SLAC-PUB-533
December 1968
(MISC)

APPLICATION OF SOMMERFELD – WATSON TRANSFORMATION TO AN ELECTROSTATICS PROBLEM*

Jon Pumplin
Stanford Linear Accelerator Center
Stanford University, Stanford, California

ABSTRACT

The electrostatics problem of a point charge between two infinite parallel conducting planes – i.e., the Green function for a parallel plate capacitor – is solved by the method of images. A Sommerfeld-Watson transformation is then used to obtain an integral representation for the potential. An asymptotic expression is derived for the region far from the charge, and the field is found to fall off exponentially.

(Submitted to Am. Journal of Physics)

---

*Work supported by the U. S. Atomic Energy Commission.

In this note, the electrostatics problem of a point charge located between two infinite conducting plates is solved by the method of images.[1] The resulting infinite series solution is then converted to an integral, using the Sommerfeld-Watson transformation – a technique whose application to the partial wave series for scattering amplitudes has been in vogue in particle physics since the work of Regge.[2] The integral expression is used to derive an approximate solution valid in the region far from the point charge, and the potential is found to fall off exponentially. Numerical calculations are made which show the asymptotic formula to be quite accurate even at distances comparable to the separation of the plates.

The geometry of the problem is shown in Fig. 1. ρ and z are cylindrical coordinates. Assume for simplicity that $z' = 0$, i.e., that the charge is located midway between the plates. The generalization to an arbitrary position will be given later. Also assume the plates to be grounded, i.e., to satisfy $\phi = 0$ where ϕ is the electrostatic potential. Then $\phi = 0$ and $\vec{E} = 0$ in the regions $z < -D/2$, $z > D/2$. Problems in which the plates have fixed net charges differ only by the addition of constant electric fields in the z direction to each of the three regions.

Our problem is to solve Laplace's equation $\nabla^2 \phi(\rho, z) = 0$ subject to the boundary conditions that $\phi(\rho, \pm D/2) = 0$ and that ϕ have no singularities in $|z| < D/2$ except for the point charge, $Q/\sqrt{\rho^2 + z^2}$, at the origin. Use of the method of images involves the following logic: the electrostatic potential due to a point charge at $z = \rho = 0$, an arbitrary charge distribution in the regions $|z| > D/2$, and no conducting plates, automatically satisfies $\nabla^2 \phi = 0$ and

$\phi \to Q/\sqrt{\rho^2 + z^2}$ at the origin; if one can specify the charge distribution so that in addition $\phi(\rho, \pm D/2) = 0$ for all $\rho \geq 0$, then ϕ must be the solution we seek, by the uniqueness theorem for electrostatics problems with boundary conditions of specified potential or charge distribution.[1]

The required image charge distribution is shown in Fig. 2: point charges Q at $z = 0, \pm 2D, \pm 4D, \ldots$ and $-Q$ at $\pm D, \pm 3D, \pm 5D, \ldots$. The condition $\phi(\rho, \pm D/2) = 0$ is clearly met, since the contribution to ϕ at an arbitrary point on one of the plates due to each charge Q is cancelled by that due to a charge $-Q$ which is the same distance away on the opposite side of the plate. The solution to our problem is thus

$$\phi(\rho, z) = Q \sum_{n=-\infty}^{\infty} (-1)^n \Big/ \sqrt{\rho^2 + (z - nD)^2} \quad . \tag{1}$$

This series converges, since its terms decrease monotonically in magnitude while alternating in sign. It is useful for practical calculation, however, only in the region $\rho \lesssim D$ – e.g., using results to be given later, one can show that to compute ϕ to 1% at $z = 0$, $\rho = 10$ D, it would be necessary to include 10^8 terms.

The Sommerfeld-Watson transformation consists in writing an infinite series such as (1) as a contour integral:

$$\phi(\rho, z) = \frac{Q}{2\pi i} \int_{c_1 + c_2} \frac{d\alpha}{\sqrt{\rho^2 + (z - \alpha D)^2}} \quad \frac{\pi}{\sin \pi\alpha} \tag{2}$$

The singularities of the integrand, consisting of poles from $1/\sin \pi\alpha$ and branch cuts from the square root, are shown in Fig. 3, together with the integration path.

