Measurement Techniques, Vol. 56, No. 9, December, 2013

APPLICATION OF NONPARAMETRIC KUIPER AND WATSON TESTS OF GOODNESS-OF-FIT FOR COMPOSITE HYPOTHESES

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UDC 519.24

This is an examination of models of statistical distributions and tables of percentage points for application of the Kuiper and Watson tests of goodness-of-fit for composite hypotheses belonging to samples with different parametric models of their probability distributions. An interactive simulation method is presented which can be used for the construction and use of a statistical distribution of a test in the course of the statistical analysis associated with hypothesis testing.

Keywords: *nonparametric goodness-of-fit testing, Kuiper and Watson tests, simple and composite tests.*

It has been shown in a study [1] of the statistical distribution and power of the nonparametric Kuiper [2] and Watson [3] tests that they have some advantage in power for the testing of simple hypotheses over the Kolmogorov, Cramer–Mises–Smirnov, and Anderson–Darling tests. This means that they can be recommended for various applications. This clear advantage does not show up in the testing of composite hypotheses, but it is evident that the Kuiper and Watson tests are appropriate for use along with these other tests of goodness-of-fit.

When testing composite hypotheses of the form H_0: $F(x) \in \{F(x, \theta), \theta \in \Theta\}$, where an estimate $\hat{\theta}$ of the scalar or vector parameter of the distribution $F(x, \theta)$ is calculated for the same sample, the nonparametric tests of goodness-of-fit lose their freedom from distributions [4]. The conditional distributions $G(S \mid H_0)$ of the test statistics during comparison of composite hypotheses depend on a number of factors [5]: the form of the observed distribution $F(x, \theta)$ corresponding to the correct tested hypothesis H_0; the type of parameters to be estimated and their number; in some cases, the specific value of the parameter (e.g., for the family of gamma- and beta-distributions); the method for evaluating the parameters. The differences in the distributions of a given statistic for testing simple and composite hypotheses are so large that neglecting this fact generally leads to incorrect use of a test and, therefore, to incorrect statistical conclusions.

Various approaches have been used for verifying composite hypotheses with the aid of the Kolmogorov, Cramer–Mises–Smirnov, and Anderson–Darling tests [6–11], as well as a computer approach and statistical modelling [12, 13], which have served as a basis for the development of recommendations for the application of nonparametric tests of goodness-of-fit [14, 15]. These results were refined and extended later [16–24], and are most fully discussed in a recent article [5].

For testing the hypothesis that a random sample obeys a continuous distribution $F(x, \theta)$ by the Kuiper test [2], one uses a statistic of the form

$$V_n = \sup_{-\infty < x < \infty} \{F_n(x) - F(x, \theta)\} - \inf_{-\infty < x < \infty} \{F_n(x) - F(x, \theta)\},$$

which is calculated with

$$V_n = D_n^+ - D_n^-, \tag{1}$$

where $F_n(x)$ is the empirical distribution function;

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$$D_n^+ = \max\left\{\frac{i}{n} - F(x_i, \theta)\right\}, \quad D_n^- = \max\left\{F(x_i, \theta) - \frac{i-1}{n}\right\};$$

and x_i are the elements of a variation series constructed from the sample.

As the limiting distribution $G(\sqrt{n}V_n \,|\, H_0)$ of the statistic $\sqrt{n}V_n$, Kuiper [2] gives the distribution function

$$G(s \,|\, H_0) = 1 - \sum_{m=1}^{\infty} 2(4m^2 s^2 - 1)\mathrm{e}^{-2m^2 s^2}.$$

In order to reduce the dependence of the distribution $G(V_n \,|\, H_0)$ of the statistic (1) on the sample size n, a modification has been proposed [25]:

$$V = V_n(\sqrt{n} + 0.155 + 0.24/\sqrt{n}). \tag{2}$$

Following [26], in [1] the following statistic has been proposed for use in the Kuiper test:

$$V_n^{\mathrm{mod}} = \sqrt{n}(D_n^+ + D_n^-) + (3\sqrt{n})^{-1}. \tag{3}$$

In the testing of simple hypotheses, the percentage points and distributions of the (2) and (3) statistics are essentially the same, and the limiting distribution of the statistic (3) can be modelled by a beta distribution of the third kind with density [1]:

$$f(x) = \frac{\theta_2^{\theta_0}}{\theta_3 \mathrm{B}(\theta_0, \theta_1)} \frac{(X)^{\theta_0 - 1}(1 - X)^{\theta_1 - 1}}{\left[1 + (\theta_2 - 1)X\right]^{\theta_0 + \theta_1}}, \tag{4}$$

where $X = (x - \theta_4)/\theta_3$ and the parameter vector $\boldsymbol{\theta} = (7.8985; 7.6865; 2.6852; 2.6373; 0.493)^{\mathrm{T}}$.

