

Application of Parameterized Hesitant Fuzzy Soft Set Theory in Decision Making

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Abstract In this paper, by combining hesitant fuzzy soft sets (HFSSs) and fuzzy parameterized, we introduce the idea of a new hybrid model, fuzzy parameterized hesitant fuzzy soft sets (FPHFSSs). The benefit of this theory is that the degree of importance of parameters is being provided to HFSSs directly from decision makers. In addition, all the information is represented in a single set in the decision making process. Then, we likewise ponder its basic operations such as AND, OR, complement, union and intersection. The basic properties such as associative, distributive and de Morgan's law of FPHFSSs are proven. Next, in order to resolve the multi-criteria decision making problem (MCDM), we present arithmetic mean score and geometry mean score incorporated with hesitant degree of FPHFSSs in TOPSIS. This algorithm can cater some existing approach that suggested to add such elements to a shorter hesitant fuzzy element, rendering it equivalent to another hesitant fuzzy element, or to duplicate its elements to obtain two sequence of the same length. Such approaches would break the original data structure and modify the data. Finally, to demonstrate the efficacy and viability of our process, we equate our algorithm with existing methods.

Keywords Fuzzy Soft Set, Fuzzy Hesitant, Fuzzy Parameterized Hesitant Fuzzy Soft Sets

1. Introduction

The concept of fuzzy sets presented by Zadeh [1] has conquered an enormous achievement in numerous fields. The extension of the fuzzy sets and one that integrated with other theories have been applied by some researchers, including the intuitionistic fuzzy sets [2]–[6], fuzzy multiset [7]–[9] and fuzzy soft sets [10]–[14]. Torra and Norakawa [15]–[16] introduced hesitant fuzzy sets (HFSs)

in which the membership of an object to a concept is presented by a series of some different values between 0 to 1. HFSs can mirror human's hesitancy further objectively than the other usual extensions of fuzzy sets. Since its introduction, researchers have used it to answer numerous decision-making issues [17]–[38]. Aside from that, some researchers integrated HFSs with some other extensions of fuzzy sets. Lv et al. [39] studied on hesitant fuzzy information measures and their clustering application. Xu & Zhang [40] made an overview on the applications of the hesitant fuzzy sets in group decision-making.

Babitha and John [41] first studied the hesitant fuzzy soft sets (HFSSs) which are the hybrid structure between HFSSs and fuzzy soft set. They proposed the basic operation of HFSSs such as union, intersection, complement and proved the De Morgan's law. Consequently, Wang et al. [42] proposed HFSSs and their operations such as "AND", "OR" complement, union and intersection and their basic law properties. Beg and Rashid [43] presented the idea of an HFSSs where adaptation to manage the conditions in which experts assess an alternative giving to finite criteria in all possible values. They also proposed the distance measure between any two elements of the HFSSs. Rezaei and Rezaei [44] proposed distance and similarity measures for HFSSs by using well-known Hamming, Euclidean, and Minkowski distance measures while Li et al. [45] extended the concept of HFSS to generalized HFSSs.

Among the significant milestones in the development of hesitant fuzzy soft sets and their generalizations is the introduction of the fuzzy parameterized aspect. The fuzzy parameterized aspect was firstly established by Cagman et al. [46] who proposed the fuzzy parameterized fuzzy soft (FPFS) sets and their basic operations and followed by others [47]–[54].

Characteristics of this work are as follows:

1. We extend the definitions of HFSS [41], [42] to the fuzzy parameterized, allowing this theory to be enhanced by weighting each parameter, namely

FPHFSSs. We investigate certain connections between two FPHFSSs and certain plain, binary-based set operations for FPHFSSs. There is also a mention of the property of operator.

2. Some of the previous methods added the maximum value, minimum value or any value to the shorter one until both have the same HFE length. These methods remove and alter data knowledge from the original data structure. In order to fill this gap, the TOPSIS algorithm is presented based on the FPHFSS's arithmetical mean and geometry mean without adding an element to HFE. This approach is simple and easy to understand.

The presentation of this article is as follows. In section 2, we call some basic concepts of hesitant fuzzy sets, fuzzy soft sets and hesitant fuzzy soft sets. In section 3, we proposed the concept of fuzzy parameterized hesitant fuzzy soft set (FPHFSSs) which is the combination of hesitant fuzzy soft set and fuzzy parameterized in which we provide the degree of importance for each alternative. We also expressed the proposed concept's basic operations namely intersection, union, and complement and then study some of their properties. In section 4, we introduce the TOPSIS based score index of FPHFSSs. Then we give numerical example of the FPHFSSs in solving decision-making problem and make a comparison analysis with other existing methods. Finally, we give the conclusion of our study and recommendation for further research.

