Application of Linear Fractional Programming Problem with Fuzzy Nature in Industry Sector

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Abstract. Several methods currently exist for solving fuzzy linear fractional programming problems under non negative fuzzy variables. However, due to the limitation of these methods, they cannot be applied for solving fully fuzzy linear fractional programming (FFLFP) problems where all the variables and parameters are fuzzy numbers. So, this paper is planning to fill in this gap and in order to obtain the fuzzy optimal solution we propose a new efficient method for FFLFP problems utilized in daily life circumstances. This proposed method is based on crisp linear fractional programming and has a simple structure. To show the efficiency of our proposed method some numerical and real life problems have been illustrated.

1. Introduction

In real world situations, sometimes, the programming models could better fit and give more insights, if we consider optimization of the ratio between the physical and/or economic quantities. Fractional programming (FP) replied to this need and efficiently used in several practical applications like cutting stock problems, ore blending problems, shipping schedules problems and different fields such as education, hospital administration, court systems, air force maintenance units, Bank branches, etc., for the past five decades; see [1–3] for more details. Meanwhile, the linear fractional programming (LFP) problems have attracted the interest of many researches due to its application in decision making such as production planning, marketing and media selection, university planning and student admissions, financial and corporate planning, health care and hospital planning, etc. see [4, 5] and references therein.

In the literature, many researchers have been recommended to solve LFP problems. Isbell and Marlow [6], first identified an example of LFP problem and solved it by a sequence of linear programming problems. Charnes and Cooper[5] considered variable transformation method to solve LFP problems. Bitran and Novaes [7] considered updated objective functions method to solve LFP problems by solving a sequence of linear programs.Martos[8], Swarup[9],Pandy and Punnen [10], Das and Mandal [11] solved the LFP problem by various types of solution procedures based on simplex method; see also [12–15] .These methods are interesting, however, in daily life circumstances, due to ambiguous information supplied by decision makers, the parameters are often illusory and it is very hard challenge for decision maker to make a

decision. In such a case, it is more appropriate to interpret the ambiguous coefficients and the vague aspirations parameters by means of the fuzzy set theory.

The concept of fuzzy set and fuzzy numbers was first introduced [16] and applied efficiently for linear optimization; see [17-32] and references therein. Furthermore, several researchers have investigated linear fractional programming problems in the fuzzy framework. For example, Sakawa and Yano[33] proposed a method to solve multi-objective linear fractional programming (MOLFP) problem under a fuzzy satisfied. Dutta et al.[34] established the sensitivity analysis in fuzzy linear fractional programming (FLP) problem. Some authors solved the FLFP problems by fuzzy goal programming approach [35–37, 39, 48]. De and Deb [40] considered a fuzzy linear fractional programming problem using sign distance ranking method where all the terms are triangular fuzzy number. Younesset al.[41] imported a design to finding bi-level multi-objective fractional integer programming problem consists of fuzzy numbers in the right-hand side of the constraints. Pop and Minasian[42], proposed a method for solving fully falsified linear fractional programming problems where all the parameters and variables are triangular fuzzy numbers. In [43, 44], they considered the same problem of [42] for solving fully fuzzy linear fractional programming problem. Veeramani and Sumathi [45] proposed solution procedure for solving fuzzy linear fractional programming problem by using fuzzy mathematical programming approach. Very recently, a number of papers have exhibited their interest to solve the FLFP problems [46-51].

In this paper, we consider a new type of fuzzy arithmetic for triangular numbers in which the coefficients of the objective function and the constraints were represented by triangular fuzzy numbers with inequality constraints utilized in daily life problem. The proposed technique is very easy and less mathematical calculation. The rest of our work is organized as follows: In Section2, we review some concept and arithmetic between two triangular of fuzzy numbers. In Section 3 formulation of FFLFP problems and use of ranking function are discussed. The new method for solving FFLFP problems is affirmed in Section 4. In Section 5, the numerical examples and the obtained results are given for illustrating the new method. In Section 6, advantages of the proposed method are discussed. Finally, the conclusion is given in Section 7.

2. Preliminaries

In this section, the basic definitions involving fuzzy sets, fuzzy numbers and operations on fuzzy numbers are outlined. For detailed information on fuzzy set theory, we refer the interested reader to [20, 52].

Definition 2.1[20] Let X denote a universal set.Then a fuzzy subset $\widetilde{A}$ of $X$is defined by its membership function $\mu_{\widetilde{A}} : X \rightarrow [0,1]$; which assigned a real number $\mu_{\widetilde{A}}(X)$ in the interval [0,1], to each element $x \in X$,where the value of $\mu_{\widetilde{A}}(X)$ at x shows the grade of membership of x in $\widetilde{A}$. A fuzzy subset $\widetilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\widetilde{A}}(X)$ and is often written $\widetilde{A}=(x, \mu_{\widetilde{A}}(X)) : x \in X$ is called a fuzzy set.

