

APPLICATION OF ORTHOGONAL TRANSFORMATIONS IN THE REVISED SIMPLEX METHOD

Evald ÜBI

Institute of Economics, Tallinn Technical University, Kopli 101, 11711 Tallinn, Estonia;
evaldy@tv.ttu.ee

Received 23 March 2000, in revised form 10 November 2000

Abstract. Unlike the classical method, orthogonal transformations are used for computing more precise simplex tables. These transformations are stored as products. With these transformations the basic matrix is turned to the triangular form, which is used instead of the reciprocal matrix. The values of initial and dual variables are found from the triangular system. The performed algorithm MULT is appropriate for solving ill-conditioned problems and it is more labour-consuming than the classical revised simplex method. Examples of solving the ill-conditioned problems are given.

Key words: linear programming, least squares method, ill-conditioned problems.

1. USE OF THE REFLECTIONS OF HOUSEHOLDER IN THE REVISED SIMPLEX METHOD

The simplex method is based on the Gauss elimination. If instead of this elimination orthogonal transformations are used, then (analogously to solving a linear system of equations) it is possible to solve problems precisely (see the examples in Sec. 3). Highly developed techniques of the least squares method (see [1]) are applied to solving problems of small size linear programming also in [2]. In the latter paper the algorithm VRMSIM is introduced to transform, analogously to the standard simplex method, all elements of the matrix A at every step.

Another approach is presented in [3], where a system of orthogonal vectors is formed for solving on computers. The classical revised simplex method is based on computing the inverse matrix. In the present paper, instead of the inverse matrix, the basic matrix in triangular form is used.

Consider the problem of linear programming

$$\begin{aligned} z = cx &\rightarrow \max, \\ Ax = b, \quad x &\geq 0, \end{aligned} \tag{1}$$

where A is an $m \times n$ matrix, b is an m -vector, c and x are n -vectors.

Assume that the variables $x(1), x(2), \dots, x(m)$ form an initial basis and denote the basic matrix by B , $B = (a(1), a(2), \dots, a(m))$. At every step of the revised simplex method the following two systems are to be solved:

$$Bx = b, \tag{2}$$

$$uB = \bar{c}, \tag{3}$$

where \bar{c} is the part of the vector c corresponding to basic variables, $u = (u(1), u(2), \dots, u(m))$ (see [4]). Let Q be an orthogonal $m \times m$ matrix such that $QB = R$ is an upper triangular matrix. It is possible to find the basic variables from the system

$$Rx = Qb, \tag{4}$$

where $R = QB$. The dual variables are sought in the form $u = tQ$, where t is an m -vector. The vector t can be found from the system

$$tR = \bar{c}. \tag{5}$$

The matrix R has already been found by solving the system (4). After that compute $u = tQ$ or $u = (Q^{-1}t^T)^T$.

It is important to stress that there is no need to compute the matrix Q ; it will be presented as a product of $m - 1$ orthogonal transformations (see [1], Ch. 10, 11). The necessary information is stored under the main diagonal of the triangular matrix R (instead of zeros) and in the m -vector W . Finding the inverse matrix Q^{-1} is not necessary either. In order to find the dual variables $u = (Q^{-1}t^T)^T$, one has to fulfill these $m - 1$ orthogonal transformations to the vector t^T in reverse order. The matrix B is transformed to the triangular form by using the reflections of Householder. To nullify the elements of the first column under the main diagonal, we need to find the hyperplane of the reflection, so that the second, third, ..., m th component of the reflection would be equal to zero. The first component of normal Y of this kind of hyperplane is as follows:

$$Y(1) = b(1, 1) + \sigma \sqrt{b^2(1, 1) + \dots + b^2(m, 1)},$$

where $\sigma = +1$ if $b(1, 1) \geq 0$, and in the opposite case $\sigma = -1$. Remember the first elements of the normals in the m -vector W : $W(1) = Y(1)$. We should prove that the other components of these normals are $Y(i) = b(i, 1), i = 2, \dots, m$.

It means that as a result of the first reflection these $m - 1$ elements do not change in the computer's memory, only the meaning is changed (see Example 1 below). The rest of the coefficients of the system (2) are transformed according to the formulas

$$\beta = - \| Y \|^2 / 2, \quad t(j) = \beta^{-1}(Y, b(j)), \quad b(j) = b(j) + t(j)Y, \quad (6)$$

where $b(j)$ denotes a column of the matrix B or the right side of this system. Analogously, the next reflection of Householder, performed with the $m - 1$ dimensional columns, nullifies the elements of the second column under the main diagonal, instead of zeros the normal of the second reflection is left, etc. After the $(m - 1)$ th reflection (the product of which is denoted by Q in the formula (4)), the system (2) has the upper triangular matrix R . The values of the basic variables are found by solving this system. The vector t can be found from the system (5) with the matrix R . To compute the vector u of dual variables, these $m - 1$ reflections of Householder are executed in reverse order.