2. Preliminaries

In this section, we recall some basic notions and definitions of hesitant fuzzy set and fuzzy soft set that will be used in this paper.

Definition 1 ([15]). Let a set X be fixed. Then a hesitant fuzzy set (HFS) H on X in terms of a function h is that when applied to X a subset of $[0,1]$ return. To be easily understood Xu and Xia expressed the hesitant fuzzy set by

$$H(x) = \{(x, h_A(x)) : x \in X\}$$

where $h_A(x)$ is a set of some different values in $[0,1]$ is called hesitant fuzzy elements (HFEs), representing the possible membership degrees of the element $x \in X$ to A . For three HFEs h , h_1 and h_2 , some operations can be described as follows:

- a) Lower bound: $h^-(x) = \min h(x)$
- b) Upper bound: $h^+(x) = \max h(x)$
- c) Complement: $h^c(x) = \{1 - \gamma \mid \gamma \in h(x)\}$;
- d) Union: $(h_1 \cup h_2)(x) = \{\gamma \in h_1(x) \cup h_2(x) \mid \gamma \geq \max\{h_1^-(x), h_2^-(x)\}\}$
- e) Intersection: $(h_1 \cap h_2)(x) = \{\gamma \in h_1(x) \cap h_2(x) \mid \gamma \leq \min\{h_1^+(x), h_2^+(x)\}\}$

Xu and Xia [55] gave other forms of union and intersection of HFEs as below:

a) Union:
$$(h_1 \cup h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$$

b) Intersection:
$$(h_1 \cap h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$$

Xia and Xu [56] defined the HFEs of h , h_1 and h_2 as follows:

- c) $h^\lambda(x) = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$
- d) $\lambda h(x) = \bigcup_{\gamma \in h} \{1 - (1 - \gamma^\lambda)\}$
- e) $(h_1 \oplus h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$
- f) $(h_1 \otimes h_2)(x) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$

Definition 2 ([57]). Let $h = \{h_1, h_2, \dots, h_n\}$ be an HFE. The following functions can be considered as the score index for HFEs:

The arithmetic mean score index:

$$S_{AM}(h) = \frac{1}{n} \sum_{i=1}^n h_i \tag{1}$$

The geometric-mean score index:

$$S_{GM}(h) = \left(\prod_{i=1}^n h_i \right)^{\frac{1}{n}} \tag{2}$$

Definition 3 ([58]). Let U be an initial universe set and E be universe set of parameters. A pair $\langle F, A \rangle$ is called a fuzzy soft set over U where $F : A \rightarrow \tilde{P}(U)$ is a mapping from A into $\tilde{P}(U)$. Here $\tilde{P}(U)$ denotes the power set of all fuzzy soft set on U and $A \subseteq E$.

Definition 4 ([41]). Let U be an initial universe set and E be universe set of parameters. A pair (F, E) is called a hesitant fuzzy soft set (HFSS) over U , if and only if $F : E \rightarrow H(U)$ defined as

$$F(x) = \{x, \varpi_\kappa(x) : x \in E\}$$

Where $H(U)$ is the set of all hesitant fuzzy subset of U over the set E , $\varpi_\kappa(x)$ is the hesitant degree of membership of x over the parameter $x \in E$.

3. Fuzzy Parameterized Hesitant Fuzzy Soft Set

In this section, we shall define fuzzy parameterized hesitant fuzzy soft set and their operations with examples.

Definition 5. Let U be an initial universe, E the set of all parameters and K a fuzzy set over E with membership function $\mu_\kappa : E \rightarrow [0,1]$ and let ϖ_κ be a hesitant fuzzy

set over U for all $x \in E$. Then a fuzzy parameterized hesitant fuzzy soft sets (FPHFSSs) over U is a set defined by function ζ_K representing a mapping

$$\zeta_K = \left\{ \left(\frac{x}{\mu_K(x)}, \varpi_K(x) \right) : x \in E, \varpi_K(x) \in H(x), \mu_K(x) \in [0,1] \right\}$$

It should be noted that the set of all FPHFSSs over U will be denoted by FPHFSS(U).