Definition 2.2[20] A fuzzy number $\widetilde{A} =(b,c,a)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\widetilde{A}}(X) = \begin{cases} \frac{(x-b)}{(c-b)}, & b \leq x \leq c, \\ \frac{(x-a)}{(c-a)}, & c \leq x \leq a, \\ 0, & \text{else.} \end{cases}$$

Definition 2.3[46] A triangular fuzzy number (b,c,a) is said to be non-negative fuzzy number iff $b \geq 0$.

Definition 2.4[46] Two triangular fuzzy number $\widetilde{A}=(b,c,a)$ and $\widetilde{B}=(e,f,d)$ are said to be equal iff $b = e, c = f, a = d$.

Definition 2.5[46] A ranking is a function $R : F(R) \rightarrow R$, where F(R) is a set of fuzzy number defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists. Let $\widetilde{A}=(b,c,a)$ is a triangular fuzzy number then $R(\widetilde{A})=\frac{b+2c+a}{4}$.

Definition 2.6[46] Let $\widetilde{A}=(b,c,a), \widetilde{B}=(e,f,d)$ be two triangular fuzzy number then:

i.$\widetilde{A}+\widetilde{B}=(b,c,a)+(e,f,d)=(b+e,c+f,a+d)$,

ii.$\widetilde{A}-\widetilde{B}=(b,c,a)-(e,f,d)=(b-d,c-f,a-e)$,

iii.Let $\widetilde{A}=(b,c,a)$ be any triangular fuzzy number and $\widetilde{B}=(e,f,d)$ be a non negative triangular fuzzy number then

$$\widetilde{A}\times\widetilde{B} = \begin{cases} (be,cf,ad), & b \geq 0 \\ (bd,cf,ad), & b < 0, a \geq 0 \\ (bd,cf,cd), & c < 0 \end{cases}$$

Definition 2.7 Let $\widetilde{A}=(b,c,a), \widetilde{B}=(e,f,d)$ be two triangular fuzzy numbers. We say that $\widetilde{A}$ is relatively less than $\widetilde{B}$, if:

i. $c < f$ or,

ii. $c = f$ and $(a-b) > (d-e)$ or,

iii.$c = f, (a-b) = (d-e)$ and $(a+b) < (d+e)$.

Note: It is clear from the Definition 2.7 that $\widetilde{A} = \widetilde{B}$ if $c = f, (a-b) = (d-e)$ and $(a+b) = (d+e)$.

3. Linear Fractional programming (LFP) problem

In this section, the general form of LFP problem is discussed. Furthermore, Charnes and Cooper's linear transformation is summarized.

$$Max \ Z(x) = \frac{c_j{}^t x_j + p}{d_j{}^t x_j + q} \tag{1}$$

subject to
$\{x \geq 0, \ Ax \leq b \Rightarrow G(x) > 0.\}$

where $j = 1,2,.....,n, A \in R^{m\times n}, b \in R^m, c, d \in R^n, p, q \in R$.For some values of x, $G(x)$ may be equal to zero. To avoid such cases, one requires that either $\{x \geq 0, \ Ax \leq b \rightarrow G(x) > 0\}$ or $\{x \geq 0, \ Ax \leq b \rightarrow G(x) < 0\}$. For convenience, assume that LFP problem satisfies the condition that

$$\{x \geq 0, \ Ax \leq b \rightarrow G(x) > 0\} \tag{2}$$

Remark 3.1 [3]

The problem 1 is said to be standard concave –convex programming problem, if $F(x)$ is concave on S with $F(\zeta) \geq 0$ for some $\zeta \in S$ and $G(x)$ is convex and positive on S.

Definition 3.1 [3]

The two mathematical programming problem $(i)MaxF(x)$, subject to $x \in S, (ii)MaxG(x)$, subject to $x \in U$ will be said to be equivalent if there is a one to one map f of the feasible area of (i), on to the feasible area of (ii), such that $F(x) = G(f(x))$ for all.

Theorem 3.1 [3] Assume that no point $(z, 0)$ with $(z, 0)$ is feasible for the following linear programming problem.

$$\begin{aligned} Max \ & c^t z + pt \\ Subject \ to \ & \\ & d^t z + qt = 1, \\ & Az - bt = 0, \\ & t > 0, z \geq 0, z \in R^n, t \in R. \end{aligned} \tag{3}$$

Then, with the condition of relation 2, the LFP problem 1 is equivalent to the linear programming problem model 3.

