Application of Lagrange Interpolation for Nonlinear Integro Differential Equations

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Abstract

A numerical method for solving nonlinear Fredholm integro differential equations of the second kind is presented. The method is based upon Lagrange functions approximation. Quadrature rule and collocation points are utilized to reduce the main problem to nonlinear system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Nonlinear integro differential equation; Lagrange interpolation; Quadrature rule; Collocation points

1 Introduction

Modeling and analysis of physical phenomena in applied sciences often generates nonlinear mathematical problems. Nonlinearity may be an inner feature of the model, i.e., evolution equations with nonlinear terms, or of the problem, i.e., nonlinear boundary conditions. The interplay between applied sciences and mathematics then leads to the development of initial and/or boundary value problems for nonlinear partial differential or integral or integro differential equations modeling real physical systems. The theory and application of integral and integro-differential equations is an important subject within applied mathematics. Integral and integro-differential equations are used as mathematical models for many and varied physical situations, and also occur as reformulations of other mathematical problems. Since many physical problems are modeled by integral and integro-differential equations, the numerical solutions of such equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations such as Haar

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wavelets, homotopy perturbation method, lagrange functions, Taylor polynomials, Chebyshev polynomials, sine-cosine wavelets, Tau method, Adomian decomposition method, hybrid Legendre and Block Pulse functions and so on.(for further details see [1-10]) In this study, we use Lagrange interpolation for solving nonlinear Fredholm integro differential equation of the form

$$
\begin{cases}
u'(x) = \int_0^1 k(x,t)\psi(t,u(t))dt + g(x), & 0 \le x \le 1, \\
u(0) = u_0,
\end{cases}
\tag{1}
$$

where, $k \in L^2[0,1)^2$, $g \in L^2[0,1)$ are known functions and $u(x)$ is the unknown function. we define

$$
F(t) = \psi(t,u(t)),
\tag{2}
$$

since

$$
u(t) = \int_0^t u'(x)dx + u_0,
\tag{3}
$$

by using (1)-(3) we obtain

$$
\begin{aligned}
F(t) &= \psi\left(t, \int_0^t u'(x)dx + u_0\right) \\
&= \psi\left(t, \int_0^t \left(\int_0^1 k(x,t)F(t)dt + g(x)\right)dx + u_0\right) \\
&= \psi\left(t, \int_0^t \int_0^1 k(x,t)F(t)dtdx + \int_0^t g(x)dx + u_0\right).
\end{aligned}
\tag{4}
$$

Now we approximate $F(t)$ as

$$
F(t) \approx \sum_{i=0}^n f_i L_i(t),
\tag{5}
$$

where,

$$
L_i(t) = \prod_{j=0, j \ne i}^n \left(\frac{t - t_j}{t_i - t_j}\right)
$$

and $f_i = F(t_i)$. Also, $L_i(t_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta. We approximate the integral of f on $[a,b]$ as:

$$
\int_a^b f(t)dt \approx \sum_{r=0}^k w_r f(t_r),
$$

In this work we take $k = n$. Hence,

$$\int_0^1 k(x,t)F(t)dt = \sum_{r=0}^{n} w_r k(x,t_r)F(t_r)$$

$$= \sum_{r=0}^{n} w_r k(x,t_r)f_r. \tag{6}$$

By substituting (6) into (4) we get

$$F(t) = \psi\left(t, \int_0^t \sum_{r=0}^{n} w_r k(x,t_r)f_r dx + \int_0^t g(x)dx + u_0\right)$$

$$= \psi\left(t, \sum_{r=0}^{n} w_r f_r \int_0^t k(x,t_r)dx + \int_0^t g(x)dx + u_0\right). \tag{7}$$