Example 1. Let $U = \{h_1, h_2, h_3, h_4\}$ be four houses, $E = \{x_1, x_2, x_3\}$ be a set of parameters x_1 for the price, x_2 for the location and x_3 for the size. Suppose $K = \left\{ \frac{x_1}{0.4}, \frac{x_2}{0.8}, \frac{x_3}{0.2} \right\}$ and $\varpi(x_i)$ are defined such that $\varpi_K(x) = \emptyset$ if $\mu_K(x) = 0$. Suppose,

$$\varpi_K(x_1) = \left\{ \frac{h_1}{\{0, 2, 0.3\}}, \frac{h_2}{\{0.5, 0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3, 0.5\}} \right\}$$

$$\varpi_K(x_2) = \left\{ \frac{h_1}{\{0.4, 0.6, 0.7\}}, \frac{h_2}{\{0.5, 0.7, 0.8\}}, \frac{h_3}{\{0.6, 0.8\}}, \frac{h_4}{\{0.7, 0.9\}} \right\}$$

$$\varpi_K(x_3) = \left\{ \frac{h_1}{\{0, 2, 0.4\}}, \frac{h_2}{\{0.6, 0.7\}}, \frac{h_3}{\{0.8, 0.9\}}, \frac{h_4}{\{0.3, 0.5\}} \right\}$$

Then the FPHFSS set is given by

$$\zeta_K = \left\{ \left(\frac{x_1}{0.4}, \left\{ \frac{h_1}{\{0, 2, 0.3\}}, \frac{h_2}{\{0.5, 0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3, 0.5\}} \right\} \right), \right. \\ \left. \left(\frac{x_2}{0.8}, \left\{ \frac{h_1}{\{0.4, 0.6, 0.7\}}, \frac{h_2}{\{0.5, 0.7, 0.8\}}, \frac{h_3}{\{0.7, 0.8\}}, \frac{h_4}{\{0.7, 0.9\}} \right\} \right), \right. \\ \left. \left(\frac{x_3}{0.2}, \left\{ \frac{h_1}{\{0, 2, 0.4\}}, \frac{h_2}{\{0.6, 0.7\}}, \frac{h_3}{\{0.8, 0.9\}}, \frac{h_4}{\{0.3, 0.5\}} \right\} \right) \right\}$$

Definition 6. Two FPHFSSs ζ_K and ζ_L are said to be equal if ζ_K is a subset of ζ_L and ζ_L is a subset of ζ_K . In other words $\zeta_K = \zeta_L$.

Definition 7. Two FPHFSSs are said to be equal, and we write $\zeta_K = \zeta_L$ if ζ_K is an FPHFSS-subset of ζ_L and ζ_L is an FPHFSS subset of ζ_K . In other words, $\zeta_K = \zeta_L$ if the following conditions are satisfied:

1. $\mu_K(x) = \mu_L(x), \forall x \in E;$
2. $\varpi_K(x) = \varpi_L(x), \forall x \in E.$

Definition 8. Let ζ_K be an FPHFSS. If $\zeta_K(x) = \emptyset$ then ζ_K is called an empty FPHFSSs denoted by ζ_\emptyset for all $x \in K$.

Definition 9. Let ζ_K be an FPHFSS. If $\zeta_K(x) = \bar{1}$. Then ζ_K is called a full FPHFSS set denoted by ζ_E for all $x \in K$.

Proposition 1. Let ζ_K, ζ_L and ζ_M be any three of FPHFSSs. Then the following results hold:

1. $\zeta_K \subseteq \zeta_E,$

2. $\zeta_\emptyset \subseteq \zeta_K,$
3. $\zeta_K \subseteq \zeta_K,$
4. $\zeta_K \subseteq \zeta_L$ and $\zeta_L \subseteq \zeta_M$ then $\zeta_K \subseteq \zeta_M,$
5. $\zeta_K = \zeta_L$ and $\zeta_L = \zeta_M$ then $\zeta_M = \zeta_K,$
6. $\zeta_K \subseteq \zeta_L$ and $\zeta_L \subseteq \zeta_K$ then $\zeta_K = \zeta_L,$

Proof: The proof is straightforward.

Definition 10. Let ζ_K be an FPHSS. Then the complement of $\zeta_K = \left\{ \left(\frac{x}{\mu_K(x)}, \varpi_K(x) \right) \right\}$ is denoted by

$$\zeta_K^c = \left\{ \left(\frac{x}{\mu_K(x)}, \varpi_K(x) \right) \right\}^c \text{ defined by } \mu_{K^c}(x) = 1 - \mu_K(x)$$

and $\varpi_{K^c}(x) = 1 - \varpi_K(x)$ where c is a fuzzy complement and \bar{c} is a hesitant fuzzy soft complement.