Let us consider now the system

$$\sum_{j=1}^m b(j)\lambda(j) = a(k), \quad (7)$$

where the column $a(k)$ is entered into the basis. As the triangular matrix $R = QB$ is above the matrix B , find first the vector $Qa(k)$. For this purpose perform $m - 1$ reflections of Householder for the column $a(k)$ by using the under part of the main diagonal of the matrix R . The vector λ is found from the triangular system

$$R\lambda = Qa(k), \quad (8)$$

where $R = QB$. The variable $x(l)$ leaving the basis is determined as usual based on the vectors λ and x . In the triangular matrix R corresponding to the new basis the columns $1, 2, \dots, l - 1$ are not changed. The rest of the columns are replaced with untransformed columns $A(l + 1), A(l + 2), \dots, A(m)$ of the matrix A , then the column $A(k)$ coming to the basis and finally the right side b are added. Next fulfill the first $l - 1$ reflections, determined by the part under the main diagonal of the matrix R , to added columns. Finally transform the added part to the triangular form and leave the normals under the main diagonal. From the triangular system find a new basic solution, etc.

2. THE DESCRIPTION OF THE ALGORITHM MULT

This algorithm solves the problem of linear programming with the revised simplex method. The $m \times n$ matrix A , the m -vector b , and the n -vector c are used for storing the coefficients of the systems (2) and (3). In addition, the $m \times (m + 1)$ matrix R is used, where under the main diagonal and in the m -vector W the normals of reflections of Householder are stored. The values of the dual variables and the solution of the system (8) are stored in the n -vector u . Two more vectors, x and IJ (the index of basic variables), are used.

2.1. The algorithm MULT($A, b, c, x, u, W, IJ, R, m, n, EP1$)

1. Insert $m, n, A, B, EP1$ (the accuracy of calculations).
2. Insert m indexes of the initial basis to the array IJ (the m columns of the matrix A corresponding to the indexes must be linearly independent).
3. Write the columns of A corresponding to basic variables to the matrix R and the vector b to the column $R(m+1)$.
4. Assign $l = 1$.
5. Perform $(m - l)$ transformations of Householder to columns $l, l+1, \dots, m, m+1$ of the matrix R , leave the normals of these transformations under the main diagonal and in the array W .
6. Find the basic variables from the system $Rx = Qb$, transformed to the triangular form in the last step.
7. Write to the column $R(m+1)$ the vector \bar{c} corresponding to basic variables.
8. Solve the system (5), write the solution to the $(m+1)$ th column of the matrix R .
9. Find the dual variables u by performing $(m - 1)$ transformations of Householder to the column $R(m+1)$; the transformations are determined by the $(m - 1)$ th, $(m - 2)$ th, ..., 2nd, 1st column of the matrix R .
10. Assign $u(i) = R(i, m+1), i = 1, \dots, m$.
11. Find $RE = \max [(u, A(j)) - c(j)] = ((u, A(k)) - c(k))$, where j is the nonbasic index.
12. If $-EP1 < RE$, then go to step 23.
13. Enter $x(k)$ into the basis, $IJ(m+1) = k$.
14. Assign $R(m+1) = A(k)$.
15. Perform $(m - 1)$ reflections of Householder to the column $R(m+1)$ (these reflections are determined by the part under the main diagonal of the matrix R and the vector W).
16. Solve the triangular system (8), write the solution to the vector u .
17. For $u(j) > 0$ find $\min \{x(j)/u(j)\} = x(j1)/u(j1)$.
18. If all $u(j) \leq 0$, then the goal function is unbounded. Stop.
19. Eliminate the l th variable $x(j1)$ from the basis, $x(j1) = 0$.
20. Assign $IJ(i) = IJ(i+1), i = l, l+1, \dots, m$.
21. Eliminate the l th column from the matrix R and replace the next columns: $R = (R(1), R(2), \dots, R(l-1), A(l+1), \dots, A(m), A(k), b)$.
22. Go to step 5.
23. The problem is solved.

2.2. Comments

The triangular systems (4) and (8) with the same matrix R for finding the vectors x and λ can be solved simultaneously.

The algorithm has been worked out for solving the system of linear equations with sparse matrixes, see [5]. However, we do not know whether it is possible to guarantee that by changing the order of basic variables of the sparse matrix A the matrix R will be sparse too. Finding an answer to this question is the task of further investigations.

Example 1.

Let us solve the following example from [6], Ch. 4 with the algorithm MULT by taking the accuracy $EP1 = 10^{-15}$.

$$z = 4x(1) + 5x(2) + 9x(3) + 11x(4) \rightarrow \max,$$

$$x(1) + x(2) + x(3) + x(4) \leq 15,$$

$$7x(1) + 5x(2) + 3x(3) + 2x(4) \leq 120,$$

$$3x(1) + 5x(2) + 10x(3) + 15x(4) \leq 100,$$

$$x \geq 0.$$

The initial basis is formed by the additional variables $x(5), x(6), x(7)$.