The Watson test [3] is used with a statistic of the form

$$U_n^2 = \sum_{i=1}^{n}\left(F(x_i, \theta) - \frac{i - 1/2}{n}\right)^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} F(x_i, \theta) - \frac{1}{2}\right)^2 + \frac{1}{12n}. \tag{5}$$

In testing simple hypotheses, the limiting distribution $G(U_n^2 \,|\, H_0)$ of the statistic (5) is given by [3]

$$G(s \,|\, H_0) = 1 - 2\sum_{m=1}^{\infty}(-1)^{m-1}\mathrm{e}^{-2m^2\pi^2 s}$$

and is approximated well over the entire range of definition by the model of an inverse gaussian distribution with density [1]

$$f(x) = \sqrt{\frac{\theta_0}{2\pi Y^3}} \exp\left(-\frac{\theta_0(Y - \theta_1)^2}{2\theta_1^2 Y}\right), \tag{6}$$

where $Y = (x - \theta_3)/\theta_2$ and the parameter vector $\boldsymbol{\theta} = (0{,}5555; 0{,}2385; 0{,}3437; 0{,}0015)^{\mathrm{T}}$.

In the following, we present models of limiting distributions and tables of percentage points for the use of the Kuiper and Watson tests for composite hypotheses according to which samples obey different parametric distribution models. It is assumed that maximum likelihood estimates are used.

Table 1 lists some distributions with respect to which composite hypotheses can be tested using the approximations constructed in this paper for the limiting distributions of the statistics of nonparametric tests of goodness-of-fit.

TABLE 1. Distributions of Random Quantities

Distribution name	Density function $f(x, \theta)$
Exponential	$\theta_0^{-1} \exp[-x/\theta_0]$
Half-normal	$2(\theta_0\sqrt{2\pi})^{-1} \exp[-x^2/(2\theta_0^2)]$
Rayleigh	$(x/\theta_0^2)\exp[-x^2/(2\theta_0^2)]$
Maxwell	$[2x^2/(\theta_0^3\sqrt{2\pi})]\exp[-x^2(2\theta_0^2)]$
Laplace	$(2\theta_0)^{-1}\exp[-\lvert x-\theta_1\rvert/\theta_0]$
Normal (Gauss)	$(\theta_0\sqrt{2\pi})^{-1}\exp[-(x-\theta_1)^2/(2\theta_0^2)]$
Log-normal	$(x\theta_0\sqrt{2\pi})^{-1}\exp[-(\ln x-\theta_1)^2/(2\theta_0^2)]$
Cauchy	$\theta_0/\{\pi[\theta_0^2+(x-\theta_1)^2]\}$
Logistic	$\frac{\pi}{\theta_0\sqrt{3}}\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\Big/\left[1+\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\right]^2$
Extreme value (maximum)	$\frac{1}{\theta_0}\exp\left\{-\frac{x-\theta_1}{\theta_0}-\exp\left(-\frac{x-\theta_1}{\theta_0}\right)\right\}$
Extreme value (minimum)	$\frac{1}{\theta_0}\exp\left\{\frac{x-\theta_1}{\theta_0}-\exp\left(\frac{x-\theta_1}{\theta_0}\right)\right\}$
Weibull	$\theta_0 x^{\theta_0-1}\theta_1^{-\theta_0}\exp\{-(x/\theta_1)^{\theta_0}\}$
Sb-Johnson *Sb*(θ_0, θ_1, θ_2, θ_3)	$\frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)}\exp\left\{-\frac{1}{2}\left[\theta_0-\theta_1\ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\}$
Sl-Johnson *Sl*(θ_0, θ_1, θ_2, θ_3)	$\frac{\theta_1}{(x-\theta_3)\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1\ln\frac{x-\theta_3}{\theta_2}\right]^2\right\}$
Su-Johnson *Su*(θ_0, θ_1, θ_2, θ_3)	$\frac{\theta_1}{\sqrt{2\pi}\sqrt{(x-\theta_3)^2+\theta_2^2}}\exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1\ln\left\{\frac{x-\theta_3}{\theta_2}+\sqrt{\left(\frac{x-\theta_3}{\theta_2}\right)^2+1}\right\}\right]^2\right\}$

