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Research Article

Application of Periodized Harmonic Wavelets towards Solution of Eigenvalue Problems for Integral Equations

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This article deals with the application of the periodized harmonic wavelets for solution of integral equations and eigenvalue problems. The solution is searched as a series of products of wavelet coefficients and wavelets. The absolute error for a general case of the wavelet approximation was analytically estimated.

1. Introduction

Mathematical models describe a variety of physical and engineering problems and processes which can be represented by integral equations (IEs). The homogeneous Fredholm IE is written as follows:

$$\lambda f(x) - \int_a^b K(x,t) f(t) dt = 0, \tag{1.1}$$

where a and b are finite numbers, the kernel $K(x,t)$ is known function, and λ and $f(x)$ are the unknown eigenvalue and associated eigenfunction. Equation (1.1) has a nontrivial solution only for some values of λ.

There exist two different methods to solve IEs numerically. The first one is to expand the equation by the appropriate set of basis functions, such as the classical orthogonal polynomials [1] or wavelets (e.g. [2, 3]), and to reduce the equation to simultaneous equations with respect to the expansion coefficients. The second method is to use the trapezoidal formula for integration [4]. Since we are interested in application of periodic harmonic wavelets (PHWs) as basis functions, we will focus our attention on the first approach.

These methods have their own advantages and disadvantages. The main advantage of our approach over the existing wavelet methods is that the wavelet expansion coefficients can be computed analytically. In addition, it will be shown that the computational cost of our approach is low and the accuracy is high. It is worth to be mentioned that the application of wavelets takes a special place in the modern computational methods thanks to quick convergence of a series of wavelets and the possibility to find the solution with a low approximation error.

The pioneering contribution into the wavelet approach for solution of IEs belongs to Beylkin et al. [5]. There were many other approaches by, for example, [2, 6, 7] towards this problem. The interest in the wavelet approach for solution of IEs is popular nowadays [8].

The most part of the existing research programs is devoted to solution of the Fredholm and Volterra-type IEs. The Galerkin and collocation methods are mainly used in such papers [2, 6, 7, 9], where besides the well-known Daubechies wavelets many other wavelets have been used, such as the Haar wavelets [2, 8, 9], CAS-wavelets [3], and so forth.

In our opinion, the attention to the PHW and its application for solution of IEs have not been sufficiently paid, although there were attempts to use this basis for solution of partial differential equations (e.g., [10–12]). The advantage of our choice is that PHWs are continuous and differentiable functions everywhere.

It is known that the wavelet approach offers an alternative route for a signal and function decomposition in the time-frequency domain. Recent applications of the wavelet transform to engineering and applied problems can be found in several studies [13]. In order to analyze some applied engineering problems, Newland proposed [13, 14] wavelets whose spectrum is confined exactly to an octave band. It was suggested that the “level” of a signal’s multiresolution would be interchangeable with its frequency band and the interpretation of the frequency content would be easier for engineers.

In addition, for the convenience of the further analysis it would be better to operate with such functions, whose Fourier transform was compact and which could, if possible, be constructed from simple functions. The wavelets considered in our paper are called PHW and they possess all mentioned properties and constitute a specific but a representative example of wavelets in general.

The main purpose of the present work is to propose for numerical solution of IEs a simple approach based on periodized harmonic wavelets. This technique is also applicable with minor changes to the Fredholm, Volterra, and integro-differential equations. In Section 2 of the paper we show that PHWs satisfy the axioms of the multiresolution analysis and can be used as basis functions in solution of IEs. An illustrative example is presented in Section 3. The generalized error estimation is given in Section 4 and it shows that the accuracy of computations is very high even when the approximation level is small.

