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APPLICATION OF HEREDITARY SYMMETRIES TO NONLINEAR EVOLUTION EQUATIONS

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1.1 STRONG SYMMETRIES

IN A topological vector space E we consider the evolution equation

$$u_t = K(u), \tag{1}$$

where the subscript t denotes the time derivative. We assume that $K: E \to E$ (possibly nonlinear) is differentiable. Throughout this paper differentiable stands for Hadamard-differentiable. We recall that a function $F: E \to \tilde{E}$ between two topological vector spaces is said to be Hadamard-differentiable [1] at $v \in E$ if there is a linear map L such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(F(v + \varepsilon x) - F(v) - \varepsilon L(x)) = 0$$

uniformly in x on each compact subset of E. $L(x)$ is then denoted by $F'(v)[x]$ and can be obtained from the directional derivative

$$F'(v)[x] = \frac{\partial}{\partial \varepsilon} F(v + \varepsilon x)\bigg|_{\varepsilon = 0}.$$

Let $u(t)$ be a solution of (1) then we are interested in *infinitesimal transformations*

$$u(t) \to u(t) + \varepsilon w(t), \varepsilon \text{ infinitesimal} \tag{2}$$

which leave (1) *form-invariant* [2]. This is the case if and only if $w(t)$ is a solution of

$$w_t = K'(u)[w] \text{ (perturbation equation).} \tag{3}$$

One solution of the perturbation equation is for example $w(t) = K(u(t))$. This follows immediately from the fact that the time derivative of $F(u(t))$ for some $F: E \to \tilde{E}$ is given by (chain rule)

$$F(u(t))_t = F'(u(t))[K(u(t))] \tag{4}$$

whenever $u(t)$ is a solution of (1).

As usual [3] a vector-valued function $S(\cdot)$ on E is said to be a *symmetry* of (1) if the infinitesimal transformation

$$u(t) \to u(t) + \varepsilon S(u(t))$$

leaves (1) form-invariant. $K(\cdot)$ is a symmetry.

* Dedicated to Prof. Dr. Heinz König on the occasion of his 50th birthday.

We endow the operators in E (i.e. continuous linear maps $E \to E$) with the pointwise convergence on E. Hence an operator-valued function $\Phi(\cdot)$ is differentiable at v if the function $v \to \Phi(v)y$ is for all $y \in E$ differentiable. An operator-valued function $\Phi(\cdot)$ on E is said to be a *strong symmetry* for (1) if $\Phi(u(t))w(t)$ is a solution of (3) whenever $u(t)$ and $w(t)$ are solutions of (1) and (3), respectively. Because of the linearity of (3) the strong symmetries are a vector space.

1.1. *Consequence.* Let $\Phi(\cdot)$ be a strong symmetry then the functions $v \to \Phi^n(v)K(v)$, $n = 0, 1, 2, \ldots,$ are symmetries.

The evolution equation (1) is called *regular* if for every time t_0 and for every initial condition $u(t_0) = u_0$, $u_0 \in E$, there is a unique solution $u(t) = u(t, u_0)$ for (1) which is differentiable with respect to u_0.

1.2. LEMMA. Let (1) be regular and let $u(t)$ be a solution of (1). Then for every time t_0 and every initial condition $w(t_0) = w_0$, $w_0 \in E$ there is a unique solution $w(t)$ of the perturbation equation.

Proof. The desired solution is given by

$$w(t) = \frac{\partial}{\partial \varepsilon} u(t, u(t_0) + \varepsilon w_0).$$

Now, let $\tilde{w}(t)$ be a second solution to the initial-value problem under consideration. Then we define $\tilde{u}(t, \tau, \varepsilon)$ to be the solution for the initial condition $\tilde{u}(\tau, \tau, \varepsilon) = u(\tau) + \varepsilon \tilde{w}(\tau)$.

Via differentiation we get

$$\frac{\partial}{\partial \varepsilon} (\tilde{u}(t, \tau, \varepsilon)_\tau)\Big|_{\varepsilon = 0} = 0. \tag{5}$$

Hence because of $w(t_0) = \tilde{w}(t_0)$ we may conclude

$$w(t) = \frac{\partial}{\partial \varepsilon} \tilde{u}(t, t_0, \varepsilon) = \frac{\partial}{\partial \varepsilon} \tilde{u}(t, t, \varepsilon) = \tilde{w}(t).$$

This shows that $w(t)$ is unique.

1.3 THEOREM. Let $\Phi(\cdot)$ be a differentiable operator-valued function on E and consider the conditions:

(i) $\Phi(\cdot)$ is a strong symmetry
(ii) $\Phi(u)_t = [K'(u), \Phi(u)]$ whenever u is a solution of (1)
(iii) $0 = \Phi'(v)K(v) - [K'(v), \Phi(v)]$ for all $v \in E$,

where $[A, B]$ denotes the commutator and where $\Phi'(v)K(v)$ stands for the operator $y \to \Phi'(v)[K(v)]y$. Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). And in case that (1) is regular all the conditions are equivalent.