Example 2. Consider example 1. By using Definition 1, we have

$$\zeta_K^c = \left\{ \left(\frac{x_1}{0.6}, \left\{ \frac{h_1}{\{0.8, 0.7\}}, \frac{h_2}{\{0.5, 0.4\}}, \frac{h_3}{\{0.7\}}, \frac{h_4}{\{0.7, 0.5\}} \right\} \right), \right. \\ \left. \left(\frac{x_2}{0.2}, \left\{ \frac{h_1}{\{0.6, 0.4, 0.3\}}, \frac{h_2}{\{0.5, 0.3, 0.2\}}, \frac{h_3}{\{0.4, 0.2\}}, \frac{h_4}{\{0.3, 0.1\}} \right\} \right), \right. \\ \left. \left(\frac{x_3}{0.8}, \left\{ \frac{h_1}{\{0.8, 0.6\}}, \frac{h_2}{\{0.4, 0.3\}}, \frac{h_3}{\{0.2, 0.1\}}, \frac{h_4}{\{0.7, 0.5\}} \right\} \right) \right\}$$

Proposition 2 Let ζ_K be an FPHSS. Then the following results hold:

1. $(\zeta_K^c)^c = \zeta_K$
2. $\zeta_\emptyset^c = \zeta_E$

Proof: The proof is straightforward.

Definition 11. Let $\zeta_K = \left\{ \frac{x}{\mu_K(x)}, \varpi_K(x) \right\}$ and

$\zeta_L = \left\{ \frac{x}{\mu_L(x)}, \varpi_L(x) \right\}$ be two FPHFSSs. The union of ζ_K

and ζ_L denoted by $\zeta_K \cup \zeta_L$ is defined by

$$\mu_{K \cup L}(x) = \max(\mu_K(x), \mu_L(x))$$

and

$$\varpi_{K \cup L}(x) = \max(\varpi_K(x), \varpi_L(x))$$

Where \cup is an s-norm and $\bar{\cup}$ is hesitant fuzzy soft union based on **Definition 1** in f).

Example 3. Consider ζ_K as in **Example 1** and let ζ_L be another FPHFSS defined as follows:

$$\zeta_L = \left\{ \left(\frac{x_1}{0.3}, \left\{ \frac{h_1}{\{0.3, 0.4, 0.5\}}, \frac{h_2}{\{0.4, 0.5, 0.6\}}, \frac{h_3}{\{0.1, 0.2, 0.3\}}, \frac{h_4}{\{0.5\}} \right\} \right), \right. \\ \left(\frac{x_2}{0.6}, \left\{ \frac{h_1}{\{0.2, 0.4\}}, \frac{h_2}{\{0.3, 0.4, 0.5\}}, \frac{h_3}{\{0.7\}}, \frac{h_4}{\{0.5, 0.7\}} \right\} \right), \\ \left. \left(\frac{x_3}{0.4}, \left\{ \frac{h_1}{\{0.2, 0.4, 0.5\}}, \frac{h_2}{\{0.3\}}, \frac{h_3}{\{0.4, 0.6\}}, \frac{h_4}{\{0.6, 0.7, 0.8\}} \right\} \right) \right\}$$

The intersection of ζ_K and ζ_L denoted by $\zeta_K \cap \zeta_L$ following **Definition 11** is given as

$$\zeta_{K \cap L}(x) = \left\{ \left(\frac{x_1}{0.4}, \left\{ \frac{h_1}{\{0.3, 0.4, 0.5\}}, \frac{h_2}{\{0.5, 0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.5\}} \right\} \right), \right. \\ \left(\frac{x_2}{0.8}, \left\{ \frac{h_1}{\{0.4, 0.6, 0.7\}}, \frac{h_2}{\{0.5, 0.7, 0.8\}}, \frac{h_3}{\{0.7, 0.8\}}, \frac{h_4}{\{0.7, 0.9\}} \right\} \right), \\ \left. \left(\frac{x_3}{0.4}, \left\{ \frac{h_1}{\{0.2, 0.4, 0.5\}}, \frac{h_2}{\{0.6, 0.7\}}, \frac{h_3}{\{0.8, 0.9\}}, \frac{h_4}{\{0.6, 0.7, 0.8\}} \right\} \right) \right\}$$

Proposition 3. Let ζ_K and ζ_L be any two FPHFSSs. Then the following results hold:

1. $\zeta_K \cup \zeta_K = \zeta_K$
2. $\zeta_\phi \cup \zeta_K = \zeta_K$
3. $\zeta_K \cup \zeta_E = \zeta_E$
4. $\zeta_K \cup \zeta_L = \zeta_L \cup \zeta_K$

Proof. The proof is straightforward.