Now, consider the two related problems

$$Max \ tF(\tfrac{z}{t})$$
Subject to
$$A(\tfrac{z}{t}) - b \leq 0,$$
$$tG(\tfrac{z}{t}) = 1,$$
$$t > 0, z \geq 0 \tag{4}$$

and

$$Max \ tF(\tfrac{z}{t})$$
Subject to
$$A(\tfrac{z}{t}) - b \leq 0,$$
$$tG(\tfrac{z}{t}) \leq 1,$$
$$t > 0, z \geq 0 \tag{5}$$

Where model (4) is obtained from model (1) by the transformation $t = \frac{1}{G(x)}, z = tx$, and model (5) differs from model (4) by replacing the equality constraints $tG(\tfrac{z}{t}) = 1$ by an inequality constraints $tG(\tfrac{z}{t}) \leq 1$.

Theorem 3.2 [3] Let for some $\zeta \in S, G(\zeta) \geq 0$, if model (1) reaches a (global) maximum at $x = x^*$, then model 5 reaches a (global) maximum at a point $(t, z) = (t^*, z^*)$ where $(\tfrac{z^*}{t^*} = x^*)$ and the objective functions at these points are equal.

Theorem 3.3 [3] If model (1) is a standard concave-convex programming problem which reaches a maximum at a point x^*, then the corresponding transformed problem model (5) attains the same maximum value at a point (t^*, z^*) where $(\tfrac{z^*}{t^*}) = x^*$. Moreover model (5) has a concave objective function and a convex feasible set. Suppose that:

$$Max \ Z(x) = \frac{F(x)}{G(x)} \tag{6}$$

subject to
$$\{x \geq 0, \ Ax \leq b \implies G(x) > 0.\}$$

Where $F(x)$ is concave and negative for each $x \in S$ and $G(x)$ is concave and positive on S, then

$$Max_{x \in S} \tfrac{F(x)}{G(x)} \iff Min_{x \in S} \tfrac{-F(x)}{G(x)} \iff Max_{x \in S} \tfrac{G(x)}{-F(x)}$$

where $-F(x)$ is convex and positive. Therefore, the problem (6) is converted into standard concave-convex programming problem transformed to the following linear programming problem:

$$Max \ tF(\tfrac{z}{t})$$
Subject to
$$A(\tfrac{z}{t}) - b \leq 0,$$
$$tG(\tfrac{z}{t}) \leq 1,$$
$$t > 0, z \geq 0 \tag{7}$$

4. Fully fuzzy linear fractional programming and Innovation for its solution

Consider the following fully fuzzy linear fractional programming (FFLFP) problem. We are going to approach m fuzzy equality constraints and n fuzzy variables where all the terms are triangular fuzzy numbers.

$$Max \ Z = \frac{\widetilde{c}^t \ \widetilde{x} + \widetilde{\alpha}}{\widetilde{d}^t \ \widetilde{x} + \widetilde{\beta}}$$
Subject to
$$\widetilde{A} \times \widetilde{x} \leq \widetilde{b},$$
$$\widetilde{x} \geq 0 \tag{8}$$

Where $\widetilde{c}^t = [\widetilde{c_j}]$ is 1 by n matrix; $\widetilde{d}^t = [\widetilde{d_j}]$ is 1 by n matrix; $\widetilde{x} = [\widetilde{x_j}]$ is n by 1 matrix; $\widetilde{A} = [\widetilde{A_{ij}}]$ is m by n matrix; $\widetilde{b} = [\widetilde{b_{ij}}]$ is a m by 1 matrix; $\widetilde{\alpha}=[\widetilde{\alpha_j}]$ and $\widetilde{\beta}=[\widetilde{\beta_j}]$ are the scalars. Here all the parameters $\widetilde{c_j},\widetilde{d_j},\widetilde{x_j}$ and $\widetilde{a_{ij}}$ are set of fuzzy numbers.

Mention: Let $\widetilde{x}$ a fuzzy optimal solution of FFLFP problem. If there exists a fuzzy number $\widetilde{y}$ where it satisfies the following conditions:
i.$\widetilde{y}$ is a non-negative fuzzy number,
ii.$\widetilde{A} \times \widetilde{y} \leq \widetilde{b}$,
iii. $\mathfrak{R}(\widetilde{c}^t \times \widetilde{x})=\mathfrak{R}(\widetilde{c}^t \times \widetilde{y})$,
iv. $\mathfrak{R}(\widetilde{d}^t \times \widetilde{x})=\mathfrak{R}(\widetilde{d}^t \times \widetilde{y})$,
then $\widetilde{y}$ is also an exact optimal solution of the problem (8) and is called a substitute optimal solution.