Equivalence of (1) and (2) is obvious, for in evaluating the integral by Cauchy's theorem, one obtains contributions only from the poles of $\frac{1}{\sin \pi\alpha}$, whose residues at $\alpha = n$ are $(-1)^n/\pi$. Now bend the integration path to $c_1' + c_2'$. This is allowed because the integrand falls exponentially for $\operatorname{Im}\alpha \to \pm\infty$. Making the substitutions $\alpha = (z + i\rho t)/D$ along c_1' and $\alpha = (z - i\rho t)/D$ along c'_2 yields

$$\phi(\rho, z) = \frac{Q}{D} \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \left(\frac{i}{\sin\left[\frac{\pi}{D}(z+i\rho t)\right]} - \frac{i}{\sin\left[\frac{\pi}{D}(z-i\rho t)\right]} \right)$$

Trigonometric identities then yield the integral representation

$$\phi(\rho, z) = \frac{4Q}{D} \cos\frac{\pi z}{D} \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \cdot \frac{\sinh \pi\rho t/D}{\cosh 2\pi\rho t/D - \cos 2\pi z/D} \tag{3}$$

An asymptotic approximation to (3) for large ρ can be obtained by using $\sinh x \approx \cosh x \approx \exp(x)\,2$, and neglecting $\cos 2\pi z/D$ in the integrand. Since the cosine is bounded by 1, these approximations require only that $\exp(2\pi\rho/D) \gg 2$; they should therefore be valid whenever $\rho \gtrsim D$. The result is

$$\phi(\rho, z) \cong \frac{4Q}{D} \cos\frac{\pi z}{D} \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \exp\left(-\frac{\pi\rho t}{D}\right)$$

Only small values of t are important in this integral because of the exponential, so when $\rho \gtrsim D$ we can replace $\sqrt{t^2-1}$ by $\sqrt{2(t-1)}$, perform the remaining integral, and obtain

$$\phi(\rho, z) \cong Q \cos\frac{\pi z}{D} \exp\left(\frac{-\pi\rho}{D}\right)\sqrt{\frac{8}{\rho D}} \qquad \text{if} \qquad \rho \gtrsim D \tag{4}$$

Figure 4 shows the rapid approach of the exact solution, calculated from (1) with the aid of a computer, to the approximate one. The potential due to the bare point charge is also shown. The equipotential surfaces and the electric field lines are shown in Fig. 5.

Generalization to the case of an arbitrary location of the point charge is not difficult.[3] If the charge Q is at z', the image charges are $+Q$ at $z' \pm 2D$, $z' \pm 4D$, ... and $-Q$ at $-z' \pm D$, $-z' \pm 3D$, ... The image solution is

$$\phi(\rho, z) = Q \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{\rho^2 + (z - z' + 2nD)^2}} - \frac{1}{\sqrt{\rho^2 + [z + z' + (2n+1)D]^2}} \right) \tag{5}$$

A Sommerfeld-Watson type of transformation can be made using

$$1/\sin\{\pi[\alpha + (z' - z)/2D]\} \cdot 1/\sin\{\pi[\alpha + (z + z' + D)/2D]\}$$

to generate the poles. The resulting integral representation is

$$\phi(\rho, z) = \frac{2Q}{D} \cos\frac{\pi z}{D} \cos\frac{\pi z'}{D} \int_1^{\infty} \frac{dt}{\sqrt{t^2 - 1}} \frac{\sinh \pi\rho t/D}{[\cosh \pi\rho t/D + \cos \pi(z' + z)/D][\cosh \pi\rho t/D - \cos \pi(z' - z)/D]} \tag{6}$$

An asymptotic formula can be obtained from (6) in the previous way:

$$\phi(\rho, z) \cong Q \cos\frac{\pi z}{D} \cos\frac{\pi z'}{D} \exp\left(\frac{-\pi\rho}{D}\right) \sqrt{\frac{8}{\rho D}} \qquad \text{if} \qquad \rho \gtrsim D \tag{7}$$

(An alternate approach to this whole problem is to separate variables in cylindrical coordinates. This leads, after considerably more effort than required by the image method, to the solution in yet another form:[4]

$$\phi(\rho, z) = \frac{4Q}{D} \sum_{n=1}^{\infty} \cos \frac{n\pi z}{D} \cos \frac{n\pi z'}{D} K_0 \left(\frac{n\pi\rho}{D}\right) \tag{8}$$

Equality of this form with (6) can be established via the integral representation[5]

$$K_0(x) = \int_1^{\infty} \frac{dt}{\sqrt{t^2 - 1}} \exp(-tx) \tag{9}$$

by interchanging the order of summation and integration. Also, (7) can be derived directly from (8) using the asymptotic expression for K_0.)

NOTES AND REFERENCES

1. See any of the standard texts, e.g., W. Panofsky and M. Phillips, Classical Electricity and Magnetism, (Addison-Wesley, Reading, Mass., 1955); J. D. Jackson, Classical Electrodynamics, (John Wiley, Inc., New York, 1962).

