Proceedings of the International MultiConference of Engineers and Computer Scientists 2012 Vol II,
IMECS 2012, March 14 - 16, 2012, Hong Kong

Application Of Optimal Homotopy Asymptotic Method For Non- Newtonian Fluid Flow In A Vertical Annulus

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Abstract— **In this paper, the flow of an incompressible non Newtonian fluid in a vertical annulus is considered. The fluid is governed by Sisko fluid model and is assumed to flow upwards under the influence of the pressure gradient and gravity. The non linear momentum equation is then solved using the optimal Homotopy asymptotic method (OHAM). The effect of the power index n, the material parameter η and the pressure gradient on the velocity and the stress are explored and presented. It is well known that the momentum flux changes its sign at the same value of the non dimensional radius for which the velocity is maximum. The same has been observed in the present study for Sisko fluids. Further, it is also observed that for negative pressure gradient, the influence of g is more on the shear thinning fluids than that of Newtonian and shear thickening fluids. Thus the second degree approximation of the solution obtained using OHAM is suffice to find analytical solutions to the above mentioned category of problems.**

***Index Terms*— Sisko fluid, vertical annulus, Optimal Homotopy Asymptotic method, Pressure gradient, momentum flux**

I. Introduction

The flow of fluid through vertical annulus is a classical problem that had attracted several researchers owing its enormous applications in real life. An analytical solution to the flow of viscous Newtonian fluid through vertical annulus can be found in the classical textbooks of Bird. et.al [1]. In real life, many fluids exhibit non- Newtonian nature. The study of the flow of such fluids is necessary due to its numerous technical and industrial applications. For instance, the flow of non Newtonian fluids through vertical annulus finds applications in the petroleum industry, oil refining industries, food industries etc. As a single fluid model cannot describe the behavior of all non Newtonian fluids, several fluid models have been proposed. Quite a number of studies were undertaken to understand the flow behavior of fluids in annulus using power law model, visco elastic model and PTT visco elastic fluid model. Studies [2] – [6] present the analytical and numerical results of such fluid flows in concentric annulus.

The fluid models considered in the above mentioned problems can model several categories of non Newtonian fluids, but cannot effectively analyze the flow of grease like fluids. It is found that the flow of such fluids in concentric annulus also has several industrial applications. These fluids have a typical behavior, that they have high viscosities at low shear rates and low viscosities at high shear rates. Hence, Sisko, in 1958 has proposed a model to understand the rheology of the grease like fluids [7]. Basing on this model, a few numerical studies were undertaken to understand the flow behavior of this fluid in concentric annulus[8, 9]. But there is a definite necessity to supplement it with an analytical solution.

The stress tensor equation for Sisko Fluid (time independent fluid) can be given as

$$S = -pI + \tau \tag{1}$$

$$\tau = \left[m + \eta \left| \sqrt{\tfrac{1}{2} + tr(A_1^2)} \right|^{n-1} \right] A_1 \tag{2}$$

where, A_1 is the first Rivlin-Erickson tensor, m, η and n are material parameters. The presence of the power index n in the constitutive equation of the sisko fluid given by equation (1) makes the Navier Stoke's momentum equation more non linear. And it is impossible to find its exact solution. Therefore, one must resort to asymptotic or numerical methods to obtain approximate solutions to the equation. The OHAM can be seen to be a much effective technique for solving the nonlinear equations without any need for linearization or any restrictive assumptions [10-14].

Hence, it is proposed to solve the problem of Sisko fluid flow in a concentric annulus, using OHAM and find the solution to the problem under steady state conditions and in the absence of body forces.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

A. *Problem Geomerty*

Consider the problem of the flow of an incompressible non Newtonian fluid governed by the Sisko model through a vertical annular region between two coaxial cylinders of radii r_i and r_o respectively as shown in Fig(1). The fluid is assumed to flow upwards i.e. in the direction opposed to the gravity due to the pressure gradient at the ends. The cylindrical coordinate system with Z-axis along the axis of symmetry of the coaxial cylinders is chosen.