TABLE 2. Upper Percentage Points and Models of the Limiting Distributions of the Statistics for the Kuiper Test with Maximum Likelihood Estimates

Distribution name	Parameters to be estimated	Percentage points			Model
		0.1	0.05	0.01	
Exponential, Rayleigh, Maxwell	Scale	1.540	1.661	1.905	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
Half-normal	Scale	1.543	1.664	1.907	B_3(11.4707; 40.7237; 7.020; 20.3675; 0.3989)
Laplace	Scale	1.469	1.587	1.825	B_3(7.8324; 8.3778; 2.6906; 2.4820; 0.4830)
	Shift	1.473	1.597	1.850	B_3(9.1630; 6.6097; 4.0210; 2.4081; 0.4900)
	Both parameters	1.278	1.365	1.541	B_3(10.0376; 7.8452; 3.4694; 1.9586; 0.4756)
Normal, log-normal	Scale	1.494	1.611	1.847	B_3(6.3057; 8.1797; 2.3279; 2.4413; 0.5370)
	Shift	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
	Both parameters	1.402	1.505	1.709	B_3(7.4917; 8.0016; 2.4595; 2.1431; 0.4937)
Cauchy	Scale or shift	1.435	1.560	1.815	B_3(3.8425; 5.9345; 2.4284; 2.1927; 0.6150)
	Both parameters	1.126	1.197	1.337	B_3(9.4267; 7.5349; 3.2515; 1.5491; 0.4700)
Logistic	Scale	1.470	1.588	1.826	B_3(9.7224; 7.8186; 3.2399; 2.4541; 0.4370)
	Shift	1.511	1.633	1.880	B_3(9.1363; 6.9693; 3.4630; 2.3985; 0.4790)
	Both parameters	1.337	1.432	1.622	B_3(14.3460; 18.6137; 3.6366; 3.9560; 0.3525)
Extreme values and Weibull	Scale[1]	1.504	1.622	1.861	*Sl*(1.2459; 4.0123; 1.3063; 0.1873)
	Shift[2]	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
	Both parameters	1.411	1.516	1.726	*Sl*(1.4012; 5.0846; 1.4465; –0.0070)

Notes: [1,2] the shape and scale of the Weibull distribution are estimated here.

Tables of the percentage points and models of the distributions of the test statistics have been constructed from modelled samples of the statistics of size $N = 1.7 \cdot 10^6$. For this N, the difference between the true distribution $G(S \mid H_0)$ for the distribution of the statistic and the modelled empirical distribution $G_N(S \mid H_0)$ has a modulus of less than 10^{-3}. The values of the test statistics were calculated from samples of pseudorandom quantities of size $n = 10^3$ generated in accordance with the observed distribution $F(x, \theta)$. In this situation, the distribution $G(S_n \mid H_0)$ is essentially the same as the limiting $G(S \mid H_0)$. It is possible to use the models given here in statistical analysis problems beginning with sample sizes $n > 25$.

The distributions $G(S \mid H_0)$ for the statistics of the Kuiper and Watson tests are best approximated by a family of beta distribution of the third kind with density (4), i.e., $B_3(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = f(x)$ and by the family of *Sl*-Johnson distributions $Sl(\theta_0, \theta_1, \theta_2, \theta_3)$ (see Table 1).

The upper percentage points and models constructed for the limiting distributions of the statistics for the Kuiper test in the case where a maximum likelihood estimate is used are shown in Table 2 for 12 of the distributions (the last three in Table 1 are omitted). The upper percentage points and models constructed for the distributions of the statistics for the Watson test are listed in Table 3 for the same distributions.

Table 4 lists the upper percentage points and models of the limiting distributions for the statistics of the nonparametric tests of goodness-of-fit in the case where composite hypotheses are being tested with respect to the *Sb*-Johnson distribution (with a maximum likelihood estimate), Table 5 gives the same for the *Sl*-Johnson distribution, and Table 6, for the *Su*-Johnson distribution. In all these cases, the distribution $G(S \mid H_0)$ of the statistics of the goodness-of-fit test is independent of the specific values of the unknown parameters of the distributions $F(x, \theta)$.