Proof. (iii) $\Rightarrow$ (ii) is a consequence of (4). Now, let u and w be solutions of (1) and (3) respectively then for arbitrary $\Phi(\cdot)$ we get via differentiation:

$$\{(\Phi(u)w)_t - K'(u)\Phi(u)w\} = (\Phi(u)_t - [K'(u), \Phi(u)])w. \tag{6}$$

Hence (ii) $\Rightarrow$ (i).

Now, if (1) is regular then in virtue of Lemma 1.2 the right-hand side of (6) vanishes for all solutions of the perturbation equation if and only if (ii) holds. So, we have proved (i) $\Rightarrow$ (ii). And (ii) $\Rightarrow$ (iii) follows from (4) and the fact that we may prescribe any initial condition for (1).

At the end of this subsection we would like to emphasize other useful properties of strong symmetries.

As usual a map $v \to p(v) \in \mathbf{R}$ is called an *integral* [4] for (1) if $p(u(t))$ is time-independent whenever $u(t)$ is a solution of (1). These integrals are potentials for conserved covariant forms. To define this notion, let E^* be the dual of E and let us denote the application of some $y \in E^*$ on $v \in E$ by $\langle y, v \rangle$. A map $\varphi: E \to E^*$ is called a *conserved covariant* if

$$\langle \varphi(u(t)), w(t) \rangle \text{ is time-independent}$$

whenever $u(t)$ and $w(t)$ are evolving according to (1) and (3), respectively. Now, let Φ^T denote the transposed of Φ, then:

1.4. *Consequence.* If $\varphi(\cdot)$ is a conserved covariant and if $\Phi(\cdot)$ is a strong symmetry then $v \to \Phi^T(v)\varphi(v)$ is again a conserved covariant.

Now, assume $\Phi(\cdot)$ to be a strong symmetry such that condition (iii) in Theorem 1.3 holds. Then the spectral properties of $\Phi(u(t))$ are not changed while $u(t)$ evolves according to (1). For regular evolution equations this can be seen from:

$$((\Phi(u) - \lambda)w)_t = K'(u)(\Phi(u) - \lambda)w.$$

In other words the perturbation equation leaves the eigenvalues of $\Phi(u(t))$ invariant. Now, let S be a symmetry, then for reasons which become obvious later on we define a solution $u(t)$ of (1) to be an *S-Φ-soliton-solution* (or rather an *N-multisoliton* with respect to S and Φ) if for some t_0 we have

$$S(u(t_0)) = \sum_{n=1}^{N} w_n(t_0), \tag{7}$$

where the $w_n(t_0)$ are *eigenvectors* of $\Phi(u(t_0))$. If any solution $w(t)$ of (3) with $w(t_0) = 0$ has to be zero (especially if the evolution equation is regular) then obviously the decomposition (7) remains valid for all times t.

1.2 HEREDITARY SYMMETRIES

We have already indicated that strong symmetries might turn out to be rather useful, but the whole problem is how to find them. Usually that is quite easy for simple evolution equations (like linear ones), but rather complicated for nonlinear ones. In this chapter we recommend a method which will allow us to pass over from strong symmetries for trivial evolution equations to those for complicated ones.

First some words on notation. Consider a bilinear operator A, a linear operator B and some $v \in E$. Then we define the operator A^* and the products AB, Av in the following way:

$$A^*: (y_1, y_2) \to A(y_2, y_1)$$

$$AB: (y_1, y_2) \to A(By_1, y_2)$$

$$Av: y \to A(v, y).$$

In other words, the products are formed with respect to the first variable of the bilinear operator. A is called *symmetric* if $A^* = A$.

If $\Phi(\cdot)$ is some operator-valued function on E, then we shall henceforth identify $\Phi'(v)$ with the bilinear operator

$$(y_1, y_2) \to \Phi'(v)[y_1]y_2.$$

Now, a differentiable operator-valued function $\Phi(\cdot)$ on E is called a *hereditary symmetry* if for all $v \in E$ the operator $[\Phi'(v), \Phi(v)]$ is symmetric. We want to put emphasis on the fact that this definition does not depend on any special evolution equation.

We consider a function $K(\cdot)$ in E and we define $L(v) = \Phi(v)K(v)$. Then with some calculation we obtain the following formula

$$\Phi'(v)L(v) - [L'(v), \Phi(v)] = \Phi(v)\{\Phi'(v)K(v) - [K'(v), \Phi(v)]\}$$
$$+ [\Phi'(v), \Phi(v)]K(v) - [\Phi'(v), \Phi(v)]^*K(v) \tag{8}$$

From this we may immediately draw the following conclusion:

1.5. THEOREM. Let $\Phi(\cdot)$ be a hereditary symmetry with

$$\Phi'(v)K(v) - [K'(v), \Phi(v)] = 0 \text{ for all } v \in E.$$

Then

$$\Phi'(v)L(v) - [L'(v), \Phi(v)] = 0 \text{ for all } v \in E,$$

where $L(v) = \Phi(v)K(v)$.