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ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Further it is assumed that the flow is steady, laminar.
Under these assumptions, the equation of continuity and the momentum equations take the form

$$\nabla . V = 0 \quad (3)$$

$$K + \frac{1}{\rho}\nabla . S = (V.\nabla)V \quad (4)$$

where K is the body force which is taken to be zero and τ is as defined in Eq.(2)
The velocity vector is given by

$$\bar{q} = \bar{q}(0,0, v(r)) \quad (5)$$

In the cylindrical coordinates, the Revlin-Erickson tensor is given by

$$A_1 = 2d = \begin{bmatrix} 2\frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial r} \\ + & 2\left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\right) & \frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ + & + & 2\frac{\partial v_z}{\partial z} \end{bmatrix} \quad (6)$$

where + denotes a symmetric entry.
By interring the velocity field given by equation (5) and on simplification we have:

$$\left|\sqrt{\frac{1}{2}tr(A_1^2)}\right|^{n-1} = \left(-\frac{dv}{dr}\right)^{n-1} \quad (7)$$

Further, using (2), the only non vanishing stress component is given by

$$\tau_{rz} = m\frac{dv}{dr} - \eta\left(-\frac{dv}{dr}\right)^n \quad (8)$$

And,

$$\nabla . S = -\frac{\partial p}{\partial z} + m\frac{d^2v}{dr^2} - \eta\frac{d}{dr}\left(\frac{dv}{dr}\right)^n + \frac{1}{r}\left(m\frac{dv}{dr} + \eta\left(\frac{dv}{dr}\right)^n\right) \quad (9)$$

Thus, by using equations (1) and (9) in momentum equation (4), we can obtain the equation of sisko fluid for our problem in the cylindrical coordinates as follows:

- r-momentum $\quad \frac{dp}{dr} = 0 \quad (10)$
- θ-momentum $\quad \frac{dp}{d\theta} = 0 \quad (11)$
- z-momentum

$$m\frac{d^2v}{dr^2} + \eta n\left(-\frac{dv}{dr}\right)^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[m\frac{dv}{dr} - \left(-\frac{dv}{dr}\right)^n\right] + \frac{dp}{dz} = 0 \quad (12)$$

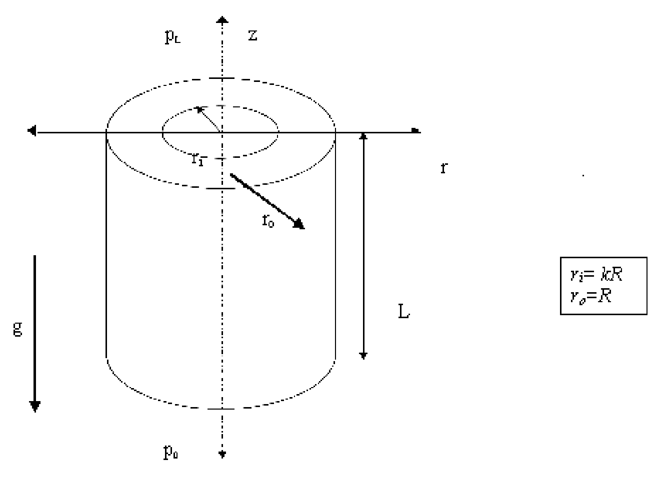

Fig. 1. Problem Geometry.

Also, the pressure term in equation (12) is taken as $p = P + \rho g z$ (refer [1]) (13)
where P is the pressure on the solid boundary and g is the acceleration due to gravity.
Using the non slip condition on the solid boundary, the boundary conditions are

$$v = 0 \; at \; r = kR$$
$$v = 0 \; at \; r = R \quad (14)$$

Using the non dimensional scheme given by

$$\overset{*}{r} = \frac{r}{R};\; \overset{*}{v} = \frac{v}{\bar{u}};\; \theta = \frac{\eta}{m}\left(\frac{\bar{u}}{R}\right)^{n-1};\; \overset{*}{z} = \frac{z}{R} \quad (15)$$

where ū is average velocity of the fluid flow

As, $\frac{dP}{dz} = \rho g = Ps\ (Constant)$ (16)

the dimensionless equation for the flow of Sisko fluid in the annular , after dropping '*' is

$$\frac{d^2v}{dr^2} + n\theta\left(-\frac{dv}{dr}\right)^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[\frac{dv}{dr} - \theta\left(-\frac{dv}{dr}\right)^n\right] + Ps = 0 \quad (17)$$