2. Periodized Harmonic Wavelets

It is known [13–15] that PHWs are defined as follows:

$$\psi\left(2^j x - k\right) = 2^{-j} \sum_{m=2^j}^{2^{j+1}-1} e^{2\pi i m (x-(k/2^j))}, \tag{2.1}$$

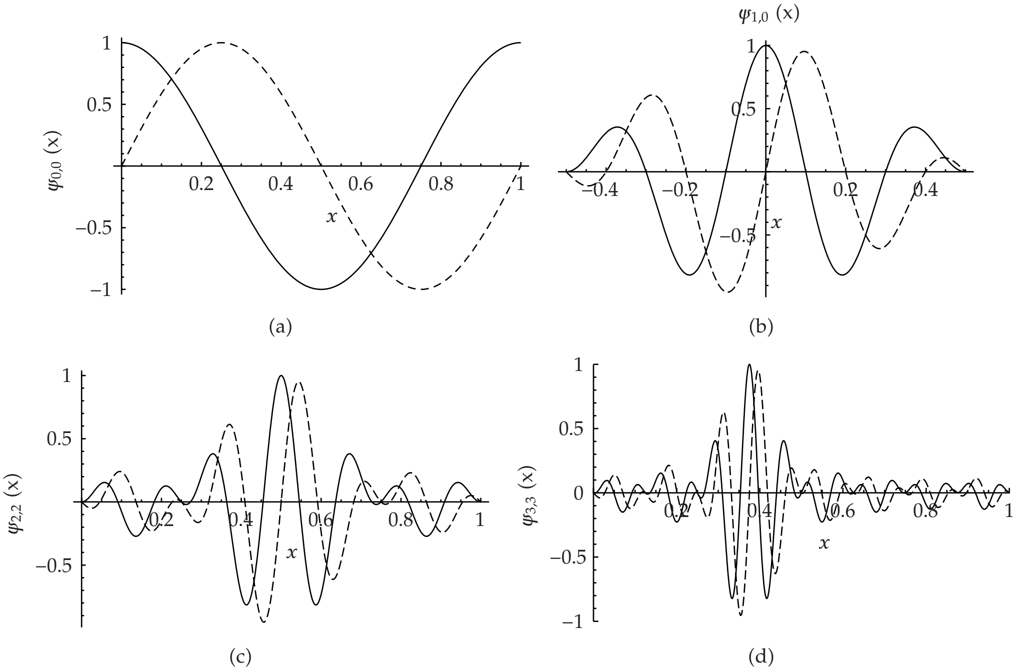

Figure 1: Real (solid line) and imaginary (dashed line) parts of the periodic harmonic wavelets $\psi_{0,0}(x)$, $\psi_{1,0}(x)$, $\psi_{2,2}(x)$, and $\psi_{3,3}(x)$.

where the scaling parameter $k = 0, \ldots, 2^j - 1$ and the dilation parameter $j = 0, \ldots, N-1$. The 1-periodicity of function (2.1) can be demonstrated as follows:

$$\psi\left(2^j(x+1) - k\right) = 2^{-j} \sum_{m=2^j}^{2^{j+1}-1} e^{2\pi i m\left((x+1) - \frac{k}{2^j}\right)} = \psi\left(2^j x - k\right). \tag{2.2}$$

The plots for several values of the scaling parameter j are shown on Figure 1 for selected positions k.

According to [14, 16], PHWs construct basis for 1-periodic functions from $L^2([0;1])$. The orthogonal projection of the function $f(x)$ onto the space of wavelets V_N of the level N is written as follows:

$$\wp_{V_N} f(x) = a_0 \varphi(x) + \sum_{j=0}^{N-1} \sum_{k=0}^{2^j-1} \left\{ a_{j,k} \psi_{j,k}(x) + \tilde{a}_{j,k} \psi^*_{j,k}(x) \right\}, \tag{2.3}$$

where the harmonic scaling function is $\varphi(x) = 1$ [14] and the "$*$" over $\psi_{j,k}$ stands for its complex conjugate. If $N \to \infty$, then $\lim_{N\to\infty} \wp_{V_N} f(x) = f(x)$ and expansion (2.3) becomes

$$f(x) = a_0 \varphi(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\{ a_{j,k} \psi_{j,k}(x) + \tilde{a}_{j,k} \psi^*_{j,k}(x) \right\}. \tag{2.4}$$

Thus, we have a set of functions, which form basis for the $L^2([0;1])$ functions. It means that we can substitute the expansion (2.3) into (1.1) and reduce it to a system of equations with respect to wavelet coefficients.