Definition 12. Let $\zeta_K = \left\{ \frac{x}{\mu_K(x)}, \varpi_K(x) \right\}$ and

$\zeta_L = \left\{ \frac{x}{\mu_L(x)}, \varpi_L(x) \right\}$ be two FPHFSSs. The intersection of ζ_K and ζ_L which is denoted by $\zeta_K \cap \zeta_L$ is defined by

$$\mu_{K \cap L}(x) = \min(\mu_K(x), \mu_L(x))$$

and

$$\varpi_{K \cap L}(x) = \min(\varpi_K(x), \varpi_L(x))$$

Where \cap is a t-norm and \cap is hesitant fuzzy intersection based on **Definition 1** in g).

Example 4. Let ζ_K as in **Example 1** and ζ_L as in **Example 3** be two FPHFSSs. The intersection of ζ_K and ζ_L which is denoted by $\zeta_K \cap \zeta_L$ following **Definition 12** is given as

$$\zeta_{K \cap L} = \left\{ \left(\frac{x_1}{0.3}, \left\{ \frac{h_1}{\{0.2, 0.3\}}, \frac{h_2}{\{0.4, 0.5, 0.6\}}, \frac{h_3}{\{0.1, 0.2, 0.3\}}, \frac{h_4}{\{0.3, 0.5\}} \right\} \right), \right. \\ \left(\frac{x_2}{0.6}, \left\{ \frac{h_1}{\{0.2, 0.4\}}, \frac{h_2}{\{0.3, 0.4, 0.5\}}, \frac{h_3}{\{0.7\}}, \frac{h_4}{\{0.5, 0.7\}} \right\} \right), \\ \left. \left(\frac{x_3}{0.2}, \left\{ \frac{h_1}{\{0.2, 0.4\}}, \frac{h_2}{\{0.3\}}, \frac{h_3}{\{0.4, 0.6\}}, \frac{h_4}{\{0.3, 0.5\}} \right\} \right) \right\}$$

Proposition 4. Let ζ_K , ζ_L and ζ_M be any three of FPHFSSs. Then the following results hold:

1. $\zeta_K \cap \zeta_K = \zeta_K$
2. $\zeta_\phi \cap \zeta_K = \zeta_\phi$
3. $\zeta_K \cap \zeta_E = \zeta_K$
4. $\zeta_K \cap \zeta_L = \zeta_L \cap \zeta_K$

Proof: The proof is straightforward.

Proposition 5. Let ζ_K , ζ_L and ζ_M be any three of FPHFSSs. Then the following results hold:

1. $(\zeta_K \cup \zeta_K)^c = \zeta_K^c$
2. $(\zeta_K \cap \zeta_L)^c = \zeta_K^c \cup \zeta_L^c$

Proof: The proof is straightforward.

Definition 13. The operation AND for two FPHFSSs ζ_K and ζ_L which is denoted by $\zeta_K \wedge \zeta_L$ is defined by $\zeta_K \wedge \zeta_L = \zeta_K \cap \zeta_L$.

Definition 14. The operation OR for two FPHFSSs ζ_K and ζ_L which is denoted by $\zeta_K \vee \zeta_L$ is defined by $\zeta_K \vee \zeta_L = \zeta_K \cup \zeta_L$.

Theorem 4. (De Morgan Law of FPHFSSs).

Let ζ_K and ζ_L be two FPHFSSs over U ; we have

1. $(\zeta_K \vee \zeta_L)^c = \zeta_K^c \wedge \zeta_L^c$,
2. $(\zeta_K \wedge \zeta_L)^c = \zeta_K^c \vee \zeta_L^c$,

Proof:

Note that,

$$\begin{aligned} \mu_{(K \cup L)^c}(x) &= (\mu_{(K \cup L)}(x))^c \\ &= 1 - \max(\mu_K(x), \mu_L(x)) \\ &= \min((1 - \mu_K(x)), (1 - \mu_L(x))) \\ &= \min(\mu_{K^c}(x), \mu_{L^c}(x)) \\ &= \mu_{K^c \cap L^c}(x) \end{aligned}$$

and

$$\begin{aligned} \varpi_{(K \cup L)^c}(x) &= (\varpi_{(K \cup L)}(x))^c \\ &= 1 - \max(\varpi_K(x), \varpi_L(x)) \\ &= \min((1 - \varpi_K(x)), (1 - \varpi_L(x))) \\ &= \min(\varpi_{K^c}(x), \varpi_{L^c}(x)) \\ &= \varpi_{K^c \cap L^c}(x) \end{aligned}$$

Theorem 5. (Associative Law of FPHFSSs)

Let ζ_K , ζ_L and ζ_M be any three FPHFSSs over U .