and the dimensionless boundary conditions are

$$v = 0 \; at \; r = k \quad (18)$$
$$v = 0 \; at \; r = 1$$

Eq. (8) after non dimensionalization is given by

$$\tau_{rz} = \frac{dv}{dr} - \theta\left(-\frac{dv}{dr}\right)^n \quad (19)$$

B. OHAM (Optimal Homotopy Asymptotic Method)

Consider the differential equation

$$L(u(x)) + g(x) + N(u(x)) = 0; \quad B(u) = 0 \quad (20)$$

where, L is a linear operator, x denotes independent variable, $u(x)$ is an unknown function, $g(x)$ is a known function, $N(u(x))$ is a nonlinear operator and B is a boundary operator.
A family of equations is constructed using (OHAM) which is given by

$$(1-p)L(\Phi(x,p) + g(x)) = H(p)\left(L(\Phi(x,p) + g(x) + N(\Phi(x,p))\right)$$
$$B(\Phi(x,p)) = 0 \quad (21)$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $\phi(x, p)$ is an unknown function, respectively. Obviously, when $p=0$ and $p = 1$, we have that

$$\Phi(x,0) = u_0(x); \quad \Phi(x,1) = u(x) \quad (22)$$

Thus, as p increases from 0 to 1, the solution $\phi(x, p)$ varies from $u_0(x)$ to the solution $u(x)$ of equation (20), Here $u_0(x)$ is obtained from Eq. (21) for $p=0$ as

$$(L(u_0(x,p)) + g(x)) = 0, B(u_0) = 0 \quad (23)$$

Auxiliary function $H(p)$ is chosen in the form

$$H(p) = pc_1 + p^2c_2 + \ldots \quad (24)$$

where the constants c_1, c_2 ,... can be determined later by the method of least squares.
Let us consider the solution of Eq. (21) in the form,

$$\Phi(x,p,c_{ii}) = u_0(x) + \sum_{r=1}^{\infty} u_r(x,c_{ii})p^r, i = 1,2..m \quad (25)$$

Now substituting Eq.(25) into Eq. (21) and equating the coefficients of like powers of p, the solutions $u_0(x)$, $u_1(x, c_1)$ … can be found. If,

$$\tilde{u}(x, c_{ii}) = u_0(x) + \sum_{i=1}^{m} u_{ii}(x, c_{ii}) \quad (26)$$

Then the constants c_i , $i = 1,2..m$ are obtained by substituting Eq. (26) into Eq. (1) which results the following residual

$$R(x, c_{ii}) = L(\tilde{u}_m(x, c_i) + g(x) + N(\tilde{u}_m(x, c_i)), i = 1,2..m \quad (27)$$

If $R=0$, then $\tilde{u}$ will be the exact solution. In nonlinear problems this condition doesn't happen. Hence, the optimal values of c_i, $i = 1,2..m$ can be obtained by many methods like the method of Least Squares, Galerkin's Method, Ritz Method, and Collocation Method . We apply the Method of Least Squares as shown below:

$$J(c_1, c_2, \ldots\ldots, c_m) = \int_a^b R^2(c_1, c_2, \ldots\ldots, c_m)dx \quad (28)$$

The constants c_i , $i = 1,2..m$ can be evaluated using

$$\frac{\partial J}{\partial c^1} = \frac{\partial J}{\partial c^2} = \cdots = \frac{\partial J}{\partial c_m} = 0 \quad (29)$$

where a and b denote the domain of the problem.

C. Solution to the problem using OHAM

To obtain the solution of Eq. (17), let us take

$$L(v) = \frac{d^2v}{dr^2} + Ps \quad (30)$$

$$N(v) = n\theta\left(-\frac{dv}{dr}\right)^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[\frac{dv}{dr} - \theta\left(-\frac{dv}{dr}\right)^n\right] \quad (31)$$

According to OHAM given by Eq.(21), we have

$$(1-p)\left(\frac{d^2v}{dr^2} + Ps\right) - H(p)\left\{\frac{d^2v}{dr^2} + Ps + n\theta\left(-\frac{dv}{dr}\right)^{n-1}\frac{d^2v}{dr^2} + \frac{1}{r}\left[\frac{dv}{dr} - \theta\left(-\frac{dv}{dr}\right)^n\right]\right\} = 0 \quad (32)$$

Let $v = v_0 + pv_1 + p^2v_2$;
and $H = pc_1 + p^2c_2$ (33)

