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Research Article

Application of Local Fractional Series Expansion Method to Solve Klein-Gordon Equations on Cantor Sets

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We use the local fractional series expansion method to solve the Klein-Gordon equations on Cantor sets within the local fractional derivatives. The analytical solutions within the nondifferential terms are discussed. The obtained results show the simplicity and efficiency of the present technique with application to the problems of the liner differential equations on Cantor sets.

1. Introduction

The Klein-Gordon equation [1] has been applied to mathematical physics such as solid-state physics, nonlinear optics, and quantum field theory. Some of the analytical methods for solving the Klein-Gordon equation include the variational iteration method [2], the tanh and the sine-cosine methods [3], the decomposition method [4], the differential transform method [5], and the homotopy-perturbation method [6].

Recently, the solutions for the fractional Klein-Gordon equation with the Caputo fractional derivative were considered in [7–9]. Golmankhaneh et al. used the homotopy-perturbation method to obtain solution for the fractional Klein-Gordon equation [7]. Kurulay [8] pointed out the solution for the fractional Klein-Gordon equation by using the homotopy analysis method. Gepreel and Mohamed [9] presented the solution for nonlinear space-time fractional Klein-Gordon equation by the homotopy analysis method.

When some domains cannot be described by smooth functions, both the classical approach and the fractional approach based on Riemann-Liouville (or Caputo) derivatives are unacceptable. In such cases, the local fractional calculus is an efficient technique for modeling these physical problems [10–23]. Using the fractional complex transform method [20], one transforms the classical Klein-Gordon equation into the Klein-Gordon equation on Cantor sets in the following form:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = F(u(x)), \tag{1}$$

subject to the initial value conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial^{\alpha}}{\partial u^{\alpha}} u(x,0) &= g(x), \end{aligned} \tag{2}$$

where the operator is the local fractional derivative operator, which is defined by [16–23]

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x-x_0)^\alpha}, \quad (3)$$

with $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$ and $u(x,t)$ and $g(x)$ are the local fractional continuous functions and $F(u(x))$ are the mixed terms of nonlinear and liner functions.

In view of (1)-(2), the linear Klein-Gordon equation on Cantor sets:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = u(x,t), \quad x > 0,\ t > 0, \quad (4)$$

subject to the initial value conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial^\alpha}{\partial u^\alpha} u(x,0) &= g(x), \end{aligned} \quad (5)$$

is under consideration, where $f(x)$ and $g(x)$ are local fractional continuous functions.

On the other hand, the local fractional series expansion method was applied to solve the wave and diffusion equations on Cantor sets [21], the local fractional Schrödinger equation in the one-dimensional Cantorian system [22], and the local fractional Helmholtz equation [23]. In this paper, our aim is to investigate a new application of this technology to solve the linear Klein-Gordon equations on Cantor sets. The paper is organized as follows. In Section 2, the idea of local fractional series expansion method is given. In Section 3, the solutions for linear Klein-Gordon equations on Cantor sets are presented. Finally, Section 4 is the conclusions.

2. The Local Fractional Series Expansion Method

In order to illustrate the idea of the local fractional series expansion method [21–23], we consider the local fractional differential operator equation in the following form:

$$u_{2t}^{2\alpha} = L_\alpha u, \quad (6)$$

where L_α is the linear local fractional operator and ϕ is a local fractional continuous function.

From (6), the multiterm separated functions with respect to x, t read as

$$u(x,t) = \sum_{i=0}^{\infty} \phi_i(t)\,\psi_i(x), \quad (7)$$

where $\phi_i(t)$ and $\psi_i(x)$ are the local fractional continuous functions.

From (7), we have

$$\phi_i(t) = \frac{t^{i\alpha}}{\Gamma(1+i\alpha)}, \quad (8)$$

so that

$$u(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \psi_i(x). \quad (9)$$

In view of (9), we obtain

$$\begin{aligned} u_t^{2\alpha} &= \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \psi_{i+2}(x), \\ L_\alpha u &= L_\alpha \left[\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \psi_i(x)\right] = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (L_\alpha \psi_i)(x). \end{aligned} \quad (10)$$

Making use of (10), we have

$$\sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} \psi_{i+2}(x) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (L_\alpha \psi_i)(x), \quad (11)$$

so that

$$\psi_{i+2}(x) = (L_\alpha \psi_i)(x). \quad (12)$$

Hence, from (12) we get

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \psi_i(x). \quad (13)$$

We now rewrite (4) in the local fractional operator form as follows:

$$u_{2t}^{2\alpha} = L_\alpha u, \quad (14)$$

subject to the initial value conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial^\alpha}{\partial u^\alpha} u(x,0) &= g(x), \end{aligned} \quad (15)$$

where the linear local fractional operator is defined as follows:

$$L_\alpha = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + I. \quad (16)$$

Hence, (16) is a special case of (6) and it is used with the linear Klein-Gordon equations on Cantor sets in next section.