Then the following results hold:

1. $\zeta_K \wedge (\zeta_L \wedge \zeta_M) = (\zeta_K \wedge \zeta_L) \wedge \zeta_M$
2. $\zeta_K \vee (\zeta_L \vee \zeta_M) = (\zeta_K \vee \zeta_L) \vee \zeta_M$

Proof:

Note that,

$$\mu_K \wedge (\mu_L \wedge \mu_M) = (\mu_K \wedge \mu_L) \wedge \mu_M$$

and

$$\varpi_K \wedge (\varpi_L \wedge \varpi_M) = (\varpi_K \wedge \varpi_L) \wedge \varpi_M$$

so that

$$\zeta_K \wedge (\zeta_L \wedge \zeta_M) = (\zeta_K \wedge \zeta_L) \wedge \zeta_M.$$

Similarly

$$\mu_K \vee (\mu_L \vee \mu_M) = (\mu_K \vee \mu_L) \vee \mu_M$$

and

$$\varpi_K \vee (\varpi_L \vee \varpi_M) = (\varpi_K \vee \varpi_L) \vee \varpi_M$$

so that

$$\zeta_K \vee (\zeta_L \vee \zeta_M) = (\zeta_K \vee \zeta_L) \vee \zeta_M.$$

Theorem 6. (Distributive Law of FPHFSSs).

Let ζ_K , ζ_L and ζ_M be any three FPHFSSs over U .

Then the following results hold:

1. $\zeta_K \wedge (\zeta_L \vee \zeta_M) = (\zeta_K \wedge \zeta_L) \vee (\zeta_K \wedge \zeta_M)$
2. $\zeta_K \vee (\zeta_L \wedge \zeta_M) = (\zeta_K \vee \zeta_L) \wedge (\zeta_K \vee \zeta_M)$

Proof:

Note that,

$$\mu_K \wedge (\mu_L \vee \mu_M) = (\mu_K \wedge \mu_L) \vee (\mu_K \wedge \mu_M)$$

and

$$\varpi_K \wedge (\varpi_L \vee \varpi_M) = (\varpi_K \wedge \varpi_L) \vee (\varpi_K \wedge \varpi_M)$$

so that

$$\zeta_K \wedge (\zeta_L \vee \zeta_M) = (\zeta_K \wedge \zeta_L) \vee (\zeta_K \wedge \zeta_M).$$

Similarly

$$\mu_K \vee (\mu_L \wedge \mu_M) = (\mu_K \vee \mu_L) \wedge (\mu_K \vee \mu_M)$$

and

$$\varpi_K \vee (\varpi_L \wedge \varpi_M) = (\varpi_K \vee \varpi_L) \wedge (\varpi_K \vee \varpi_M)$$

so that

$$\zeta_K \vee (\zeta_L \wedge \zeta_M) = (\zeta_K \vee \zeta_L) \wedge (\zeta_K \vee \zeta_M).$$

4. Applications

In this part, we will explain how to utilize this algorithm in solving MCDM. We extend the theory of HFSSs by giving importance weight for each parameter in decision making process. We utilize the TOPSIS algorithm with arithmetic mean and geometric mean score to solve problem in FPHFSSs environment. It should be noted that some of the existing operations, arrangement and various measures of hesitant fuzzy sets need the hesitant fuzzy elements to have exactly the same length. In practice, however, the length of the hesitant fuzzy element may vary. The approach proposed in [56] applied those elements to the shorter hesitant fuzzy element, rendering its equivalent to another hesitant fuzzy element, or repeating its elements to obtain two sequence of the same length. These

approaches would break the original data structure and alter the data details [39]. In this paper, we presented an algorithm based on the two score functions of HFE by Farhadinia [57]. Based on this argument, we materialize with our algorithm for FPHFSS given as below.

Algorithm: The TOPSIS based score index of FPHFSSs.

Step 1. Transform the FPHFSSs into fuzzy decision matrix. The FPHFSSs matrix of decision K can be represented as

$$K = \begin{bmatrix} \mu_1 h_{11} & \mu_2 h_{12} & \dots & \mu_n h_{1n} \\ \mu_1 h_{21} & \mu_1 h_{22} & \dots & \mu_n h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 h_{m1} & \mu_1 h_{m2} & \dots & \mu_n h_{mn} \end{bmatrix}$$

Step 2. Calculate the score index matrix of the hesitant fuzzy element of FPHFSSs. the arithmetic mean score

$$\widehat{S}_{AM} = \left(\frac{1}{n} \sum_i^{i=l} \mu h_i \right)^{\psi(h_i)} \tag{3}$$

the geometry mean score

$$\widehat{S}_{GM} = \left(\left(\prod_{i=1}^l \mu h_i \right)^{\frac{1}{l}} \right)^{\psi} \tag{4}$$

where l is the number of elements in HFE and $\psi(h_i) = 1 - \frac{1}{l(h_i)}$ is a value of hesitant degree.