Collocating (7) at the points $t_j, j = 0, 1, ..., n$ gives

$$f_j = \psi\left(t_j, \sum_{r=0}^{n} w_r f_r \int_0^{t_j} k(x,t_r)dx + \int_0^{t_j} g(x)dx + u_0\right), \tag{8}$$

now we take $z = \frac{x}{t_j}$, therefore (8) yields

$$f_j = \psi\left(t_j, t_j \sum_{r=0}^{n} w_r f_r \int_0^1 k(t_j z, t_r)dz + t_j \int_0^1 g(t_j z)dz + u_0\right), \tag{9}$$

system (9) gives $n + 1$ nonlinear equations which can be solved for $f_j, j = 0, 1, ..., n$ by Newton iterative method. We note that the integrals in (9) may be evaluated numerically. So, by using quadrature rule $u'(x)$ can be approximated as

$$u'(x) = \int_0^1 k(x,t)F(t)dt + g(x)$$

$$= \sum_{r=0}^{n} w_r f_r k(x,t_r) + g(x). \tag{10}$$

Also, we get the desired approximation for $u(t)$ as follows

$$u(t) = \int_0^t u'(x)dx + u_0$$

$$= \sum_{r=0}^{n} w_r f_r \int_0^t k(x,t_r)dx + \int_0^t g(x)dx + u_0. \tag{11}$$

2   Illustrative Examples

Now for numerical implementation of the presented method, we choose five numerical examples, for which the exact solution is known for comparison with the approximate solution.

Example 1.

$$\begin{cases} u'(x) = \frac{5}{4} - \frac{1}{3}x^2 + \int_0^1 (x^2 - t)(u(t))^2 dt, \\ u(0) = 0, \end{cases}$$

with the exact solution $u(t) = t$.

Example 2.

$$\begin{cases} u'(x) = -\frac{1}{2}e^{x+2} + \frac{3}{2}e^x + \int_0^1 e^{x-t}(u(t))^3 dt, \\ u(0) = 1, \end{cases}$$

with the exact solution $u(t) = e^t$.

Example 3.

$$\begin{cases} u'(x) = g(x) + \int_0^1 \cos(x - t)(u(t))^2 dt, \\ u(0) = 0, \end{cases}$$

with the exact solution $u(t) = \sin(t)$.
Also, $g(x) = \cos(x) - \frac{2}{3}\sin(x) + \frac{1}{2}\sin(x - 1) + \frac{1}{4}\sin(x + 1) - \frac{1}{12}\sin(x - 3)$.

Example 4.

$$\begin{cases} u'(x) = 1 - \frac{1}{3}x^3 + \int_0^1 x^3(u(t))^2 dt, \\ u(0) = 0, \end{cases}$$

with the exact solution $u(t) = t$.

Example 5.

$$\begin{cases} u'(x) = 1 - \frac{1}{2}x + \frac{x}{2e} + \int_0^1 xte^{-(u(t))^2}dt, \\ u(0) = 0, \end{cases}$$

with the exact solution $u(t) = t$.

Table 1 shows the computed error $|e| = |u_{exact}(t) - u_n(t)|$ with Simpson quadrature rule for examples 1-5 with $n=6$.

Table 1

t	Example 1	Example 2	Example 3	Example 4	Example 5
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0	6×10^{-6}	2×10^{-6}	0.0	1×10^{-7}
0.2	0.0	1×10^{-5}	6×10^{-6}	0.0	7×10^{-7}
0.3	0.0	2×10^{-5}	1×10^{-5}	0.0	1×10^{-6}
0.4	0.0	3×10^{-5}	1×10^{-5}	0.0	3×10^{-6}
0.5	0.0	4×10^{-5}	2×10^{-5}	0.0	4×10^{-6}
0.6	0.0	5×10^{-5}	2×10^{-5}	0.0	7×10^{-6}
0.7	0.0	6×10^{-5}	3×10^{-5}	0.0	9×10^{-6}
0.8	0.0	7×10^{-5}	4×10^{-5}	0.0	1×10^{-5}
0.9	0.0	9×10^{-5}	4×10^{-5}	0.0	1×10^{-5}
1.0	0.0	1×10^{-4}	5×10^{-5}	0.0	1×10^{-5}

3 Conclusion

In this paper, Lagrange interpolation and Simpson quadrature rule were used to solve nonlinear Fredholm integro differential equations. The presented approach leads to solve nonlinear system of equations which may easily be solved by Newton iterative method. Numerical results state that the method has good accuracy and remarkable performance. Also, approximate solutions may be more accurate using larger n and more precise quadrature rules.

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