Consider the model (8) and let $\widetilde{x}^* = (x^*, y^*, z^*)$ be an optimal solution of this FFLFP. Furthermore, let all the parameters $\widetilde{x}, \widetilde{c}, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{d}, \widetilde{b}$ and $\widetilde{z}$ are represented by triangular fuzzy numbers $(x, y, z), (p, q, r), (\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3), (u, v, w), (b_1, b_2, b_3)$ and (z_1, z_2, z_3) respectively. Then we can rewrite the mentioned FFLFP as follows:

$$Max(z_1, z_2, z_3) = \frac{(p,q,r)^t \times (x,y,z)+(\alpha_1,\alpha_2,\alpha_3)}{(u,v,w)^t \times (x,y,z)+(\beta_1,\beta_2,\beta_3)}$$
$$Subject\ to$$
$$(b, c, a) \times (x, y, z) \leq (b_1, b_2, b_3),$$
$$(x, y, z) \geq 0.$$

Therefore, by definition 2.6 it i easy to say that x^*, y^*, z^* are the optimal solutions of the following crisp problems:

(P_1) $\qquad Max\ z_1 = \frac{px+\alpha_1}{ux+\beta_1}$

$\qquad\qquad Subject\ to \quad bx \leq b_1$
$\qquad\qquad\qquad\qquad\qquad x \geq 0.$

(P_2) $\qquad Max\ z_2 = \frac{qy+\alpha_2}{vy+\beta_2}$

$\qquad\qquad Subject\ to \quad cy \leq b_2$
$\qquad\qquad\qquad\qquad\qquad y \geq 0.$

(P_3) $\qquad Max\ z_3 = \frac{rz+\alpha_3}{wz+\beta_3}$

$\qquad\qquad Subject\ to \quad az \leq b_3$
$\qquad\qquad\qquad\qquad\qquad z \geq 0.$

Hence, from above demonstrations, the steps of our method can be written as follows:
Step 1.Write the FFLFP problem as follows:

$$Max\ \widetilde{z} = \frac{\widetilde{c_j}\widetilde{x_j}+\widetilde{\alpha}}{\widetilde{d_j}\widetilde{x_j}+\widetilde{\beta}}$$
Subject to:
$$\widetilde{a_{ij}}\widetilde{x_j} \leq \widetilde{b_i}$$
$$\widetilde{x_j} \geq 0.$$

Step 2.If all the terms represent the triangular fuzzy numbers, then write the FFLFP problem as follows:

$$Max\ \widetilde{z} = \frac{(p_j,q_j,r_j) \times (x_j,y_j,z_j)+(\alpha_1,\alpha_2,\alpha_3)}{(u_j,v_j,w_j) \times (x_j,y_j,z_j)+(\beta_1,\beta_2,\beta_3)}$$
Subject to:
$$(b_{ij}, c_{ij}, a_{ij}) \times (x_j, y_j, z_j) \leq (b_i, g_i, h_i)$$
$$(x_j, y_j, z_j) \geq 0.$$

Step 3. To determine the optimal value of the above problem, transform both the objective function and constraints into its equivalent crisp problem. Then, the problem may be written as in the form of three separate problems (P_1), (P_2) and (P_3) of crisp linear fractional programming problem.

Step 4. Start with the problem (P_n); $(n = 1, 2, 3)$ go to Step 5.

Step 5. Set $n = 1$.

Step 6. The above problems are crisp linear fractional programming problem, which can be solved by conventional method which was discussed in Section 3.

Step 7. If $n = 3$, go to Step 8, otherwise $n = n + 1$, go to Step 6.

Step 8. Write the solution of FFLFP problems in the form of $\widetilde{x}$ and obtain the optimal solution as $\widetilde{z} = (z_1, z_2, z_3)$.

Step 9. Finally, by using Definition 2.7, compare the results.

5. Application of our proposed method

In this section, we take some real life problems and proved ability of our proposed method:

Example 5.1. In TATA Hospital Jamshedpur, India has two nutritional experiments (Vitamin A and Calcium) with two products Milk (glass) and Salad (500mg) with profit around 6 dollars and around 2 dollars per unit respectively. However, the cost for each one unit of the above product is around 1 and around 1 dollars respectively. Consider that a fixed cost of around 2 dollars as added to the cost function. Determine the maximum profit of these two products. Here, the environmental coefficients such as profit (due to market situations), cost (due to market conditions), vitamin A and calcium (due to the presents of the suppliers) are imprecise numbers with triangular possibility distributions over the planning horizon due to incomplete information. For example, the profit of the product A is (4, 6, 8) dollars. Similarly, the other parameters and variables are assumed to be triangular fuzzy numbers. Hence, above problem can be formulated as the following FFLFP problem:

Table 1: Information of Example 5.1

Nutrient	Milk (glass)	Salad (500mg)	Min nutrient required
Vitamin A	1	1	7
Calcium	2	3	17

Solution: In this case, let x_1 and x_2 to be the amount of units of Vitamin A and Calcium to be produced. Then the above problem can be formulated as:

$$Max \ \widetilde{z} = \frac{\widetilde{6x_1} + \widetilde{2x_2}}{\widetilde{x_1} + \widetilde{x_2} + \widetilde{2}}$$