2. T. Regge, Nuovo Cimento 54, 951 (1959).

3. The most general problem of this type, a point charge inside a grounded conducting box, can still be solved by images. If the charge is at (x', y', z') and the sides of the box at $x = \pm D_x/2$, $y = \pm D_y/2$, and $z = \pm D_z/2$, then

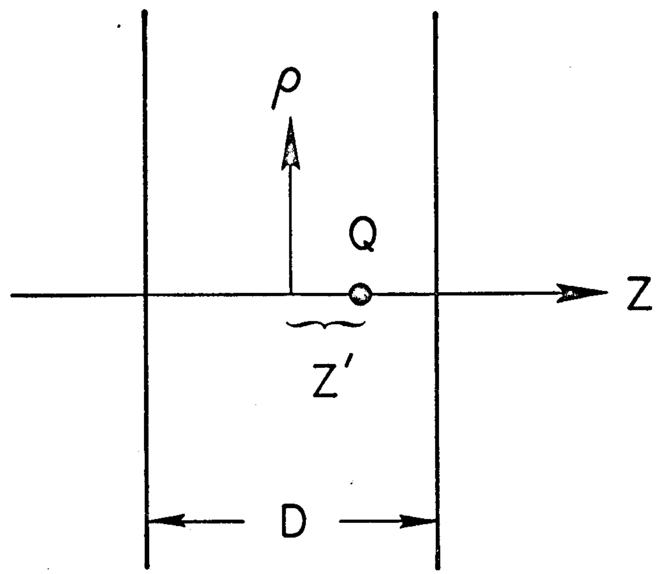
$$\phi(x,y,z) = Q \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ \left[(x - x' + 2\ell D_x)^2 + (y - y' + 2m D_y)^2 + (z - z' + 2n D_z)^2 \right]^{-1/2} - \left[\left(x + x' + (2\ell + 1) D_x\right)^2 + \left(y + y' + (2m+1) D_y\right)^2 + \left(z + z' + (2n+1) D_z\right)^2 \right]^{-1/2} \right\}$$

4. See J. D. Jackson, (Ref. 1) problem 3.13, p.97; C. Fong and C. Kittel, American Journal of Physics 35, 1091 (1967).

5. Handbook of Mathematical Functions, edited by M. Abramowitz and I. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55.

FIGURE CAPTIONS

1. Definition of coordinates.
2. Image solution for the case $z' = 0$ (point charge midway between the plates).
3. Original and deformed contours for (2).
4. Exact and approximate solutions for $z = z' = 0$.
5. Equipotentials and electric field lines.