Step 0

$x(5)$	$x(6)$	$x(7)$	$R(4)$
1	0	0	15
0	1	0	120
0	0	1	100

Step 1

$x(5)$	$x(6)$	$x(7)$	$R(4)$	t	u	W	x
-1	0	0	-15	0	0	1	15
0	-1	0	-120	0	0	1	120
0	0	1	100	0	0	0	100

Step 2

$x(5)$	$x(6)$	$x(4)$	$A(4)$	$QA(4)$	λ
-1	0	0	1	-1	1
0	-1	0	2	-2	2
0	0	1	15	15	15

$x(5)$	$x(6)$	$x(4)$	$R(4)$
-1	0	1	15
0	-1	2	120
0	0	15	100

$x(5)$	$x(6)$	$x(4)$	$R(4)$	t	u	W	x
-1	0	-1	-15	0	0	1	8.33
0	-1	-2	-120	0	0	1	106.67
0	0	15	100	0.73	0.73	0	6.67

Step 3

$x(5)$	$x(6)$	$x(4)$	$A(1)$	$QA(1)$	λ
-1	0	-1	1	-1	0.8
0	-1	-2	7	-7	6.6
0	0	15	3	3	0.2

$x(6)$	$x(4)$	$x(1)$	$R(4)$
0	1	1	15
1	2	7	120
0	15	3	100

$x(6)$	$x(4)$	$x(1)$	$R(4)$	t	u	W	x
-1	-2	-7	-120	0	2.25	0	37.92
1	15.03	3.06	100.78	0.73	0	-1	4.58
0	15	-0.80	-8.31	-2.21	0.58	0	10.42

Step 4

$x(6)$	$x(4)$	$x(1)$	$A(3)$	$QA(3)$	λ
-1	-2	-7	1	-3	-1.08
1	15.03	3.06	3	10.04	0.58
0	15	-0.80	10	-0.33	0.42

$x(6)$	$x(1)$	$x(3)$	$R(4)$
-1	1	1	15
1	7	3	120
0	3	10	100

$x(6)$	$x(1)$	$x(3)$	$R(4)$	t	u	W	x
-1	-7	-3	-120	0	-1.86	0	46.43
1	3.16	9.80	99.61	1.26	0	-1	7.14
0	3	2.21	17.39	-1.53	0.71	0	7.86

The optimal values of basic variables, $x(6) = 325/7$, $x(1) = 50/7$, $x(3) = 55/7$, are determined with an accuracy of 10^{-13} from the system

$$-x(6) - 7x(1) - 3x(3) = -120,$$

$$3.16x(1) + 9.80x(3) = 99.61,$$

$$2.21x(3) = 17.39,$$

whose coefficients are presented with an accuracy of 0.01. Exact solution of the problem is $x(1) = 50/7$, $x(2) = x(4) = 0$, $x(3) = 55/7$.

3. THE RESULTS OF CALCULATIONS

All computations were performed on an IBM-4381 using FORTRAN codes. For all variables double precision was used.

Example 2.

$$\begin{aligned} z &= x(1) + x(2) + x(3) + x(4) \rightarrow \max, \\ (1+t)x(1) + x(2) + x(3) + x(4) &\leq 4+t, \\ x(1) + x(3) + x(4) &\leq 3, \\ x(1) + x(4) &\leq 2, \\ x &\geq 0. \end{aligned}$$

The maximum value of objective functions is $z = 4 + t$, $t > 0$. The algorithm MULT found in the case of $t = 0.0000000001$ the solution $x = (0; 2.000000000100000; 0; 2)$. It should be noted that on the last step the estimate of the nonbasic variable is equal to 10^{-10} .

Example 3.

Let us consider a linear problem with the Hilbert matrix $H(m)$, see [2]. The optimal solution was found for $m = 3$ with the accuracy $\delta = 10^{-13}$, for $m = 7$ with the accuracy $\delta = 10^{-9}$, and for $m = 11$ with the accuracy $\delta = 10^{-3}$. If $m \geq 12$, then in solving the systems (4) and (5) the pivot element is smaller than computing accuracy and the algorithm does not reach the solution.

REFERENCES

1. Lawson, C. and Hanson, R. *Solving Least Squares Problems*. Prentice-Hall, New Jersey, 1974.
2. Übi, E. Conversion of the simplex tableau by the Givens rotation. *Proc. Estonian Acad. Sci. Phys. Math.*, 1991, **40**, 75–79.
3. Bulavskij, V. *Solving Linear Programming Problems*. Nauka, Moscow, 1977 (in Russian).
4. Hu, T. *Integer Programming and Network Flows*. Addison-Wesley, New Jersey, 1970.
5. George, A. and Liu, J. *Computer Solution of Sparse Positive Definite Systems*. Prentice-Hall, New Jersey, 1981.
6. Wagner, H. *Principles of Operations Research*. Prentice-Hall, New Jersey, 1969.

ORTOGONAALSETE TEISENDUSTE KASUTAMINE MODIFITSEERITUD SIMPLEKSMEEODIS

Evald ÜBI

On kirjeldatud simpleksmeetodi varianti, kus Gaussi elimineerimismeetodi asemel kasutatakse ortogonaalseid teisendusi. Viimased on küll töömahukamad, kuid võimaldavad ülesannet täpsemalt lahendada.