This together with Theorem 1.3 leads to:

1.6. COROLLARY. If $\Phi(\cdot)$ is a hereditary symmetry and a strong symmetry for the regular evolution equation

$$u_t = K(u)$$

then $\Phi(\cdot)$ is a strong symmetry for the following evolution equations

$$u_t = (\Phi(u))^n K(u), n = 0, 1, 2, \ldots.$$

Another consequence of (8) deals with the following

1.7. *Situation.* Consider a subalgebra $\mathscr{A}$ of all functions from E to $L(E, E)$ containing the functions $\Phi(\cdot)$, $K'(\cdot)$ and $\Phi'(\cdot)K(\cdot)$ and such that $\Phi(\cdot)$ is not a zero-divisor of $\mathscr{A}$ (which is certainly the case if $\mathscr{A}$ has no zero-divisors).

1.8. THEOREM. Let $\Phi(\cdot)$ be a hereditary symmetry, assume that $\mathscr{A}$ is as in 1.7 and assume that for $L(v) = \Phi(v)K(v)$ we have $\Phi'(v)L(v) - [L'(v), \Phi(v)] = 0 \; \forall \, v \in E$. Then

$$\Phi'(v)K(v) - [K'(v), \Phi(v)] = 0 \text{ for all } v \in E.$$

1.3 EXAMPLES

We consider for E the complex linear space $C^\infty(\mathbf{R})$ or $\mathscr{S}(\mathbf{R})$, where $C^\infty(\mathbf{R})$ are the infinitely often differentiable functions on $\mathbf{R}$ and were $\mathscr{S}(\mathbf{R}) \subset C^\infty(\mathbf{R})$ are the fast decreasing functions.

We assume that E is endowed with a Hausdorff-topology such that all the following operators are continuous. By D we denote the differential operator and D^{-1} stands for

$$D^{-1}(w)(x) \underset{\text{def}}{=} \int_a^x w(\xi)\,d\xi \quad \forall w \in E$$

where $a \in \mathbf{R}$ if $E = C^\infty(\mathbf{R})$ and $a = -\infty$ if $E = \mathscr{S}(\mathbf{R})$. If $\varphi \in C^\infty(\mathbf{R})$ and $v \in E$ then $\varphi(v)$ denotes the following multiplication operator:

$$(\varphi(v)w)(x) = \varphi(v(x))w(x) \quad \forall w \in E.$$

We have found that the following operators $\Phi_n(u)$ are hereditary symmetries:

(H0) $\Phi_0(u) = \varphi(u)$

(H1) $\Phi_1(u) = u_x D^{-1}\varphi(u)$

(H2) $\Phi_2(u) = \Psi(u) + u_x D^{-1}\varphi(u)$

(H3) $\Phi_3(u) = D + \alpha(u_x D^{-1} + u)$

(H4) $\Phi_4(u) = D^2 + (\gamma + 2\beta u + \alpha u^2) + u_x D^{-1}(\beta + \alpha u)$,

where φ and Ψ are arbitrary elements of $C^\infty(\mathbf{R})$ and where α, β, γ are scalars. The proof can be found in the appendix.

Further hereditary symmetries we can easily obtain by restricting the linear structure in E to multiplication with real scalars.

In this case

(H5) $\Phi_5(u) = iuD^{-1}\mathrm{Re}(\bar{u}\cdot)$

(H6) $\Phi_6(u) = iD + i\alpha u D^{-1}\mathrm{Re}(\bar{u}\cdot), \alpha \in \mathbf{R}$

are hereditary symmetries. Here, of course, the bar means complex conjugation and $\mathrm{Re}(\bar{u}\cdot)$ stands for the real-linear operator

$$w \to \tfrac{1}{2}(\bar{u}w + u\bar{w}) = \text{Real part of}\,(\bar{u}w).$$

The restriction to real-linear structure is essential, otherwise these operators are neither differentiable nor linear.

2. APPLICATIONS

The heritage of the equation $u_t = u_x$.

2.1. *The descendants*

In this chapter fast decreasing solutions of nonlinear partial differential equations are treated. We show that the results of our symmetry considerations lead in a very natural way to recursion formulas for symmetries and conservation laws. Furthermore, our theory hides a new complete description of the soliton solutions for the equations under consideration.

Let $\mathscr{S}_-(\mathbf{R})$ denote the space of those C^∞-functions vanishing rapidly at $-\infty$. The space $\mathscr{S}_-(\mathbf{R})$ is embedded in the dual of $\mathscr{S}(\mathbf{R})$ via the following bilinear form:

$$\langle y, v\rangle = \begin{cases} \int_{\mathbf{R}} y(x)v(x)\,\mathrm{d}x \text{ in case that } \mathscr{S}(\mathbf{R}) \text{ is regarded as a complex-linear space} \\ \int_{\mathbf{R}} (y_1(x)v_1(x) + y_2(x)v_2(x))\,\mathrm{d}x \text{ in case that } \mathscr{S}(\mathbf{R}) \text{ is regarded as a real linear space,} \end{cases} \tag{9}$$

where v_1, v_2, y_1, y_2 are real functions such that $v_1 + iv_2 = v \in \mathscr{S}(\mathbf{R})$ and $y_1 + iy_2 = y \in \mathscr{S}_-(\mathbf{R})$. The space $\mathscr{S}(\mathbf{R})$ is endowed with a topology such that the functionals given by $\mathscr{S}_-(\mathbf{R})$ and all the following operators are continuous.