1180A1

Fig. 1

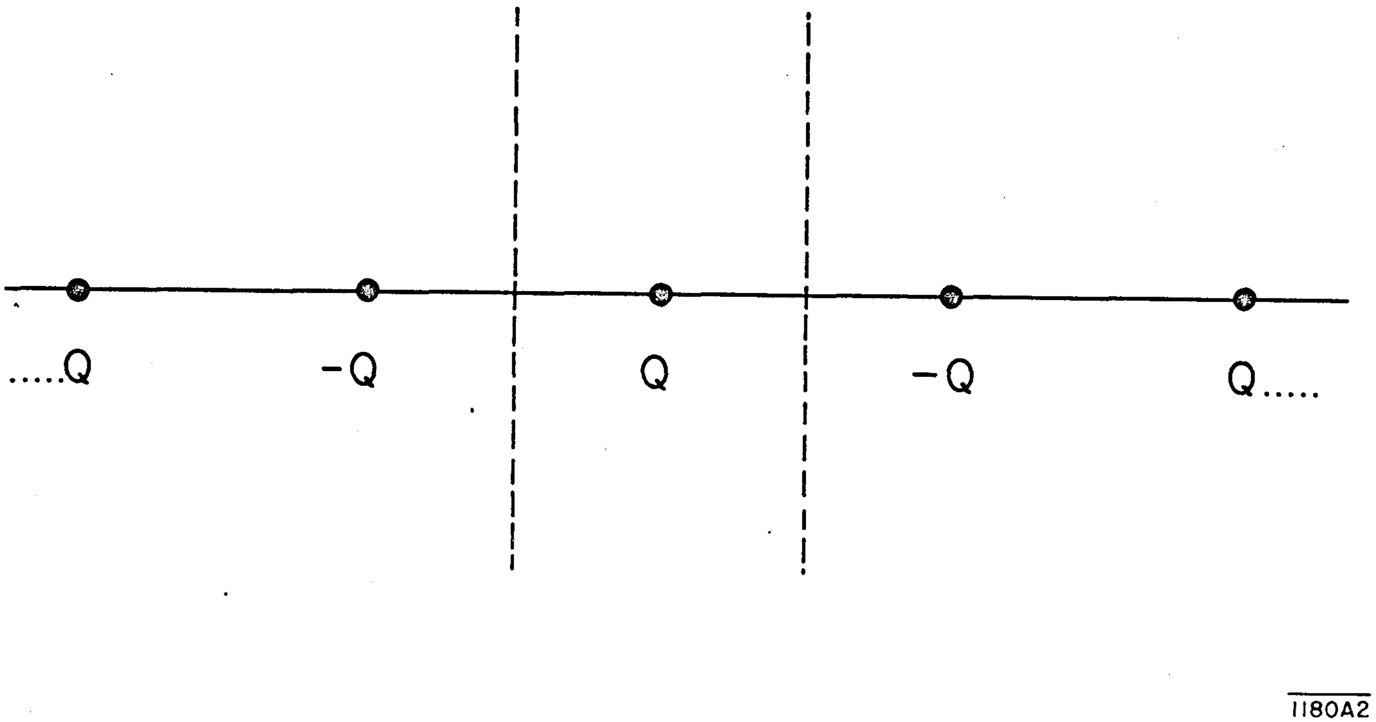


Fig. 2