TABLE 3. Upper Percentage Points and Models of the Limiting Distributions of the Statistics for the Watson Test with Maximum Likelihood Estimates

Distribution name	Parameters to be estimated	Percentage points			Model
		0.1	0.05	0.01	
Exponential, Rayleigh, Maxwell	Scale	0.129	0.159	0.230	B_3(4.0419; 2.9119; 10.5931; 0.5000; 0.0096)
Half-normal	Scale	0.131	0.161	0.232	B_3(4.9988; 3.8721; 15.1781; 0.6900; 0.0059)
Laplace	Scale	0.115	0.144	0.214	B_3(9.2136; 3.8610; 30.5491; 0.7010; 0.0015)
	Shift	0.111	0.139	0.209	B_3(7.4479; 3.2650; 30.7784; 0.6227; 0.0063)
	Both parameters	0.071	0.084	0.114	B_3(9.0116; 5.3554; 17.3201; 0.3908; 0.0038)
Normal, log-normal	Scale	0.122	0.151	0.221	B_3(8.8122; 3.7536; 29.8074; 0.7171; 0.0019)
	Shift	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
	Both parameters	0.096	0.116	0.164	B_3(3.5230; 4.4077; 9.2281; 0.4785; 0.0104)
Cauchy	Scale or shift	0.105	0.133	0.203	*Sl*(2.7778; 1.5065; 0.2690; 0.0049)
	Both parameters	0.052	0.061	0.081	B_3(8.3558; 4.8650; 12.0768; 0.1930; 0.0049)
Logistic	Scale	0.115	0.144	0.214	B_3(9.2136; 3.8610; 30.5491; 0.7010; 0.0015)
	Shift	0.119	0.148	0.218	B_3(3.9730; 3.9414; 13.2655; 0.6637; 0.0090)
	Both parameters	0.081	0.098	0.135	B_3(4.2608; 4.6784; 9.3054; 0.3810; 0.0084)
Extreme values and Weibull	Scale[1]	0.122	0.151	0.221	B_3(8.8122; 3.7536; 29.8074; 0.7171; 0.0019)
	Shift[2]	0.129	0.159	0.230	B_3(4.9988; 3.8721; 15.1781; 0.6792; 0.0061)
	Both parameters	0.097	0.118	0.165	*Sl*(1.2863; 1.6736; 0.0927; 0.0052)

Notes: [1,2] as in Table 2.

TABLE 4. Upper Percentage Points and Models for the Limiting Distributions of the Statistics of Nonparametric Tests of Goodness-of-Fit for Testing of Hypotheses with Respect to *Sb*-Johnson Distributions with Use of Maximum Likelihood Estimates

Parameters to be estimated	Percentage points			Model
	0.1	0.05	0.01	
Kuiper test				
θ_0	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
θ_1	1.494	1.611	1.847	B_3(6.3057; 8.1797; 2.3279; 2.4413; 0.5370)
θ_0, θ_1	1.402	1.505	1.709	B_3(7.4917; 8.0016; 2.4595; 2.1431; 0.4937)
Watson test				
θ_0	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
θ_1	0.122	0.151	0.221	B_3(8.8122; 3.7536; 29.8074; 0.7171; 0.0019)
θ_0, θ_1	0.096	0.116	0.164	B_3(3.5230; 4.4077; 9.2281; 0.4785; 0.0104)

TABLE 5. Upper Percentage Points and Models for the Limiting Distributions of the Statistics of Nonparametric Tests of Goodness-of-Fit for Testing of Hypotheses with Respect to *Sl*-Johnson Distributions with Use of Maximum Likelihood Estimates

Parameters to be estimated	Percentage points			Model
	0.1	0.05	0.01	
Kuiper test				
θ_0	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
θ_1	1.512	1.631	1.872	B_3(6.7423; 8.0549; 2.4935; 2.4976; 0.5250)
θ_2	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
θ_0, θ_1	1.402	1.505	1.709	B_3(7.4917; 8.0016; 2.4595; 2.1431; 0.4937)
θ_0, θ_2	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
θ_1, θ_2	1.402	1.505	1.709	B_3(7.4917; 8.0016; 2.4595; 2.1431; 0.4937)
θ_0, θ_1, θ_2	1.402	1.505	1.709	B_3(7.4917; 8.0016; 2.4595; 2.1431; 0.4937)
Watson test				
θ_0	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
θ_1	0.124	0.153	0.223	B_3(3.4122; 4.9262; 9.6902; 0.7643; 0.0087)
θ_2	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
θ_0, θ_1	0.096	0.116	0.164	B_3(3.5230; 4.4077; 9.2281; 0.4785; 0.0104)
θ_0, θ_2	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
θ_1, θ_2	0.096	0.116	0.164	B_3(3.5230; 4.4077; 9.2281; 0.4785; 0.0104)
θ_0, θ_1, θ_2	0.096	0.116	0.164	B_3(3.5230; 4.4077; 9.2281; 0.4785; 0.0104)

When the distributions of the statistics of the nonparametric tests of goodness-of-fit depend on the values of the parameter or parameters of the distribution with which goodness-of-fit is being tested, the problem can be solved in the following way. Since the estimates of the parameters become known in the course of the analysis, the distribution of the statistics required for testing the hypothesis cannot be found in advance.