Step 3. Determine the positive ideal solution (PIS) and negative ideal solution (NIS)

$$A^+ = \left\{ x_j, \max \left\langle \widehat{S}_{AM_{ij}} \right\rangle \mid j = 1, 2, \dots, n \right\} \tag{5}$$

and

$$A^- = \left\{ x_j, \min \left\langle \widehat{S}_{AM_{ij}} \right\rangle \mid j = 1, 2, \dots, n \right\} \tag{6}$$

Step 4. Calculate distance from the PIS and NIS to score values.

$$d_i^+ = \sqrt{\sum_{j=1}^n \left| A_j^+ - \widehat{S}_{AM_{ij}} \right|^2}, \quad |i = 1, 2, \dots, n \tag{7}$$

and

$$d_i^- = \sqrt{\sum_{j=1}^n \left| \widehat{S}_{AM_{ij}} - A_j^- \right|^2}, \quad |i = 1, 2, \dots, n \tag{8}$$

Step 5. Calculate the CI of every alternative

$$CI_i = \frac{d_i^-}{d_i^- + d_i^+} \tag{9}$$

Step 6. Rank the alternatives.

Table 1. Fuzzy decision matrix of FPHFSSs

	C_1	C_2	C_3	C_4
P_1	0.15{0.3,0.4,0.5}	0.3{0.1,0.7,0.8,0.9}	0.2{0.2,0.4,0.5}	0.35{0.3,0.5,0.6,0.9}
P_2	0.15{0.3,0.5}	0.3{0.2,0.5,0.6,0.7,0.9}	0.2{0.1,0.5,0.5,0.8}	0.35{0.3,0.4,0.7}
P_3	0.15{0.6,0.7}	0.3{0.6,0.9}	0.2{0.3,0.5,0.7}	0.35{0.4,0.6}
P_4	0.15{0.3,0.4,0.7,0.8}	0.3{0.2,0.4,0.7}	0.2{0.1,0.8}	0.35{0.6,0.8,0.9}
P_5	0.15{0.1,0.3,0.6,0.7,0.9}	0.3{0.4,0.6,0.7,0.8}	0.2{0.7,0.8,0.9}	0.35{0.3,0.6,0.7,0.9}

4.1. Numerical Examples

This section provides a numerical illustration of the viability of the proposed score index-TOPSIS approach (modified from Xu & Xia, 2011a) in the FPHFSSs decision-making problems. A comparative analysis is provided to validate its reasonableness and usefulness with other existing methods.

Energy is an important element for community social and economic growth. Considering five energy projects to be invested $P_i(i=1,2,3,4,5)$ the decision-makers will assess five potential projects anonymously from the four following criteria, technological (C_1), environmental (C_2), socio-political (C_3), and economic (C_4). The modified part is where the degree of importance for each criteria is directly decided by the decision makers.

We present our methods to solve the MCDM problem above using the steps presented before

Step 1. Construct the FPHFSSs into fuzzy decision matrix as shown in Table 1.

Step 2. Find the score index of the hesitant fuzzy element of FPHFSSs according to eq.(3). The arithmetic mean score of FPHFSSs are shown in Table 2.

Table 2. Arithmetic mean score matrix of FPHFSSs

	C_1	C_2	C_3	C_4
P_1	0.0814	0.2109	0.1025	0.2311
P_2	0.0949	0.1940	0.1189	0.2106
P_3	0.1209	0.2598	0.1260	0.2475
P_4	0.0958	0.1718	0.1342	0.2932
P_5	0.0889	0.2109	0.1724	0.2460

Step 3. Define the PIS and NIS based on arithmetic

mean score as shown in Table 3 using equation (5) and (6).

Table 3. PIS and Nis arithmetic mean

	C_1	C_2	C_3	C_4
A^+	0.1209	0.2598	0.1724	0.2932
A^-	0.0814	0.1718	0.1025	0.2106

Step 4 and step 5. To determine the separation d^+ and d^- of each alternative P_i from the PIS and the NIS, the eq.(7) and eq.(8) shall be used. Then, calculate the relative closeness CI of each alternative using Equation (9).