Then, using Eq. (33) in Eq.(32), we have,

$$(1-p)(v_0'' + pv_1'' + p^2v_2'' + Ps) - (pc_1 + p^2c_2)\{(v_0'' + pv_1'' + p^2v_2'' + Ps) + n\theta(-v_0' - pv_1' - p^2v_2')^{n-1}(v_0'' + pv_1'' + p^2v_2'') + \frac{1}{r}[v_0' + pv_1' + v_2' - \theta(-v_0' - pv_1' - p^2v_2')^n]\} = 0 \quad (34)$$

Solution to equation (34) or shear thinning fluids: (i.e) n=0.
Coefficient of p^0 :

$$v_0'' = -2Ps$$
$$v_0(k) = 0, \quad v_0(1) = 0 \quad (35)$$

Coefficient of p^1 :

$$v_1'' = 2*Ps + 2*c_1*Ps - \frac{c_1\theta}{x} + \frac{c_1 v_0'}{x} + v_0'' + c_1 v_0''$$
$$v_1(k) = 0, \quad v_1(1) = 0 \quad (36)$$

……

Similarly, collecting the coefficient p^2 and solving Eq. (35), (36).., gives a second order approximation to the velocity as

$$v(r) = v_0 + \ v_1 + \ v_2 \quad (37)$$

The constants c_0 and c_1 are calculated using Eq. (28) and Eq. (29) with a=k and b=1.
Using equation (37) in (19) gives the non dimensional stress component.
For n=1 corresponding to the Newtonian case and for n=2 corresponding to the shear thickening fluids, OHAM equations are written and are solved using Mathematica software for different values of the pressure gradient (Ps), the material parameter (η). The results are presented through graphs.

III. CONCLUSIONS

Figures (2) to (4) show Velocity and Shear Stress profiles for different values of the non dimensional material parameter (θ) and power index (n). We can observe that the change in momentum flux changes sign at the same value of r* for which the velocity is maximum. Figures (5) to (10) displays the variation of velocity and shear stress profiles for Shear thinning fluids, Newtonian and shear thickening fluids. It is seen that the effect of gravity on the flow of Newtonian fluid is insignificant (Figure (6)) when compared to that of the shear thinning (figure (5)) and shear thickening fluids (figure (7)). It is also observed that the strong shear thickening effects increase for increase in the power index n. (Fig (8) - (10). These results are also similar to the ones that are observed by Khan et.al in their study of the heat transfer of Sisko fluid in an annular pipe [15].