Thus, the distributions of the statistics of the tests to be used must be found in an interactive mode in the course of the statistical analysis. Of course, this requires a developed program with parallel processing (as in our case) to speed up the simulation and make use of the available computational resources. Under these conditions, the time to construct (with the required accuracy) the distributions $G_N(S_n | H_0)$ of the test statistics needed for testing the hypothesis and to determine the attained level of significance $P\{S_n \geq S^*\}$, where S^* is the value of the statistic calculated from the sample, is not very large against the background of a complete solution to the statistical analysis problem.

The following example demonstrates the accuracy with which the available level of significance can be determined as a function of the sample volume N of the empirical distribution of the statistic modelled in an interactive model.

Example. We now test the composite hypothesis that the following sample of size $n = 100$ is described by an inverse gaussian distribution with the density of Eq. (6):

0.945	1.040	0.239	0.382	0.398	0.946	1.248	1.437	0.286	0.987
2.009	0.319	0.498	0.694	0.340	1.289	0.316	1.839	0.432	0.705
0.371	0.668	0.421	1.267	0.466	0.311	0.466	0.967	1.031	0.477
0.322	1.656	1.745	0.786	0.253	1.260	0.145	3.032	0.329	0.645

TABLE 6. Upper Percentage Points and Models for the Limiting Distributions of the Statistics of Nonparametric Tests of Goodness-of-Fit for Testing of Hypotheses with Respect to *Su*-Johnson Distributions with Use of Maximum Likelihood Estimates

Parameters to be estimated	Percentage points			Model
	0.1	0.05	0.01	
Kuiper test				
θ_0	1.540	1.662	1.908	B_3(5.5932; 7.6149; 2.1484; 2.3961; 0.5630)
θ_1	1.512	1.631	1.872	B_3(6.7676; 8.3605; 2.3501; 2.4976; 0.5142)
θ_2	1.491	1.612	1.857	B_3(7.5884; 8.1397; 2.6781; 2.4982; 0.4882)
θ_3	1.517	1.638	1.885	B_3(8.1449; 7.2651; 3.0338; 2.4418; 0.4880)
θ_0, θ_1	1.402	1.505	1.709	B_3(8.1449; 7.2650; 3.0338; 2.1431; 0.5015)
θ_0, θ_2	1.393	1.496	1.703	B_3(7.5234; 7.3134; 2.7694; 2.1076; 0.5035)
θ_0, θ_3	1.390	1.496	1.713	B_3(8.0187; 7.7542; 2.7862; 2.1751; 0.4800)
θ_1, θ_2	1.414	1.525	1.749	B_3(8.6702; 7.5387; 2.9284; 2.2036; 0.4600)
θ_1, θ_3	1.375	1.475	1.675	B_3(8.6702; 7.5387; 2.9284; 2.0887; 0.4740)
θ_2, θ_3	1.350	1.447	1.640	B_3(9.0132; 7.9999; 2.8585; 2.0644; 0.4635)
θ_0, θ_1, θ_2	1.324	1.422	1.621	B_3(10.7806; 8.4043; 3.2432; 2.1461; 0.4150)
θ_0, θ_1, θ_3	1.333	1.431	1.629	B_3(10.3455; 8.0495; 3.5687; 2.1993; 0.4463)
θ_0, θ_2, θ_3	1.296	1.388	1.575	B_3(10.3223; 7.7893; 3.3393; 2.0021; 0.4358)
θ_1, θ_2, θ_3	1.299	1.394	1.584	B_3(10.5957; 8.2600; 3.2334; 2.0676; 0.4194)
θ_0, θ_1, θ_2, θ_3	1.235	1.321	1.494	B_3(9.9689; 7.3418; 3.4037; 1.8225; 0.4438)
Watson test				
θ_0	0.127	0.157	0.228	B_3(3.6769; 4.4438; 9.8994; 0.6805; 0.0082)
θ_1	0.124	0.153	0.223	B_3(3.4122; 4.9262; 9.6902; 0.7643; 0.0087)
θ_2	0.117	0.146	0.215	B_3(6.0296; 3.7175; 22.6978; 0.7115; 0.0057)
θ_3	0.121	0.150	0.220	B_3(7.4154; 3.9208; 22.4649; 0.6800; 0.0022)
θ_0, θ_1	0.096	0.116	0.164	B_3 (3.5230; 4.4077; 9.2281; 0.4785; 0.0104)
θ_0, θ_2	0.093	0.114	0.161	B_3 (4.0651; 4.8643; 9.5614; 0.4903; 0.0078)
θ_0, θ_3	0.092	0.113	0.162	B_3 (4.4170; 4.9456; 10.4292; 0.5005; 0.0067)
θ_1, θ_2	0.099	0.123	0.181	B_3 (5.5181; 4.1815; 16.0852; 0.5478; 0.0055)
θ_1, θ_3	0.089	0.108	0.151	B_3 (5.7461; 4.4051; 13.9768; 0.4528; 0.0060)
θ_2, θ_3	0.084	0.101	0.141	B_3 (5.9952; 4.3409; 13.8757; 0.4020; 0.0060)
θ_0, θ_1, θ_2	0.077	0.093	0.131	B_3 (5.5809; 4.9570; 14.1052; 0.4540; 0.0060)
θ_0, θ_1, θ_3	0.080	0.097	0.137	B_3 (5.8959; 4.4478; 14.5923; 0.4132; 0.0060)
θ_0, θ_2, θ_3	0.072	0.087	0.121	B_3 (6.1780; 4.6712; 14.5568; 0.3791; 0.0060)
θ_1, θ_2, θ_3	0.072	0.087	0.121	B_3 (6.1780; 4.6712; 14.5568; 0.3791; 0.0060)
θ_0, θ_1, θ_2, θ_3	0.062	0.074	0.101	B_3 (7.3816; 4.4215; 14.1896; 0.2616; 0.0053)