The obvious extensions of the operators $\Phi(\cdot)$ to $\mathscr{S}_-(\mathbf{R})$ are also denoted by $\Phi(\cdot)$, and $\Phi(u)^\sharp$ stands for its adjoint with respect to (9); i.e.

$$\langle \Phi^\sharp y, v\rangle = \langle y, \Phi v\rangle \quad \forall y \in \mathscr{S}_-(\mathbf{R}), v \in \mathscr{S}(\mathbf{R}).$$

2.1. *Remark.* For all the operators $\Phi(\cdot)$ given by (H1) to (H6) we have

$$[D, \Phi(v)] = \Phi'(v)[v_x] \quad \forall v \in \mathscr{S}_-(\mathbf{R}). \tag{10}$$

Hence, they are all strong symmetries for the trivial evolution equation: $u_t = u_x$.

Actually (10) remains valid for an algebra of operators. It does not hold if $\mathscr{S}(\mathbf{R})$ is replaced, say by $C^\infty(\mathbf{R})$, because in this case D and D^{-1} do not commute. But with some skillful handling of the matter one can steer around this difficulty. So, very many of the following results do go over to the $C^\infty(\mathbf{R})$-case. This will be shown in a subsequent paper.

Now, choose as $\Phi(\cdot)$ any of the operators given by (H1)–(H6) and consider the evolution equations

$$u_t = K_n(u), \quad n = 0, 1, 2, \ldots \tag{11}$$

where

$$K_n(u) = \Phi(u)^n u_x. \tag{12}$$

The equation $u_t = K_{n+1}(u)$ will be called the n-th generalization of $u_t = K_1(u)$. Because of translation invariance $u \to u_x$ is always a symmetry for these equations.

2.2. *Consequence.* By Theorem 1.5 and Remark 2.1 $\Phi(u)$ is a strong symmetry for all the equations given by (11). Hence all the $K_m(u)$ are symmetries for all those equations.

For the construction of conserved covariants we need a nontrivial one to start with. This is not always possible (example 1). But quite often we shall have *energy conservation*, which means that

$$\langle u, u\rangle \text{ is time-independent} \tag{13}$$

for solutions $u(t)$ of the evolution equation under consideration. (Actually sometimes the physical meaning of this conserved quantity is the conservation of the number of particles).

2.3. *Consequence.* Assume that we have energy conservation for the evolution equation $u_t = K_n(u)$. Then u (or rather the linear functional given via (9) by u) is a conserved covariant. Hence by 1.4 all the

$$G_m(u) = \Phi^{*m}(u)u, \quad m = 0, 1, 2, \ldots \tag{14}$$

are conserved covariants.

It is quite easy to find the corresponding integrals (potentials) $H_m(u)$ if the $G_m(u)$ are fullfilling the following *integrability condition*

$$\langle G_m'(u)[v], w\rangle = \langle G_m'(u)[w], v\rangle \quad \forall v, w \in \mathscr{S}(\mathbf{R}). \tag{15}$$

Then putting $H_m'(u) = G_m(u)$ we obtain

$$H_m(u) = \int_0^1 \langle G_m(\lambda u), u\rangle \,\mathrm{d}\lambda. \tag{16}$$

Now, let us turn our attention to *solitons* of the form:

$$u_x = \sum_{k=1}^{N} w_k, \text{ where } \Phi(u)w_k = \lambda_k w_k \tag{17}$$

We recall that a solution u of $u_t = K_n(u)$ is for all times t of this form if it has this decomposition for one time t_0. We discuss first the case $N = 1$. Then we have

$$u_t = \Phi(u)^n u_x = \lambda_1^n w_1 = \lambda_1^n u_x. \tag{18}$$

Hence $u(x, t)$ has to be a *travelling-wave* solution

$$u(x, t) = \sigma(x + \lambda_1^n t)$$

with velocity λ_1^n.

Now, we consider a special solution of $u_t = K_n(u)$ having the property that it decomposes asymptotically ($|t| \to \infty$) into N travelling-waves with different velocities such that all the energy is carried by the asymptotic waves [5]. Then in virtue of (18) and because of the fact that $\Phi(u)$ is a local operator the eigenvectors of $\Phi(u)$ are asymptotically the eigenvectors of $\Phi(\sigma)$ (σ a travelling wave) since the overlap of the waves vanishes rapidly.

2.4. *Consequence.* The solutions of $u_t = K_n(u)$ ($n \geqslant 1$) which decompose asymptotically into N travelling waves such that all the energy is carried by those asymptotic waves are described by the following system of ordinary differential equations:

$$u_x = \sum_{k=1}^{N} w_k, \Phi(u)w_k = \lambda_k w_k,$$

where the λ_k, $k = 1, \ldots, N$ are the n-th roots of the corresponding velocities.