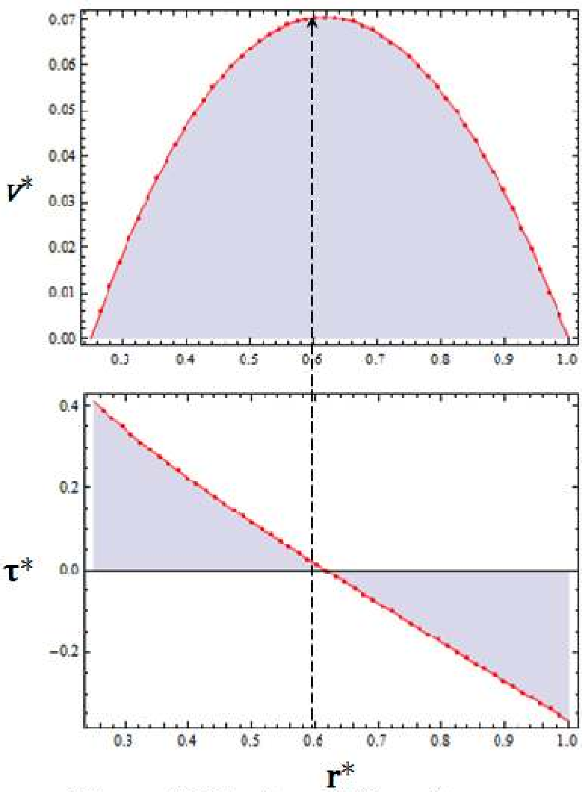

Figure : 2 Velocity and Shear Stress profiles for θ=0 and n=0

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

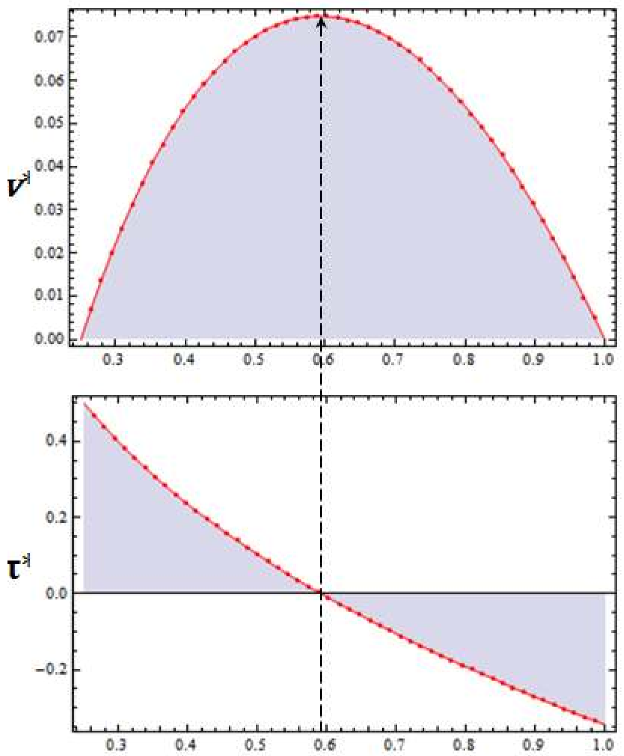

Figure : 3Velocity and Shear Stress profiles for θ=0 and n=1

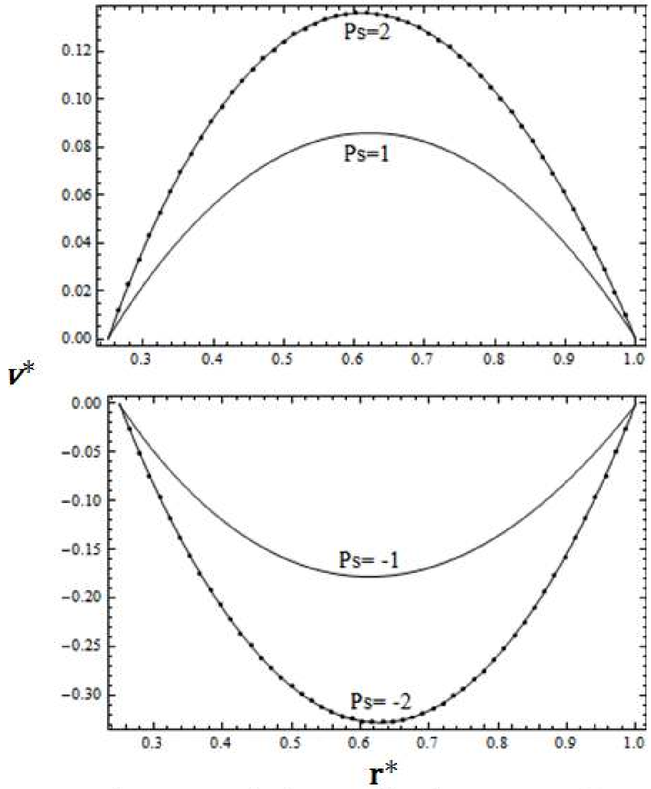

Figure : 5 Velocity profiles for θ=0.2 and n=0

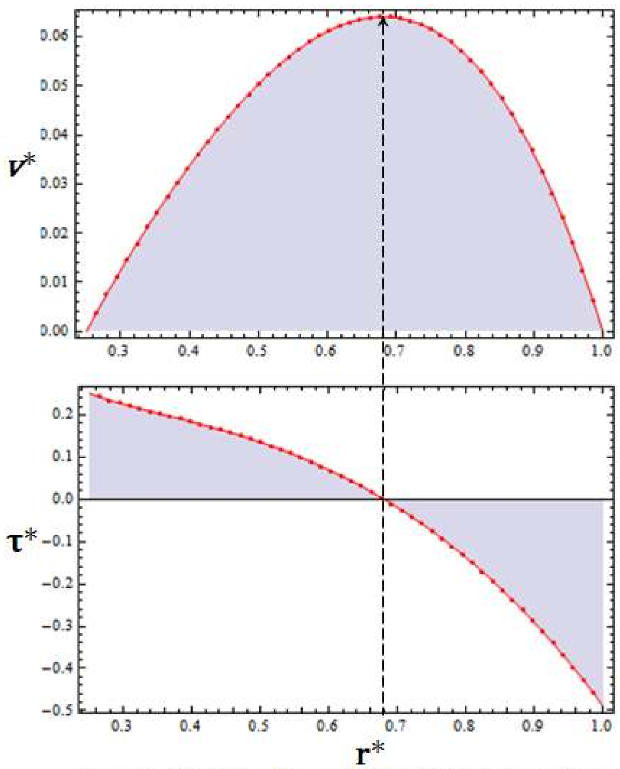

