA)
      C(1) = (X(3) - X(2))/(2.*AREA)
      C(2) = (X(1) - X(3))/(2.*AREA)
      C(3) = (X(2) - X(1))/(2.*AREA)
  141 CONTINUE
      SUMX2=0.0
      SUMXY=0.0
      SUMY2=0.0
      DO 140 K=1,3
      SUMX2 = SUMX2 + X(K)*X(K)
      SUMXY = SUMXY + X(K)*Y(K)
  140 SUMY2 = SUMY2 + Y(K)*Y(K)
      DO 150 J=1,3
      DO 150 I=1,3
      SE(I,J)=0.0D0
      PE(I,J)=0.0D0
      IF(IAXI.EQ.0) GO TO 139
      SE(I,J)=SE(I,J)+(6.2831853*((XKX*B(I)*B(J))+(YKY*C(I)*C(J)))*
     2ORX*AREA)
      GO TO 138
  139 CONTINUE
      IF(ICON.EQ.0) GO TO 137
      IF(YKY) 135,136,135
  135 CONTINUE
      SE(I,J)=SE(I,J)+((XKX*B(I)*B(J))+(YKY*C(I)*C(J))+(UXX/2)*(A(I)*B(J)
     2)+A(J)*B(I))+(VYY/2)*(A(I)*C(J)+A(J)*C(I))+((UXX*UXX)/(4*XKX)+(VYY*VYY)/(4
     3*VYY)/(4*YKY))*((A(I)*A(J))+(B(I)*B(J)*SUMX2/12)+((B(I)*C(J)+B(J)*C(I))
     4C(I))*SUMXY/12)+(C(I)*C(J)*SUMY2/12)))*AREA
      GO TO 138
  136 CONTINUE
      SE(I,J)=SE(I,J)+((XKX*B(I)*B(J))                  +(UXX/2)*(A(I)*B(J)
     2)+A(J)*B(I))+(VYY/2)*(A(I)*C(J)+A(J)*C(I))+((UXX*UXX)/(4*XKX)
     3               )*((A(I)*A(J))+(B(I)*B(J)*SUMX2/12)+((B(I)*C(J)+B(J)*C(I))
     4C(I))*SUMXY/12)+(C(I)*C(J)*SUMY2/12)))*AREA
```

```
      GO TO 138
  137 CONTINUE
      SE(I,J) = SE(I,J)+((XKX*B(I)*B(J)) +(YKY*C(I)*C(J)))*AREA
  138 CONTINUE
      IF (ITIME.EQ.0) GO TO 150
      IF(IAXI.EQ.0) GO TO 149
      PE(I,J) = PE(I,J)+((TKT*AREA*UKX*6.2831853)/12.)*((12.*A(I)*A(J))+
     2(B(I)*B(J)*SUMX2)+((B(I)*C(J)+B(J)*C(I))*SUMXY)+(C(I)*C(J)*SUMY2))
      GO TO 150
  149 CONTINUE
      PE(I,J) = PE(I,J)+((TKT*AREA)/12.)*((12.*A(I)*A(J))+(B(I)*B(J)*SUM
     2X2)+((B(I)*C(J)+B(J)*C(I))*SUMXY)+(C(I)*C(J)*SUMY2))
  150 CONTINUE
      DO 165 K=1,3
  165 RE(K) = FF*AREA/3.0
      RETURN
      END
```

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