Examples.

(1) *Burgers equation.* Application of

$$\Phi(u) = D + (u_x D^{-1} + u) \text{ (special case of (H3))} \tag{19}$$

to u_x leads to Burgers equation [6]:

$$u_t = \Phi(u)u_x = u_{xx} + 2uu_x. \tag{20}$$

Neither solitons nor polynomial conserved covariants can be found. But all the $\Phi(u)^n u_x$ are symmetries for this equation.

(2) *Korteweg–de Vries equation.* Consider

$$\Phi(u) = D^2 + 4u + 2u_x D^{-1} \text{ (special case of (H4))}. \tag{21}$$

Then the first equation in the series $u_t = K_n(u) = \Phi(u)^n u_x$ ($n \geqslant 1$) is the well-investigated KdV-equation:

$$u_t = u_{xxx} + 6uu_x. \tag{22}$$

From

$$D\Phi(u)^* = \Phi(u)D \tag{23}$$

we obtain for the symmetries $K_m(u)$:

$$\langle D^{-1}K_m(u), K_n(u)\rangle = \langle u, \Phi^{m+n}(u)Du\rangle = -\langle u, D\Phi^*(u)^{m+n}u\rangle = -\langle u, \Phi^{n+m}(u)Du\rangle.$$

Hence

$$\langle D^{-1}K_m(u), K_n(u)\rangle = 0 \text{ for all } n,m, \tag{24}$$

which implies that $\langle u, u\rangle$ is time-independent for any solution of the KdV or one of its generalizations. So, by 2.3 all the

$$G_m(u) = \Phi^* (u)^m u = D^{-1}\Phi(u)^m u_x = D^{-1}K_m(u) \tag{25}$$

are conserved covariants for all KdV-equations. The first three of this series are:

$$G_0(u) = u$$

$$G_1(u) = u_{xx} + 3u^2$$

$$G_2(u) = u_{xxxx} + 5u_x^2 + 10uu_{xx} + 10u^3.$$

All the $G_m(u)$ do have potentials. These potentials are the well known integrals of the KdV. All these results are known (for example [7], [8]). the recursion formula (25) was discovered by A. Lenard.

According to consequence 2.4 $u(t)$ is an N-soliton solution if there is some t_0 such that

$$u_x(t_0) = \sum_{k=1}^{N} w_k; \Phi(u(t_0))w_k = \lambda_k w_k; k = 1, \ldots, N. \tag{26}$$

Using (23), this is shown to be equivalent to:

$$u(t_0) = \sum_{k=1}^{N} y_k, \Phi^*(u(t_0))y_k = \lambda_k y_k, \quad k = 1, \ldots, N. \tag{27}$$

An easy calculation leads to the usual soliton description [5], [7].

(3) *The modified KdV-equation.* Let us consider another special case of (H4):

$$\Phi(u) = D^2 + 4u_x D^{-1} u + 4u^2. \tag{28}$$

Then the modified KdV-equation [9]

$$u_t = u_{xxx} + 6u_x u^2 \tag{29}$$

is the first one of the series $u_t = K_n(u) (n \geqslant 1)$. Again the $K_m(u)$ are symmetries for all these equations. And because of

$$D\Phi^{\#}(u) = \Phi(u)D, \tag{30}$$

we can go through the same analysis as in the last example. The

$$G_m(u) = D^{-1} K_m(u)$$

are conserved covariants. The first three are

$$G_0(u) = u$$
$$G_1(u) = u_{xx} + 2u^3$$
$$G_2(u) = u_{xxxx} + 10u_{xx}u^2 + 10u_x^2 u + 6u^5.$$

The function $u(t)$ is an N-soliton solution if there is some t_0 such that (26) or (27) does hold with respect to $\Phi(u)$ given by (28).

2.6. *Remark.* Any linear combination of (28) and (21) is again a hereditary symmetry. So, for any evolution equation having as right hand side a linear combination of the right hand sides of the KdV and the modified KdV the same "theory" goes through.

(4) *The Zakharov–Shabat equation.* Now, let us restrict the linear structure in $\mathscr{S}(\mathbf{R})$ to the reals. We consider the hereditary symmetry (H6):

$$\Phi(u) = -iD + 4iuD^{-1}\mathrm{Re}(\bar{u}\cdot). \tag{31}$$

Then the Zakharov–Shabat equation [10]

$$u_t = -iu_{xx} + 2iu^2\bar{u} \tag{32}$$

is the first one in the series $u_t = K_{n+1}(u)$ $(n \geqslant 1)$. All the $K_m(u)$, $m = 0, 1, \ldots$, are symmetries for (32) or any of its generalizations. It is easily shown that energy conservation:

$$\langle u, u \rangle = \text{time independent}$$

holds for the Zakharov–Shabat equation. Hence, all the

$$G_m(u) = \Phi^{\#}(u)^m u \tag{33}$$

are conserved covariants for this equation. Because of

$$\Phi^{\#}(u) = \Phi(iu) \tag{34}$$

and

$$\Phi(iu)u = -iu_x \tag{35}$$

we have the following relation between the conserved covariants and the symmetries:

$$G_{m+1}(u) = -K_m(iu). \tag{36}$$

The first four in this series of conserved covariants are

$$\begin{aligned} G_0(u) &= u \\ G_1(u) &= -iu_x \\ G_2(u) &= -u_{xx} + 2\bar{u}u^2 \\ G_3(u) &= iu_{xxx} - i2(u^2\bar{u})_x - 2iu(\bar{u}u_x - u\bar{u}_x). \end{aligned}$$