Subject to:

$$\widetilde{x_1} + \widetilde{x_2} \leq \widetilde{7}$$
$$\widetilde{2x_1} + \widetilde{3x_2} \leq \widetilde{17}$$
$$\widetilde{x_1}, \widetilde{x_1} \geq 0.$$

Let us we take $\widetilde{x_1} = (y_1, z_1, x_1)$, $\widetilde{x_2} = (y_2, z_2, x_2)$, $\widetilde{z} = (z_1, z_2, z_3)$. Now we consider the coefficients $\widetilde{7} = (3, 7, 11)$, $\widetilde{17} = (7, 17, 27)$, $\widetilde{6} = (4, 6, 8)$, $\widetilde{2} = (1, 2, 3)$, $\widetilde{3} = (2, 3, 4)$ and $\widetilde{1} = (0, 1, 2)$. The problem can be written as follows:

$$Max \ (z_1, z_2, z_3) = \frac{(4,6,8)(y_1, z_1, x_1) + (1,2,3)(y_2, z_2, x_2)}{(0,1,2)(y_1, z_1, x_1) + (0,1,2)(y_2, z_2, x_2) + (1,2,3)}$$

Subject to:

$$(0, 1, 2)(y_1, z_1, x_1) + (0, 1, 2)(y_2, z_2, x_2) \leq (3, 7, 11)$$
$$(1, 2, 3)(y_1, z_1, x_1) + (2, 3, 4)(y_2, z_2, x_2) \leq (7, 17, 27)$$
$$y_1, y_2, z_1, x_1, x_2 \geq 0.$$

The fully fuzzy linear fractional programming problems $(P_1), (P_2)$ and (P_3) will be:

(p_1) $Maz\ z_1 = \frac{4y_1+y_2}{1}$

    $s.t.$ $y_1 + 2y_2 \leq 7$

           $y_1, y_2 \geq 0$

(p_2) $Maz\ z_2 = \frac{6z_1+2z_2}{z_1+z_2+2}$

    $s.t.$ $z_1 + z_2 \leq 7$

          $2z_1 + 3z_2 \leq 17$

            $z_1, z_2 \geq 0$

(p_3) $Maz\ z_3 = \frac{8x_1+3x_2}{2x_1+2x_2+3}$

    $s.t.$ $2x_1 + 2x_2 \leq 11$

         $3x_1 + 4x_2 \leq 27$

           $x_1, x_2 \geq 0$

The problems $(P_1), (P_2)$ and (P_3) are crisp LFP problems, which can be solved by the classical methods discussed in Section 3.

Then, we obtain the fuzzy optimal solution as:

$\widetilde{x_1} = (y_1, z_1, x_1) = (5.5, 7, 7)$ and $\widetilde{x_2} = (y_2, z_2, x_2) = (0, 0, 0)$.

and the optimal value of the problem as:

$\widetilde{Z}_1 = (Z^L, Z^M, Z^U) = (3.14, 4.6, 28)$.

In Veeramani and Sumathi methods [51] the optimal value of the problems are:

$\widetilde{Z}_2 = (Z^L, Z^M, Z^U) = (3.14, 4.4, 22)$.

In Stanojevic and Stancu method [54], the fuzzy optimal values are:

$\widetilde{Z}_3 = (Z^L, Z^M, Z^U) = (3.2, 4.5, 28)$.

By comparing the results of the proposed method with two existing methods [51,54], based on ranking function and the ordering by Definition 2.5, we can conclude that our result is more effective, because:

$\widetilde{Z}_1 = 10.08 > \widetilde{Z}_3 = 10.05 > \widetilde{Z}_2 = 8.48$

By comparing proposed method results with existing method [51,54], based on Definition 2.7, we conclude that our result is more efficient than other existing method.

$3.14 = (Z^L)_{(proposed\ method)} = (Z^L)_{(method\ of[51])} < (Z^L)_{(method\ of\ [54])}$

$7.74 = (Z^L + Z^M)_{(proposed\ method)} > (Z^L + Z^M)_{(method\ of\ [54])} = 7.7 > (Z^L + Z^M)_{(method\ of\ [51])} = 7.54$