3. Analytical Solutions for Linear Klein-Gordon Equations on Cantor Sets

In this section, we present the nondifferentiable solutions for linear Klein-Gordon equations on Cantor sets.

Example 1. Let us consider the Klein-Gordon equations on Cantor sets in the following form:

$$u_{2t}^{2\alpha} = L_\alpha u, \quad (17)$$

subject to the initial value conditions:

$$u(x,0)=0,\quad \frac{\partial^\alpha}{\partial u^\alpha}u(x,0)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{18}$$

From (12) and (18), we can structure the following iterative formulas:

$$\begin{aligned}\psi_{i+2}(x)&=\left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}}+\psi_i\right)(x),\\ \psi_0(x)&=0,\\ \psi_{i+2}(x)&=\left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}}+\psi_i\right)(x),\\ \psi_1(x)&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}.\end{aligned} \tag{19}$$

Hence, we can calculate

$$\begin{aligned}\psi_0(x)&=0,\\ \psi_1(x)&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)},\\ \psi_2(x)&=0,\\ \psi_3(x)&=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)},\\ \psi_4(x)&=0,\\ \psi_5(x)&=1+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)},\\ &\vdots\end{aligned} \tag{20}$$

and so on.

Therefore, we have

$$u(x,t)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{i=1}^{\infty}\frac{t^{(2i-1)\alpha}}{\Gamma(1+(2i-1)\alpha)}+\sum_{i=1}^{\infty}\frac{t^{(1+2i)\alpha}}{\Gamma(1+(1+2i)\alpha)}, \tag{21}$$

and the corresponding graph is illustrated in Figure 1.

Example 2. We consider the following Klein-Gordon equations on Cantor sets:

$$u_{2t}^{2\alpha}=L_\alpha u, \tag{22}$$

subject to the initial value conditions:

$$u(x,0)=\frac{x^\alpha}{\Gamma(1+\alpha)},\quad \frac{\partial^\alpha}{\partial u^\alpha}u(x,0)=0. \tag{23}$$

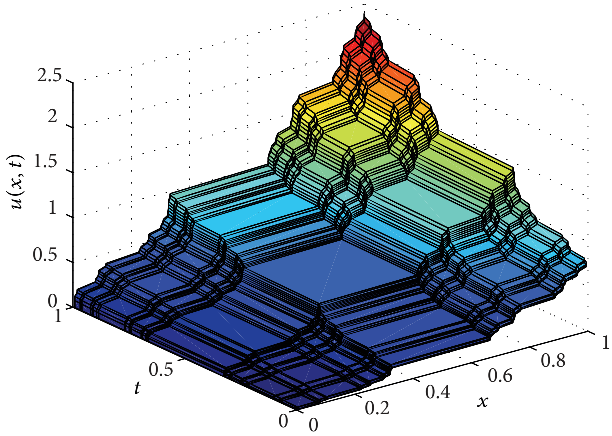

FIGURE 1: The plot of $u(x,t)$ for the parameter $\alpha=\ln 2/\ln 3$.

From (12) and (23), we get the following iterative formulas:

$$\begin{aligned}\psi_{i+2}(x)&=\left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}}+\psi_i\right)(x),\\ \psi_0(x)&=\frac{x^\alpha}{\Gamma(1+\alpha)},\\ \psi_{i+2}(x)&=\left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}}+\psi_i\right)(x),\\ \psi_1(x)&=0.\end{aligned} \tag{24}$$

Hence, we get

$$\begin{aligned}\psi_0(x)&=\frac{x^\alpha}{\Gamma(1+\alpha)},\\ \psi_1(x)&=0,\\ \psi_2(x)&=\frac{x^\alpha}{\Gamma(1+\alpha)},\\ \psi_3(x)&=0,\\ \psi_4(x)&=\frac{x^\alpha}{\Gamma(1+\alpha)},\\ \psi_5(x)&=0,\\ &\vdots\end{aligned} \tag{25}$$

and so on.

Hereby, we obtain the solution of (22):

$$u(x,t)=\frac{x^\alpha}{\Gamma(1+\alpha)}\sum_{i=0}^{\infty}\frac{t^{2i\alpha}}{\Gamma(1+2i\alpha)}, \tag{26}$$

and the corresponding graph is depicted in Figure 2.