All these covariants do have potentials. These potentials are the well-known integrals for the Zakharov–Shabat equation [10]. Again the N-soliton solutions of the Zakharov–Shabat equation (and its generalizations) are characterized by

$$u_x = \sum_{k=1}^{N} w_k, \; \Phi(u)w_k = \lambda_k w_k, \quad k = 1, \ldots, N.$$

2.2. *The ancestors*

Let E be equal to $\mathscr{S}(\mathbf{R})$ or $\mathscr{S}_-(\mathbf{R})$. By $\mathscr{A}(D, D^{-1})$ we denote the algebra of operator-valued functions on E generated by functions of the following type:

$$v \to D, v \to D^{-1}, v \to f(v^{(n)}), v \to g(v^{(-n)}),$$

where the f and g occuring in the last two multiplication operators are allowed to be arbitrary entire analytic functions, and where $v^{(n)}$ and $v^{(-n)}$ are denoting the n-th derivative and the n-th integral of v.

Then we have the (purely algebraic) fact:

2.7. *Remark.* $\mathscr{A}(D, D^{-1})$ has no zero divisors.
We briefly indicate a proof of this remark. Let us fix a bounded nonempty open set $\Omega \subset \mathbf{R}$. Then we consider the algebra of operators on $\mathscr{D}(\Omega)$ (C^∞-functions with support in Ω) generated by D, D^{-1} and by multiplication with entire analytic functions $f(x)$. This algebra will be denoted by $A(D, D^{-1})$. Then we put $d(D) = 1$, $d(D^{-1}) = -1$, $d(x) = -1$, $d(0) = -\infty$, and we extend this to a *degree* $d: A(D, D^{-1}) \to \mathbf{Z} \cup \{-\infty\}$, that is a multiplicative function with (i)$d(h) = -\infty \Leftrightarrow h =$ 0, and $d(h_1 + h_2) \leqslant \max(d(h_1), d(h_2))$. The multiplicativity of the degree implies that $A(D, D^{-1})$ has no zero divisors. Now take $T_1(\cdot) \neq 0$ and $T_2(\cdot) \neq 0$ out of $\mathscr{A}(D, D^{-1})$. For any m there is some $v_m \in E$ such that its restriction to Ω is equal to the polynomial x^m. Depending on $T_1(\cdot)$ and $T_2(\cdot)$ there is a sufficiently large m such that $T_1(x^m)$ and $T_2(x^m)$ (restriction of $T_1(v_m)$ and $T_2(v_m)$ to $\mathscr{D}(\Omega)$) are both nonzero elements of $A(D, D^{-1})$. Hence, $T_1(x^m) \cdot T_2(x^m) \neq 0$ and this implies that the product of $T_1(\cdot)$ and $T_2(\cdot)$ is not equal to zero.

Now, an immediate application of Theorem 1.8 leads to:

2.8. *Consequence.* Let $\Phi(v) \in \mathscr{A}(D, D^{-1})$ be a hereditary symmetry and let $K(\cdot)$ be a vector-valued map on E such that $K'(\cdot)$ and $\Phi'(\cdot)K(\cdot)$ are elements of $\mathscr{A}(D, D^{-1})$. Assume furthermore that for $L(v) = \Phi(v)K(v)$ we have

$$\Phi'(v)L(v) - [L'(v), \Phi(v)] = 0 \quad \forall v \in E. \tag{37}$$

Then $\Phi(\cdot)$ is a strong symmetry for the evolution equation

$$u_t = K(u). \tag{38}$$

Examples.

(1) *The sine-Gordon equation.*
Let the function space E be equal to $\mathscr{S}_-(\mathbf{R})$. We look for solutions $K(u)$ of

$$\Phi(u)K(u) = u_x, \tag{39}$$

where $\Phi(u)$ is the hereditary symmetry given by (28). $K(u)$ must be of the form $\dot\Psi(\int_{-\infty}^x u(\xi)\,\mathrm{d}\xi)$, with Ψ being a function of one variable and $\dot\Psi(z)$ standing for $(\mathrm{d}/\mathrm{d}z)\Psi(z)$. Insertion of this into (39) leads to

$$\Phi(u)K(u) = u^2(\dddot\Psi + 4\dot\Psi) + u_x(4\Psi + \ddot\Psi) - 4u_x\Psi(0).$$

Hence,

$$\Psi(\xi) = -\tfrac{1}{4}\cos(2\xi) + \beta\sin(2\xi).$$

But β must be zero since we require $K(u) \in \mathscr{S}_-(\mathbf{R})$.