Figure : 4 Velocity and Shear Stress profiles for θ=0.2 and n=2

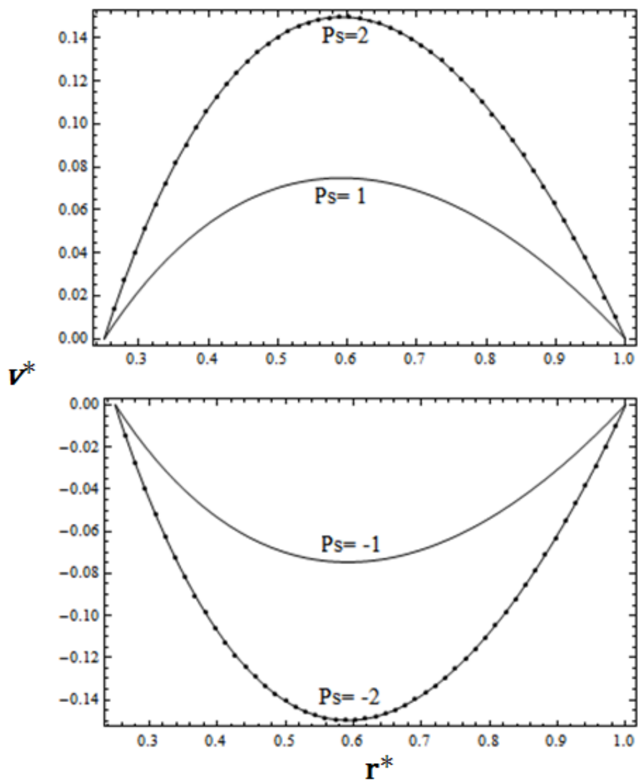

Figure : 6 Velocity profiles for θ=0 and n=1

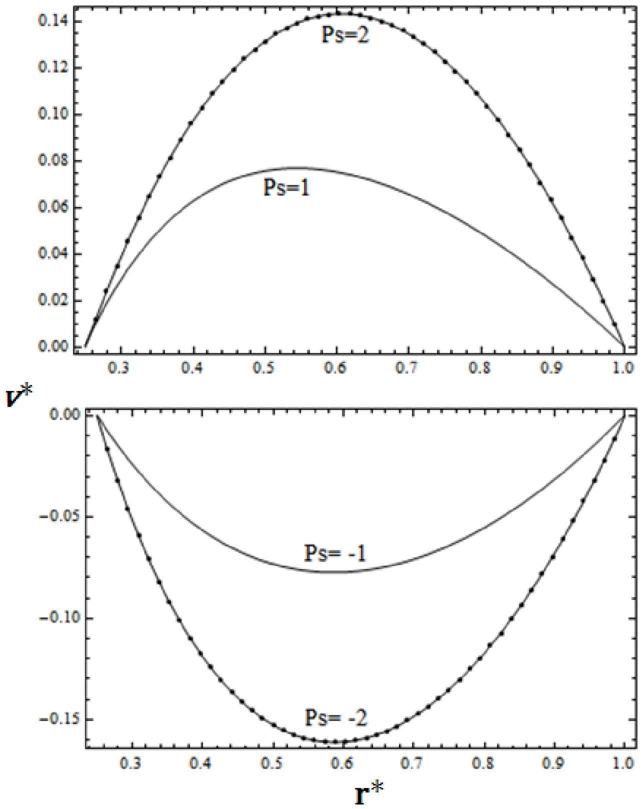

Figure : 7 Velocity profiles for θ=0.2 and n=2

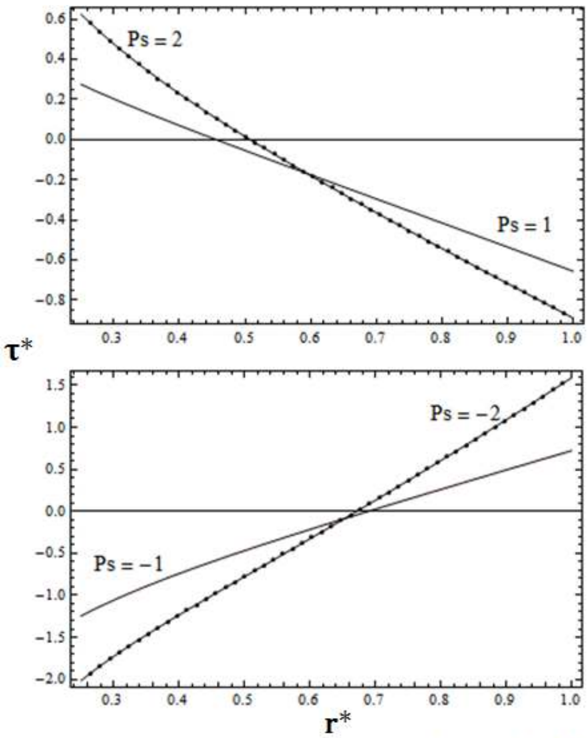

Figure : 8 Shear Stress profiles for θ=0.2 and n=0

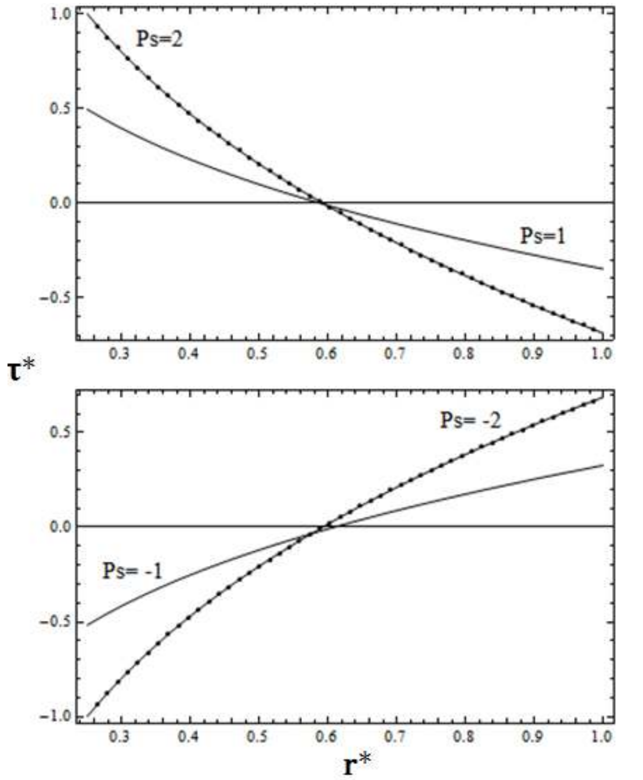

Figure : 9 Shear Stress profiles for θ=0 and n=1