Example 3. We present the following Klein-Gordon equations on Cantor sets:

$$u_{2t}^{2\alpha}=L_\alpha u, \tag{27}$$

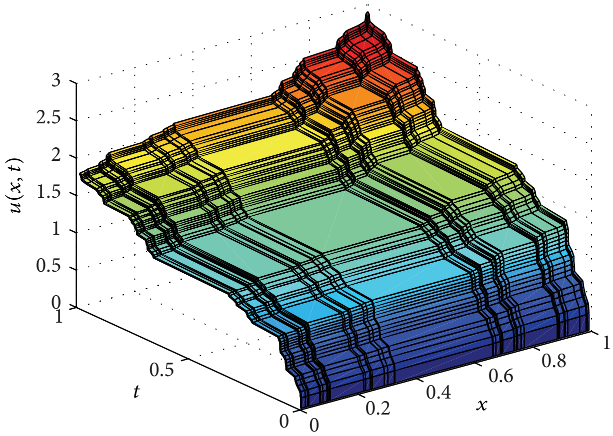

FIGURE 2: The plot of $u(x,t)$ for the parameter $\alpha = \ln 2/\ln 3$.

subject to the initial value conditions:

$$u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad \frac{\partial^{\alpha}}{\partial u^{\alpha}} u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{28}$$

From (12) and (27)-(28), we get the following iterative formulas:

$$\begin{aligned} \psi_{i+2}(x) &= \left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}} + \psi_i\right)(x), \\ \psi_0(x) &= \frac{x^{2\alpha}}{\Gamma(1+\alpha)}, \\ \psi_{i+2}(x) &= \left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}} + \psi_i\right)(x), \\ \psi_1(x) &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \end{aligned} \tag{29}$$

From (29) we obtain

$$\begin{aligned} \psi_0(x) &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ \psi_1(x) &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ \psi_2(x) &= 1 + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ \psi_3(x) &= 1 + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ \psi_4(x) &= 1 + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ \psi_5(x) &= 1 + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &\vdots \end{aligned} \tag{30}$$

and so on.

Therefore, we obtain the exact solution of (27)

$$u(x,t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(t^{\alpha}) + E_{\alpha}(t^{\alpha}) - \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1, \tag{31}$$

and its graph is shown in Figure 3.

Example 4. The Klein-Gordon equation on Cantor sets is presented as

$$u_{2t}^{2\alpha} = L_{\alpha} u, \tag{32}$$

and the initial value conditions are written as

$$u(x,0) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \quad \frac{\partial^{\alpha}}{\partial u^{\alpha}} u(x,0) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}. \tag{33}$$

From (12) and (27)-(28), the following iterative formulas are as follows:

$$\begin{aligned} \psi_{i+2}(x) &= \left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}} + \psi_i\right)(x), \\ \psi_0(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_{i+2}(x) &= \left(\frac{\partial^{2\alpha}\psi_i}{\partial x^{2\alpha}} + \psi_i\right)(x), \\ \psi_1(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}. \end{aligned} \tag{34}$$

From (29), we give

$$\begin{aligned} \psi_0(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_1(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_2(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_3(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_4(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ \psi_5(x) &= \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ &\vdots \end{aligned} \tag{35}$$

and so on.

Therefore, we give the exact solution of (32):

$$u(x,t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(t^{\alpha}), \tag{36}$$

and its graph is shown in Figure 4.

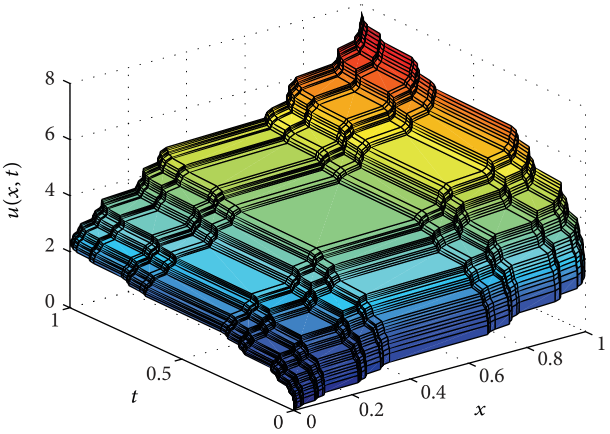

FIGURE 3: The plot of $u(x, t)$ for the parameter $\alpha = \ln 2/\ln 3$.

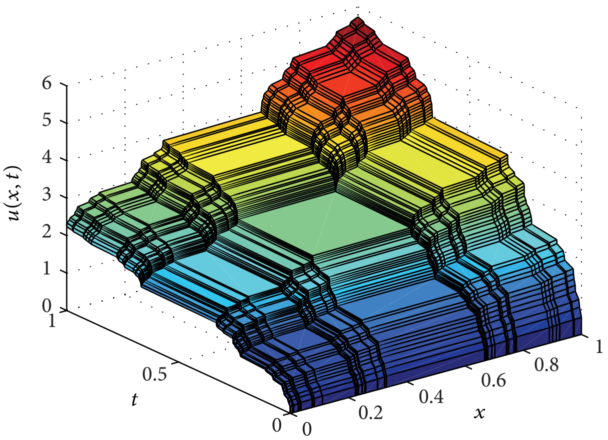

FIGURE 4: The plot of $u(x, t)$ for the parameter $\alpha = \ln 2/\ln 3$.

4. Conclusions

In this work the Klein-Gordon equations on Cantor sets within the local fractional differential operator had been analyzed using the local fractional series expansion method. The nondifferentiable solutions for local fractional Klein-Gordon equations were obtained. The present method is a powerful mathematical tool for solving the local fractional linear differential equations.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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