3. Eigenvalues and Eigenfunctions

Let us consider the homogeneous Fredholm integral equation

$$\hat{f}(\hat{x}) - \lambda \int_0^{\pi} \cos\left(\hat{x} + \hat{t}\right) \hat{f}\left(\hat{t}\right) d\hat{t} = 0. \tag{3.1}$$

In order to deal with 1-periodic functions, it is convenient to introduce new variables as follows: $\hat{x} = 2\pi x$ and $\hat{t} = 2\pi t$. And we get the new equation

$$f(x) - 2\pi\lambda \int_0^{0.5} \cos[2\pi(x + t)] f(t) dt = 0, \tag{3.2}$$

which we will solve by the collocation method. Denote the collocation points by

$$a \le x_1 < x_2 < \cdots < x_l < \cdots \le b. \tag{3.3}$$

Recalling the decomposition of a real periodic function (2.3) on the space of PHW, we have

$$\mathcal{P}_{V_1} f(x) = a_0 + \psi_{0,0}(x) + \tilde{a}_{0,0}\psi^*_{0,0}(x), \tag{3.4}$$

for $N = 1$. Also, $a_{0,0} = \tilde{a}_{0,0}$ for a real function. The corresponding choice of the collocation points $\{x_l\}$ leads us to a system of linear algebraic equations with the parameter λ and unknowns $\{a_0; a_{0,0}; \tilde{a}_{0,0}\}$ as

$$a_0 + a_{0,0}e^{2\pi i x_l} + \tilde{a}_{0,0}e^{-2\pi i x_l} - 2\pi\lambda \int_0^{1/2} \cos[2\pi(x_l + t)]\left(a_0 + a_{0,0}e^{2\pi i t} + \tilde{a}_{0,0}e^{-2\pi i t}\right) dt = 0. \tag{3.5}$$

The solution of this system of equations gives us two pairs of coefficients $\{0; 0.5; 0.5\}$ and $\{0; -i/2; i/2\}$. Thus, we can find parameters $\lambda_1 = 2/\pi, \lambda_2 = -2/\pi$ and the eigenfunctions $\hat{f}_1(\hat{x}) = \cos\hat{x}$, $\hat{f}_2(\hat{x}) = \sin\hat{x}$.

We obtained the projection of the solution of the unknown eigenfunctions f_1, f_2 on the first level of approximation. Note that the obtained projection for $N = 1$ coincides with the analytical solution. If we have continued to search for the solution on the other levels of approximation, the connection coefficients $\{a_{j,k}\}$ would be zeros.

4. Approximation Properties of Multiresolution Spaces

Let us now consider the approximation error for the periodic wavelets. Let $f(x) \in L^2([0;1])$ and assume that its periodic expansion (2.4) is P times differentiable everywhere. Denote the approximation error as follows:

$$e_N^{\text{per}}(x) = f(x) - \mathcal{P}_{V_N} f(x), \quad x \in [0;1], \tag{4.1}$$

where $\mathcal{P}_{V_N} f(x)$ is the orthogonal projection of $f(x)$ onto the space of PHW. The symbol "per" over e_N assumes that the error is a periodic function. The derivation of the value of $e_N^{\text{per}}(x)$ is presented in the following theorem.

Theorem 4.1. *The approximation error* (4.1) *is bounded by the exponential decay* $|e_N^{per}(x)| = \mathcal{O}(2^{-NP})$.

Proof. Using the wavelet periodic expansion (2.4), we find that

$$\mathcal{P}_{V_N} f(x) = \sum_{k=0}^{\infty} a_{\varphi,k}\varphi(x-k) + \sum_{j=0}^{N-1}\sum_{k=0}^{2^j-1} a_{j,k}\psi_{j,k}(x). \tag{4.2}$$

At any given scale, the projection of the function on the subspace of wavelets of the certain scale approaches to the function as the number of zero wavelet moments P tends to infinity, that is, $N \to \infty$ and we get $f(x)$ itself:

$$f(x) = \sum_{k=0}^{\infty} a_{\varphi,k}\varphi(x-k) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j,k}\psi_{j,k}(x). \tag{4.3}$$

Then, by subtracting (4.2) from (4.3), we obtain an expression for the error e_N^{per} in terms of the wavelets at scales $j \geq N$:

$$e_N^{\text{per}}(x) = \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} a_{j,k}\psi_{j,k}(x). \tag{4.4}$$

Define

$$C_\psi = \max_{x\in I_{j,k}}\left|\psi\left(2^j x - k\right)\right| = \max_{y\in[0,D-1]}|\psi(y)|. \tag{4.5}$$