$(3.14, 4.6, 28) = (\widetilde{Z}_1)_{(proposed\ method)} > (\widetilde{Z}_2)_{(method\ of\ [54])} = (3.2, 4.5, 28) > (\widetilde{Z}_3)_{(method\ of\ [51])} = (3.14, 4.4, 22)$
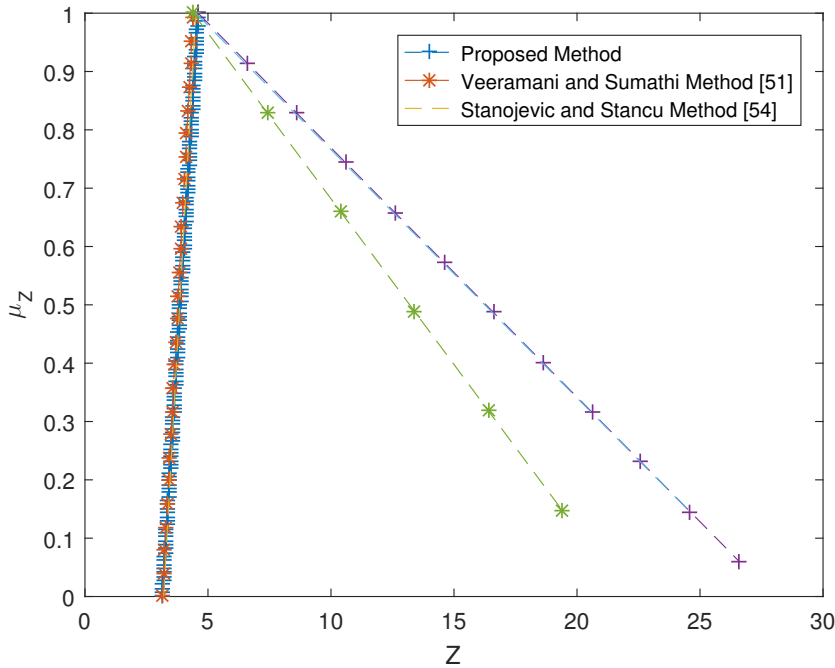

Figure 1: Membership function of the optimal solution for proposed method and existing method

In figure 1, we compare the membership function for the proposed method and the existing methods [51, 54]. Graph (Fig. 1) show that the modified technique yields better values of most of the membership functions and individual objective functions in comparison of existing methods [51,54]. It is clear that both the approaches are close to one another but the modified methodology is efficient and requires less computation than earlier technique in terms of considering the solution preferences by the decision maker at each level. In the Fig. we mention Z is a objective function and Z_π is a membership functions.
The following real life problem which considered by Veeramani and Sumathi [51] is used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach.

Example 5.2: Dali Company is the leading producer of soft drinks and low-temperature foods in Taiwan. Currently, Dali plans to develop the South-East Asian market and broaden the visibility of Dali products in the Chinese market. Notably, following the entry of Taiwan to the World Trade Organization, Dali plans to seek strategic alliance with prominent international companies and introduced international bread to lighten the embedded future impact. In the domestic soft drinks market, Dali produces tea beverages to meet demand from four distribution centers in Taichung, Chiayi, Kaohsiung, and Taipei, with production being based at three plants in Changhua, Touliu, and Hsinchu. According to the preliminary environmental information, Table.2 summarizes the potential supply available from the given three plants. The forecast demand from the four distribution centers as is shown Table 3. The profit of the company gained by each route is presented in Table 4. Table 5 summarizes the unit shipping cost for each route for the upcoming season. The environmental coefficient and related parameters generally are imprecise numbers with triangular possibility distributions over the planning horizon due to incomplete or unobtainable information. For example, the unavailable supply of the Changhua plant is (7.2, 8, 8.8) thousand dozen bottles, the forecast demand of the Taichung distribution center is (6.2, 7, 7.8) thousand dozen bottles, profit per dozen bottles for Changhua to Taichung is (8, 10, 10.8) dollars and the transportation cost per dozen bottles for Changhua to Taichung is (8, 10, 10.8) dollars. The management of Dali is initiating a study to maximize the profit as much as possible.

Table 2: Supply of the plants

Source	Changhua	Touliu	Hsinchu
Supply(thousand dozen bottles)	(7.2,8,8.8)	(12,14,16)	(10.2,12,13.8)

Table 3: Demand of the destinations

Destination	Taichung	Chiayi	Kaohsiung	Taipei
Demand(thousand dozen bottles)	(6.2,7,7.8)	(8.9,10,11)	(6.5,8,9.5)	(7.8,9,10.2)

The real world problem can be modeled to the following FFLFP problem:

$$Maz\ z = \frac{\left\{\begin{array}{l}(8,\ 10,\ 10.8)\,x_{11} + (20.4,\ 22,\ 24)\,x_{12} + (8,\ 10,\ 10.6)\,x_{13} + (18.8,\ 20,\ 22)\,x_{14} + \\ (14,\ 15,\ 16)\,x_{21} + (18.2,\ 20,\ 22)\,x_{22} + (10,\ 12,\ 13)\,x_{23} + (6,\ 8,\ 8.8)\,x_{24} + \\ (18.4,\ 20,\ 21)\,x_{31} + (9.6,\ 12,\ 13)\,x_{32} + (7.8,\ 10,\ 10.8)\,x_{33} + (14,\ 15,\ 16)\,x_{34}\end{array}\right\}}{\left\{\begin{array}{l}(1.5,\ 2,\ 2.5)\,x_{11} + (4,\ 5,\ 6)\,x_{12} + (1.3,\ 2,\ 2.5)\,x_{13} + (3,\ 4,\ 5)\,x_{14} + \\ (2.5,\ 3,\ 4)\,x_{21} + (2,\ 3,\ 4)\,x_{22} + (2.3,\ 3,\ 4)\,x_{23} + (1.5,\ 2,\ 2.5)\,x_{24} + \\ (3,\ 4,\ 5)\,x_{31} + (2,\ 3,\ 4)\,x_{32} + (1.5,\ 2,\ 2.7)\,x_{33} + (2,\ 3,\ 4)\,x_{34}\end{array}\right\}}$$