Table 4.Distance from PIS and NIS CI

	d^+	d^-	CI
P_1	0.1127	0.0442	0.4360
P_2	0.1212	0.0308	0.2846
P_3	0.0651	0.1059	1.7334
P_4	0.0992	0.0896	0.9935
P_5	0.0751	0.0879	1.2578

Step 6. Based on the value of CI the ranking is $P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$.

4.2. Comparison Analysis

In this section, we will compare the ranking results with other existing methods. Table 5 compares the ranking of alternatives given by our proposed method and other existing methods.

Table 5. The comparison ranking of alternatives

Methods	Ranking
Proposed methods	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ TOPSIS - arithmetic mean. $P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ TOPSIS - geometry mean.
Liu and Wang [59]	$P_3 \succ P_2 \succ P_4 \succ P_1 \succ P_5$ The preference weighted generalize distance with distance parameter = 1 and preference parameter = 0.1 $P_3 \succ P_2 \succ P_1 \succ P_4 \succ P_5$ The preference weighted generalize distance with distance parameter = 2 and preference parameter = 0.1 $P_3 \succ P_5 \succ P_2 \succ P_4 \succ P_1$ The preference weighted generalize distance with distance parameter = 6 and preference parameter = 0.1
Sun et al. [22]	Similarity like positive correlation decision making factor = 0.7. Weight of the distance like positive correlation decision making factor = 0.6. $P_5 \succ P_1 \succ P_2 \succ P_4 \succ P_3$, Rc (η) $P_5 \succ P_1 \succ P_4 \succ P_3 \succ P_2$, Rc (δ) $P_5 \succ P_1 \succ P_2 \succ P_4 \succ P_3$ Rc (ξ) $P_5 \succ P_1 \succ P_2 \succ P_4 \succ P_3$, Rc (ρ)
Li et al. [60]	Weighted distance measure with preference. $P_5 \succ P_3 \succ P_4 \succ P_1 \succ P_2$ for $\lambda = 1, a = 0.1, \beta = 0.9$. $P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ for $\lambda = 1, a = 0.5, \beta = 0.5$.
Xu and Xia[55]	$P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ generalized hesitant weighted distance with $\lambda = 1$ $P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ generalized hesitant weighted Hausdorff distance with $\lambda = 1$ $P_3 \succ P_5 \succ P_4 \succ P_1 \succ P_2$ generalized hybrid hesitant weighted distance with $\lambda = 1$.

This table shows some of the rankings provided by our proposed methods and another established method of selecting the energy site. The distance between the hesitant evaluation values and the ideal reference intervals is uncertain on the basis of Liu and Wang [46]. For example, the hesitant evaluations for { 0.7, 0.6 } and { 0.7, 0.5 } with the ideal reference [0.7,0.5] are the same even if the hesitant values are different. While the ranking given by Sun et al. [22] contradicts with the ranking from other methods, in which the P_3 is ranked in the final position although mostly P_3 is placed first by other methods. The ranking given by Li et al [60] is quite compatible with some other algorithms, even so their algorithms are complicated for decision-making because it is difficult to determine the value of λ, a and β . Xu and Xia [55] presented the following rules for proper operation: the shorter one is expanded by inserting a minimum value, maximum value, or any value thereof until it has the same length as the longer one. The choice of this value depends mainly on the risk preferences of decision-makers. Optimists are expected to achieve desirable results and may bring maximum value and pessimists predict to contribute minimum value. Obviously, the original data structure will be broken and the data details modified [30].

5. Conclusions

In this paper, we consider the parameterized hesitant fuzzy soft set which includes the combination of the hesitant fuzzy set and fuzzy soft sets where an important degree is given for each element in the set of parameters. The introduction of elements of fuzzy parameters are used to avoid the degree of importance to which criteria are created from the rating of each alternative. Decision makers should decide which criteria are more important than others. Next, we study some of the algorithm’s properties. The complement, union and intersection, AND and OR operation have been defined on the FPHFSSs. In addition, the algorithm given can cater for certain methods which added or repeated the hesitant fuzzy elements to the same length as this technique could destroy the original information. Finally, we provided an example which demonstrates that this theory can be used to solve MCDM problems. Comparisons are made to show feasibility and viability of our proposed method. We hope that our work could enhance the study on hesitant fuzzy soft sets which could further be applied in many other areas such as data analysis and forecasting. At the same time, it is anticipated that the fuzzy parameterized concept can extend to other generalization of fuzzy sets such as interval fuzzy set,

intuitionistic fuzzy set, hesitant fuzzy set and others.

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