TABLE 7. Levels of Significance for the Kuiper and Watson Statistics with Different Sample Sizes N

Statistic	10^3	10^4	10^5	10^6
$V_n^{mod} = 1.1113$	0.479	0.492	0.493	0.492
$U_n^2 = 0.05200$	0.467	0.479	0.483	0.482

0.374	0.236	2.081	1.198	0.692	0.599	0.811	0.274	1.311	0.534
1.048	1.411	1.052	1.051	4.682	0.111	1.201	0.375	0.373	3.694
0.426	0.675	3.150	0.424	1.422	3.058	1.579	0.436	1.167	0.445
0.463	0.759	1.598	2.270	0.884	0.448	0.858	0.310	0.431	0.919
0.796	0.415	0.143	0.805	0.827	0.161	8.028	0.149	2.396	2.514
1.027	0.775	0.240	2.745	0.885	0.672	0.810	0.144	0.125	1.621

The shape parameters θ_0 and θ_1 and the scale parameter θ_2 are estimated from the sample, while the shift parameter $\theta_3 = 0$ is assumed known. The maximum likelihood estimates of the parameters found for this sample are $\hat{\theta}_0 = 0.7481$, $\hat{\theta}_1 = 0.7806$, $\hat{\theta}_2 = 1.3202$. The calculated values of the statistics are $V_n^{mod} = 1.1113$ for Kuiper and $U_n^2 = 0.05200$ for Watson. The distributions of the test statistics in this case depend on θ_0 and θ_1 [20, 22] but not on θ_2, and must be found for $\theta_0 = 0.7481$, $\theta_0 = 0.7806$.

The attained levels of significance for the tests $P\{V_n^{mod} \geq 1.1113\}$ and $P\{U_n^2 \geq 0.05200\}$ were obtained for different accuracies of modelling of the distributions of the statistics (for different sizes N of the modelled samples of statistics) and are listed in Table 7.

Thus, the models of distributions of statistics and tables of percentage points given here make it possible to apply the Kuiper and Watson tests correctly when testing composite hypotheses regarding a series of parametric models for distributions. The interactive method employed here offers the possibility of correctly using the tests when the distribution of the test statistics corresponding to correctness of the test hypothesis H_0 is not known prior to the time it is used.

This work was supported by the Ministry of Education and Science of the Russian Federation as part of the government task (Project 8.1274.2011) and the Federal Targeted Program on Scientific and Teaching Staff for an Innovative Russia (Agreement No. 14.V37.21.0860).

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