Thus the solution is:

$$K(u) = \tfrac{1}{2}\sin\left(\int_{-\infty}^x u(\xi)\,\mathrm{d}\xi\right). \tag{40}$$

Now, consequence 2.8 tells us that $\Phi(u)$ is a strong symmetry for the equation

$$u_t(x,t) = \tfrac{1}{2}\sin\left(2\int_{-\infty}^x u(\xi,t)\,\mathrm{d}\xi\right) \tag{41}$$

This is the well-known sine-Gordon equation [11], [12].

The analysis for symmetries, conserved covariants and soliton-solutions proceeds exactly as in the case of the modified KdV, (since this is by (39) the second generalization of the sine-Gordon equation). An infinite sequence of symmetries is given by

$$S_m(u) = \Phi(u)^m u_x, \quad m = 0, 1, \ldots. \tag{42}$$

And in case that the solution $u(t)$ is in $\mathscr{S}(\mathbf{R})$ the

$$G_m(u) = D^{-1}S_m(u) \tag{43}$$

are conserved covariants. $u(t)$ is a soliton solution if there is some t_0 such that

$$u_x(t_0) = \sum_{k=1}^N w_k, \; \Phi(u(t_0))w_k = \lambda_k w_k, \text{ with } \lambda_k \neq 0, \quad k = 1, \ldots, N. \tag{44}$$

But, because of $u_t = \Phi(u)^{-1}u_x$ the relation between the λ_n and the speeds of the asymptotic travelling-waves is for this equation given by:

$$c_k = \frac{1}{\lambda_k}. \tag{45}$$

2. *The sinh-Gordon equation.*
We can perform the same calculation with the operator

$$\Phi(u) = D^2 - 4u_xD^{-1}u - 4u_2 \tag{46}$$

instead of (28). Then the solution of $\Phi(u)K(u) = u_x$ is

$$K(u) = \tfrac{1}{2}\sinh\left(2\int_{-\infty}^{x} u(\xi)\,\mathrm{d}\xi\right), \tag{47}$$

which leads to the evolution equation

$$u_t(x, t) = \tfrac{1}{2}\sinh\left(2\int_{-\infty}^{x} u(\xi, t)\,\mathrm{d}\xi\right). \tag{48}$$

This equation has no soliton solutions, since it is an ancestor (second degree) of the soliton-free type of the modified KdV [11]:

$$u_t = u_{xxx} - 6u_xu. \tag{49}$$

The symmetries and conserved covariants are given by (42) and (43), where $\Phi(u)$ must be replaced by (46).

APPENDIX

Here we shall give the proof that (H0)–(H6) define hereditary symmetries. For two bilinear operators we shall write $A \simeq B$ iff $A - B$ is a symmetric bilinear operator. Instead of $A \simeq B$ we sometimes write $A(v, w) \simeq B(v, w)$, where v, w are understood as arbitrary elements of E. In the following we shall depend very much on integration by parts:

$$vD^{-1}w - D^{-1}v_xD^{-1}w = D^{-1}vw.$$

By differentiation we obtain $\Phi_0'(u)(v, w) = \varphi_u(u)vw$, where $\varphi_u(u)$ denotes the partial derivative with respect to u. From this we immediately see that $[\Phi_0'(u), \Phi_0(u)] = 0$.
Further differentiation yields $\Phi_1'(u)(v, w) = v_xD^{-1}\varphi(u)w + u_xD^{-1}\varphi_u(u)vw$.
And from this we get

$$\begin{aligned}[\Phi_1(u)', \Phi_1(u)](v, w) = {} & \left(u_x(D^{-1}\varphi(u)v)\right)_x D^{-1}\varphi(u)w + u_xD^{-1}\varphi_u(u)wu_xD^{-1}\varphi(u)v \\ & - u_xD^{-1}\varphi(u)v_xD^{-1}\varphi(u)w - \underline{u_xD^{-1}\varphi(u)u_xD^{-1}\varphi_u(u)vw}.\end{aligned} \tag{A1}$$

Via differentiation and integration by parts the first term becomes:

$$\begin{aligned} & \underline{u_{xx}(D^{-1}\varphi(u)v)(D^{-1}\varphi(u)w)} + u_x\varphi(u)vD^{-1}\varphi(u)w \\ & \simeq \underline{u_xD^{-1}\varphi(u)v\varphi(u)w} + u_xD^{-1}(\varphi(u)v)_xD^{-1}\varphi(u)w \\ & \simeq + u_xD^{-1}\varphi_u(u)u_xvD^{-1}\varphi(u)w + u_xD^{-1}\varphi(u)v_xD^{-1}\varphi(u)w.\end{aligned}$$