Since $\max_{x\in I_{j,k}}|\psi_{j,k}(x)| = 2^{j/2}C_\psi$ and according to the Theorem of decay of wavelet coefficients [17], it is

$$\left|a_{j,k}\psi_{j,k}(x)\right| \leq C_P 2^{-jP}\max_{\xi\in I_{j,k}}\left|f^{(P)}(\xi)\right|C_\psi. \tag{4.6}$$

Recall that

$$\operatorname{supp}(\varphi_{j,k}) = I_{j,k} = \left[\frac{k}{2^j}; \frac{k+D-1}{2^j}\right]. \tag{4.7}$$

Hence, there are at most $D-1$ intervals $I_{j,k}$ containing a given value of x. Thus, for any x only $D-1$ terms in the inner summation in (4.4) are nonzero. Let I_j be a union of all these intervals, that is,

$$I_j(x) = \bigcup_{\{l: x \in I_{j,l}\}} I_{j,l}, \tag{4.8}$$

and let

$$\mu_j^P(x) = \max_{\xi \in I_j(x)} \left| f^P(\xi) \right|. \tag{4.9}$$

Then we can find a common bound for all terms in the inner sum:

$$\sum_{k=-\infty}^{\infty} \left| a_{j,k}\varphi_{j,k} \right| \le C_\varphi C_P 2^{-jP}(D-1)\mu_j^P(x). \tag{4.10}$$

The outer sum over j can be evaluated using the fact that

$$\mu_N^P(x) \ge \mu_{N+1}^P(x) \ge \mu_{N+2}^P(x) \ge \dots, \tag{4.11}$$

and we establish the bound

$$\begin{aligned} \left| e_N^{\text{per}}(x) \right| &\le C_\varphi C_P (D-1)\mu_N^P(x) \sum_{j=N}^{\infty} 2^{-jP} \\ &= C_\varphi C_P (D-1)\mu_N^P(x) \frac{2^{-NP}}{1-2^{-P}}. \end{aligned} \tag{4.12}$$

Thus, we see that for an arbitrary, but fixed x, the approximation error will be bounded as follows:

$$\left| e_N^{\text{per}}(x) \right| = \mathcal{O}\left(2^{-NP}\right), \tag{4.13}$$

where $\mathcal{O}$ only denotes an upper bound. This is an exponential decay with respect to the resolution N. Furthermore the greater number of vanishing moments P of a periodic wavelet increases the rate of the decay. □

Let us compare the approximation error of wavelets with the error of the Fourier approximation for N terms. In order to do this, we need to introduce a smooth function of the order q.

Definition 4.2. A smooth function is a function that has continuous derivatives up to some desired order q over some domain. A function can, therefore, be said to be smooth over a restricted interval such as $[a;b]$.

According to [18], we can find that the approximation error of the Fourier series is

$$\left|e^{F}(q,N)\right| = \max_{a\le x\le b}\left|F(N,x)-f(x)\right| = \mathcal{O}\left(N^{-q-0.5}\right). \tag{4.14}$$

This is also an exponential decay with respect to the number of terms in the series and the level of smoothness of a function. In order to give a more detailed comparison of these two methods, it is necessary to consider specific examples.

5. Concluding Remarks

In this work we have proposed PHWs as basis functions for solution of IEs. The approach was verified by solving a test problem and its approximation error was analytically estimated for periodized wavelets. The assumption of 1-periodicity of solution does not restrict the generality of the problem, since we can always make the substitutional change of variables.

There are several important facts to remember about the wavelet approximation.

(1) The goal of the wavelet expansion of a function or signal is to obtain the coefficients of the expansion $a_{j,k}$.

(2) The second goal is to have the most zero coefficients or very small. This is called a sparse representation and it is very important in applications for statistical estimation and detection, data compression, noise reduction, and fast algorithms.

(3) The fact that the error is restricted to a small neighborhood of the discontinuity is the result of the "locality" of wavelets. The behavior of $f(x)$ at one location affects only the coefficients of wavelets close to that location.

(4) Most of the linear part of $f(x)$ is represented exactly.

We can infer from the example that the present approach is applicable to a large class of problems, where the expected solution is a periodic function. It should be also mentioned that any differential equation can be transformed into an integral equation. It means that it might be solved a large class of eigenvalue equations derived by differential equations.

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