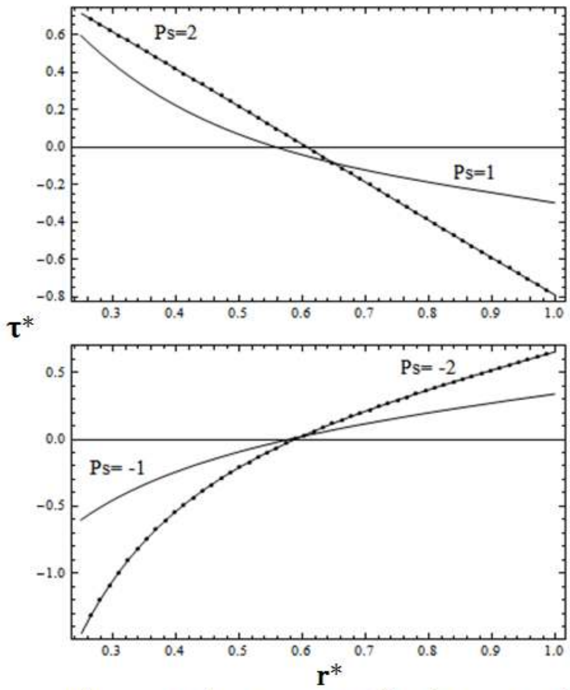

Figure : 10 Shear Stress profiles for θ=0.2 and n=2

Acknowledgements:
The authors wish to thank Dr. D. JayaKrishna, Assistant Professor, Dept of Mechanical Engineering, Bits Pilani Hyderabad Campus for his valuable contribution.

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ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)