Subject to:

Table 4: Profit of the company

Source	Destination			
	Taichung	Chiayi	Kaohsiung	Taipei
Changhua	(8,10,10.8)	(20.4,22,24)	(8,10,10.6)	(18.8,20,22)
Touliu	(14,15,16)	(18.2,20,22)	(10,12,13)	(6,8,8.8)
Hsinchu	(18.4,20,21)	(9.6,12,13)	(7.8,10,10.8)	(14,15,16)

Table 5: Shipping costs

Source	Destination			
	Taichung	Chiayi	Kaohsiung	Taipei
Changhua	(1.5,2,2.5)	(4,5,6)	(1.3,2,2.5)	(3,4,5)
Touliu	(2.5,3,4)	(2,3,4)	(2.3,3,4)	(1.5,2,2.5)
Hsinchu	(3,4,5)	(2,3,4)	(1.5,2,2.7)	(2,3,4)

$x_{11} + x_{12} + x_{13} + x_{14} \leq (7.2, 8, 8.8)$
$x_{21} + x_{22} + x_{23} + x_{24} \leq (12, 14, 16)$
$x_{31} + x_{32} + x_{33} + x_{34} \leq (10.2, 12, 13.8)$
$x_{11} + x_{21} + x_{31} \leq (6.2, 7, 7.8)$
$x_{12} + x_{22} + x_{32} \leq (8.9, 10, 11.1)$
$x_{13} + x_{23} + x_{33} \leq (6.5, 8, 9.5)$
$x_{14} + x_{24} + x_{34} \leq (7.8, 9, 10.2)$
x_{ij}, where $i = 1, 2, 3; j = 1, 2, 3, 4$.

We solved the above problem by using our proposed method. The transformed linear programming problems are solved by classical methods and we get the fuzzy optimal value of the problem as:

$$\widetilde{Z}_1 = (Z^L, Z^M, Z^U) = (2.26, 4.64, 9.48).$$

In Veeramani and Sumathi methods [51] the optimal value of the problems are:
$\widetilde{Z}_2 = (Z^L, Z^M, Z^U) = (2.2, 4.35, 7.39)$.

In Stanojevic and Stancu methods [54], the fuzzy optimal values are:
$\widetilde{Z}_3 = (Z^L, Z^M, Z^U) = (2.3, 4.54, 7.96)$.

By comparing the results of the proposed method with two existing methods [51,54], based on ranking function and the ordering by Definition 2.5, we can conclude that our result is more effective, because:
$\widetilde{Z}_1 = 5.25 > \widetilde{Z}_3 = 4.83 > \widetilde{Z}_2 = 4.57$
By comparing proposed method results with existing method [51,54], based on Definition 2.7, we conclude that our result is more efficient than other existing method.
$2.3 = (Z^L)_{(\text{method of }[54])} > 2.26 = (Z^L)_{(\text{proposed method})} > 2.2 = (Z^L)_{(\text{method of }[51])}$
$6.9 = (Z^L + Z^M)_{(\text{proposed method})} > (Z^L + Z^M)_{(\text{method of }[54])} = 6.84 > (Z^L + Z^M)_{(\text{method of }[51])} = 6.55$
$(2.26, 4.64, 9.48) = (\widetilde{Z}_1)_{(\text{proposed method})} > (\widetilde{Z}_2)_{(\text{method of}[54])} = (2.3, 4.54, 7.96) > (\widetilde{Z}_3)_{(\text{method of }[51])} = (2.2, 4.35, 7.39)$