The second term cancels out with the third term of (A1), and the first term together with the second one of (A1) is symmetric. In this calculation we have underlined those expressions which are clearly symmetric. Hence

$$[\Phi_1'(u), \Phi_1(u)] \simeq 0. \tag{A2}$$

Now, we calculate the following commutators:

$$\begin{aligned}[\Phi_1'(u), \Psi(u)](v, w) &= (\Psi(u)v)_x D^{-1}\varphi(u)w + \underline{u_x D^{-1}\varphi_u(u)\Psi(u)vw} \\ &\quad - \Psi(u)v_x D^{-1}\varphi(u)w - \underline{\Psi(u)u_x D^{-1}\varphi_u(u)vw} \\ &\simeq \Psi_u(u)vu_x D^{-1}\varphi(u)w \end{aligned} \tag{A3}$$

$$\begin{aligned}[\Psi'(u), \Phi_1(u)](v, w) &= \Psi_u(u)wu_x D^{-1}\varphi(u)v - \underline{\Phi_1(u)\Psi_u(u)vw} \\ &\simeq \Psi_u(u)wu_x D^{-1}\varphi(u)v. \end{aligned} \tag{A4}$$

Hence,

$$[\Psi'(u), \Phi_1(u)] + [\Phi_1'(u), \Psi(u)] \simeq 0 \tag{A5}$$

This together with (A2) yields:

$$[\Phi_2'(u), \Phi_2(u)] \simeq 0. \tag{A6}$$

One easily obtains

$$[\Psi'(u), D](v, w) \simeq \Psi_u(u)v_x w \tag{A7}$$

$$[\Phi_1'(u), D](v, w) \simeq u_x D^{-1}\varphi_u(u)v_x w - v_x\varphi(u)w. \tag{A8}$$

Specialization and addition leads to

$$[(u_x D^{-1} + u)', D] \simeq 0, \tag{A9}$$

Hence $[\Phi_3'(u), \Phi_3(u)] \simeq 0$.

For the proof that (H4) defines a hereditary symmetry we calculate:

$$[\Psi'(u), D^2](v, w) \simeq \Psi_u(u)v_{xx}w \tag{A10}$$

and

$$\begin{aligned}[\Phi_1'(u), D^2](v, w) &\simeq v_{xxx}D^{-1}\varphi(u)w + u_x D^{-1}\varphi_u(u)v_{xx}w - D^2 v_x D^{-1}\varphi(u)w \\ &\simeq u_x D^{-1}\varphi_u(u)v_{xx}w - 2v_{xx}\varphi(u)w - v_x(\varphi(u)w)_x \\ &\simeq u_x D^{-1}\varphi_u(u)v_{xx}w - 2v_{xx}\varphi(u)w - v_x\varphi_u(u)u_x w \\ &\simeq - u_x D^{-1}v_x(\varphi_u(u))_x w - 2v_{xx}\varphi(u)w. \end{aligned} \tag{A11}$$

Now, put $\varphi(u) = \alpha u + \beta$ and $\Psi(u) = \gamma + 2\beta u + \alpha u^2$ then $(\varphi_u(u))_x = 0$ and $2\varphi(u) = \Psi_u(u)$. Addition of (A11) and (A10) gives for this special case that $[\Psi'(u) + \Phi_1'(u), D^2] \simeq 0$. This together with (H2) proves that (H4) defines a hereditary symmetry.

Now, let us restrict the linear structure to the reals. The derivative of $\Phi_5(u)$ is then

$$\Phi_5'(u)(v, w) = ivD^{-1}\mathrm{Re}(\bar{u}w) + iuD^{-1}\mathrm{Re}(\bar{v}w).$$

The crucial commutators are calculated to be:

$$
\begin{aligned}
[\Phi_5'(u), \Phi_5(u)](v, w) = & - u(D^{-1}\mathrm{Re}(\bar{u}v))(D^{-1}\mathrm{Re}(\bar{u}w)) && \text{symmetric} \\
& \left.\begin{aligned} & - iuD^{-1}\mathrm{Re}(iw\bar{u}D^{-1}\mathrm{Re}(\bar{u}v)) \\ & - iuD^{-1}\mathrm{Re}(iv\bar{u}D^{-1}\mathrm{Re}(\bar{u}w)) \end{aligned}\right\} && \text{symmetric} \\
& - iuD^{-1}\mathrm{Re}(i\bar{u}uD^{-1}(\bar{v}w)) && \text{symmetric} \\
& \simeq 0.
\end{aligned} \tag{A12}
$$

$$
\begin{aligned}
[\Phi_5'(u), iD] = & - v_x D^{-1}\mathrm{Re}(\bar{u}w) - iuD^{-1}\mathrm{Re}(i\bar{v}_x w) \\
& + DvD^{-1}\mathrm{Re}(\bar{u}w) + \underline{DuD^{-1}\mathrm{Re}(\bar{v}w)} \\
& \simeq \tfrac{1}{2}\big(v(\bar{u}w + u\bar{w}) + uD^{-1}(\bar{v}_x w - v_x \bar{w})\big) \\
& \simeq \tfrac{1}{2}\big(vu\bar{w} + uD^{-1}\bar{v}_x w - uv\bar{w} + uD^{-1}v\bar{w}_x\big) \\
& \simeq 0.
\end{aligned} \tag{A13}
$$

Combination of these results shows that Φ_5 and Φ_6 are hereditary symmetries.

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