In Fig.2, we compare the membership function for the proposed method and the existing methods [51, 54]. Graph (Figs. 2) show that the modified technique yields better values of most of the membership functions and individual objective functions in comparison of existing methods [51,54]. It is clear that both the approaches are close to one another but the modified methodology is efficient and requires less
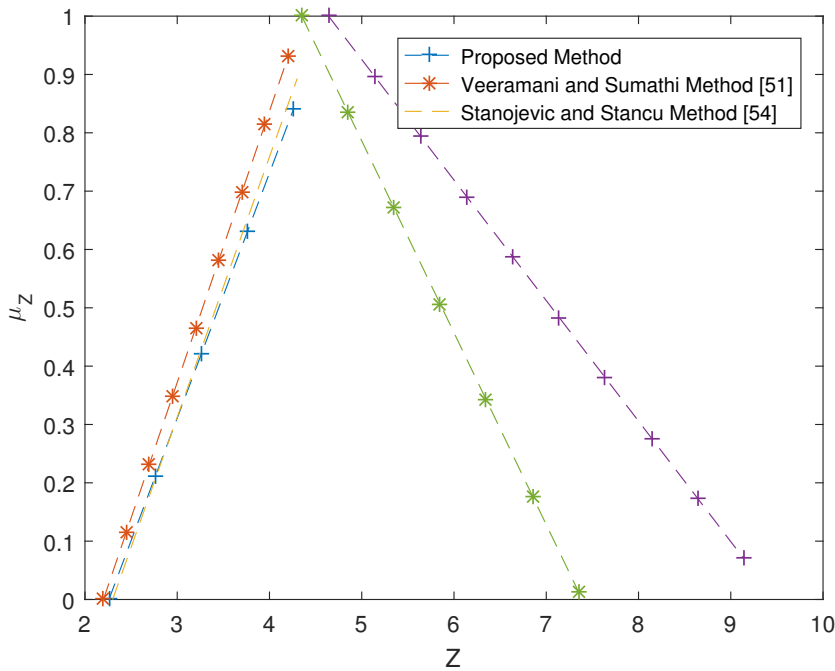
Figure 2: Membership function of the optimal solution for proposed method and existing method

computations than earlier technique in terms of considering the solution preferences by the decision maker at each level. In the Fig. we mention Z is a objective functions and Z_π is a membership functions.

The following example is used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach.

Example 5.3. Consider the following FFLFP problem:

$$Max \ Z = \frac{(3,5,7)\times(y_1,z_1,x_1)+(2,3,4)\times(y_2,z_2,x_2)}{(4,5,6)\times(y_1,z_1,x_1)+(1,2,3)\times(y_2,z_2,x_2)+(0,1,2)}$$

Subject to:

$$(2,3,4)(y_1,z_1,x_1) + (3,5,7)(y_2,z_2,x_2) \le (11,15,19)$$
$$(4,5,6)(y_1,z_1,x_1) + (3,5,7)(y_2,z_2,x_2) \le (8,10,12)$$
$$y_1, y_2, z_1, x_1, x_2 \ge 0.$$

We solved the above problem by using our proposed method. The transformed linear programming problems are solved by the classical methods, which are discussed in Section 3.

$$\widetilde{Z}_1 = (Z^L, Z^M, Z^U) = (0.67, 1.23, 2.51).$$

In Veeramani and Sumathi methods [51] the optimal value of the problems are:
$$\widetilde{Z}_2 = (Z^L, Z^M, Z^U) = (0.7, 1.15, 2.6).$$

In Stanojevic and Stancu method [54], the fuzzy optimal values are:
$$\widetilde{Z}_3 = (Z^L, Z^M, Z^U) = (0.5, 1.2, 2.01).$$

By comparing the results of the proposed method with two existing methods [51,54], based on ranking function and the ordering by Definition 2.5, we can conclude that our result is more effective, because:

$\widetilde{Z}_1 = 1.41 > \widetilde{Z}_2 = 1.40 > \widetilde{Z}_3 = 1.22$

By comparing proposed method results with existing method [51,54], based on Definition 2.7, we conclude that our result is more efficient than other existing method.

$$1.9 = (Z^L + Z^M)_{\text{(proposed method)}} > (Z^L + Z^M)_{\text{(method of [51])}} = 1.85 > (Z^L + Z^M)_{\text{(method of [54])}} = 1.7$$

$$(0.67, 1.23, 2.51) = (\widetilde{Z}_1)_{\text{(proposed method)}} = (\widetilde{Z}_2)_{\text{(method of [51])}} = (0.7, 1.15, 2.6) > (\widetilde{Z}_3)_{\text{(method of [54])}} = (0.5, 1.2, 2.01)$$

Now if we analyze the optimal solution of the considered above two real life problems, we note that our proposed method was 100 percent successful, numerically. But, the method of [51] failed in 15 percent of the problems, numerically. Therefore, our proposed method and its solutions are also more reliable than the mentioned method. Furthermore, the iteration and elapsed times of the method of[51] are greater than our proposed method.

6. Conclusion

In the past few years, a growing interest has been shown in fuzzy linear fractional programming and currently there are several methods for solving FLFP. However, to the best of our knowledge, a few efficient optimal solution has been found in fully fuzzy linear fractional programming (FFLFP). In this paper, a new efficient method for FFLFP has been proposed, in order to obtain the fuzzy optimal solution. Furthermore, the limitations of other existing methods have been pointed out. To show the efficiency of the proposed method, some numerical examples have been illustrated.

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