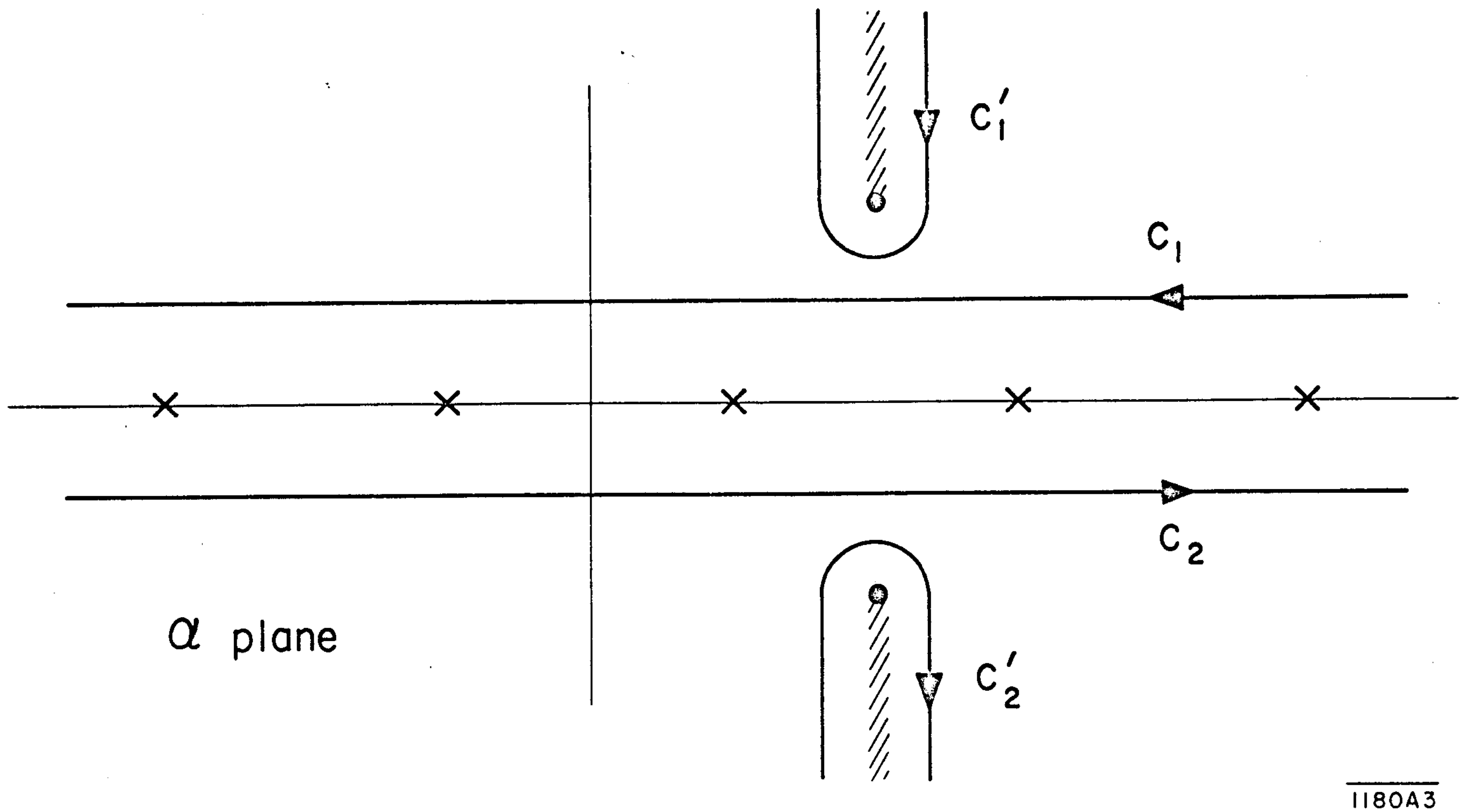
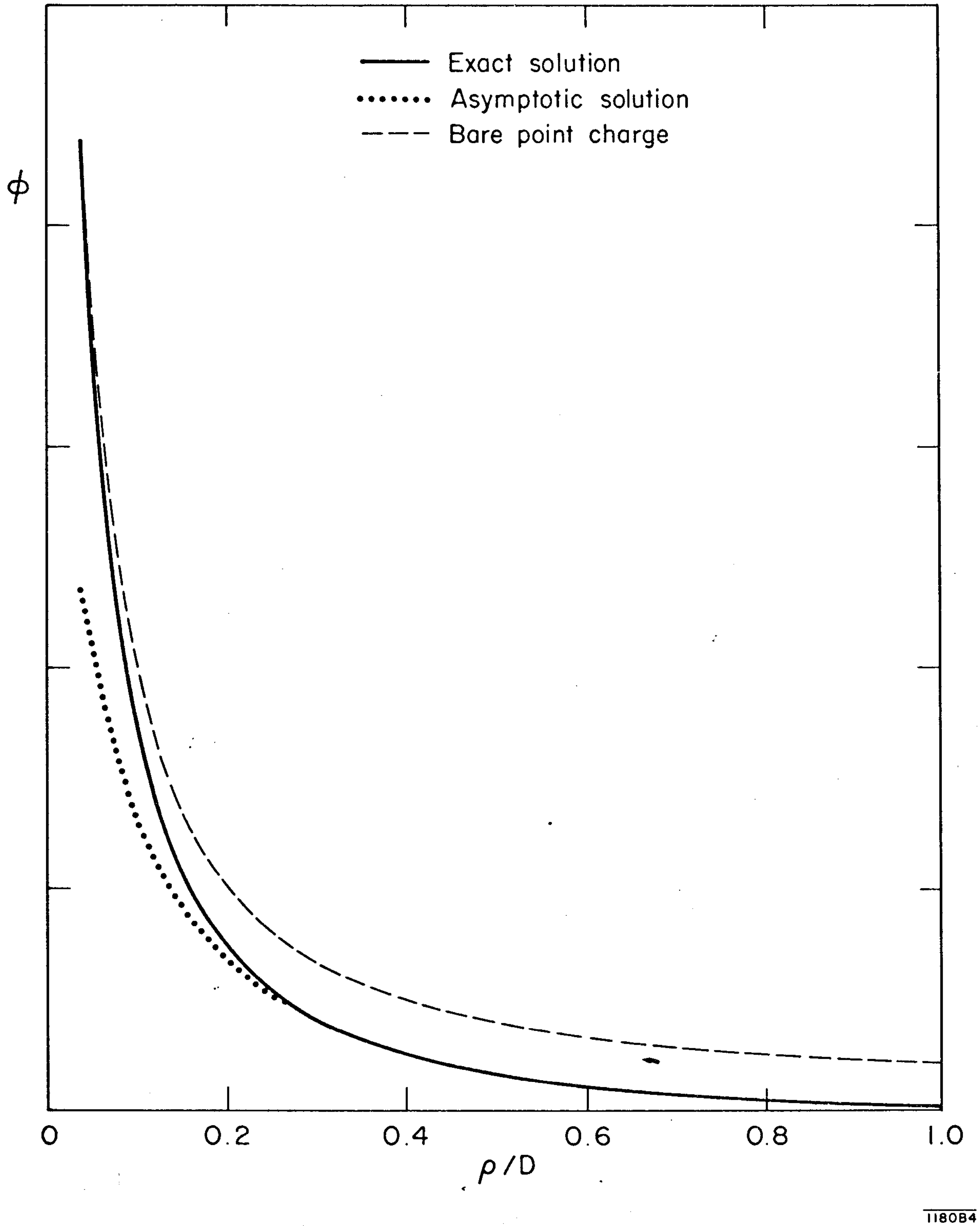
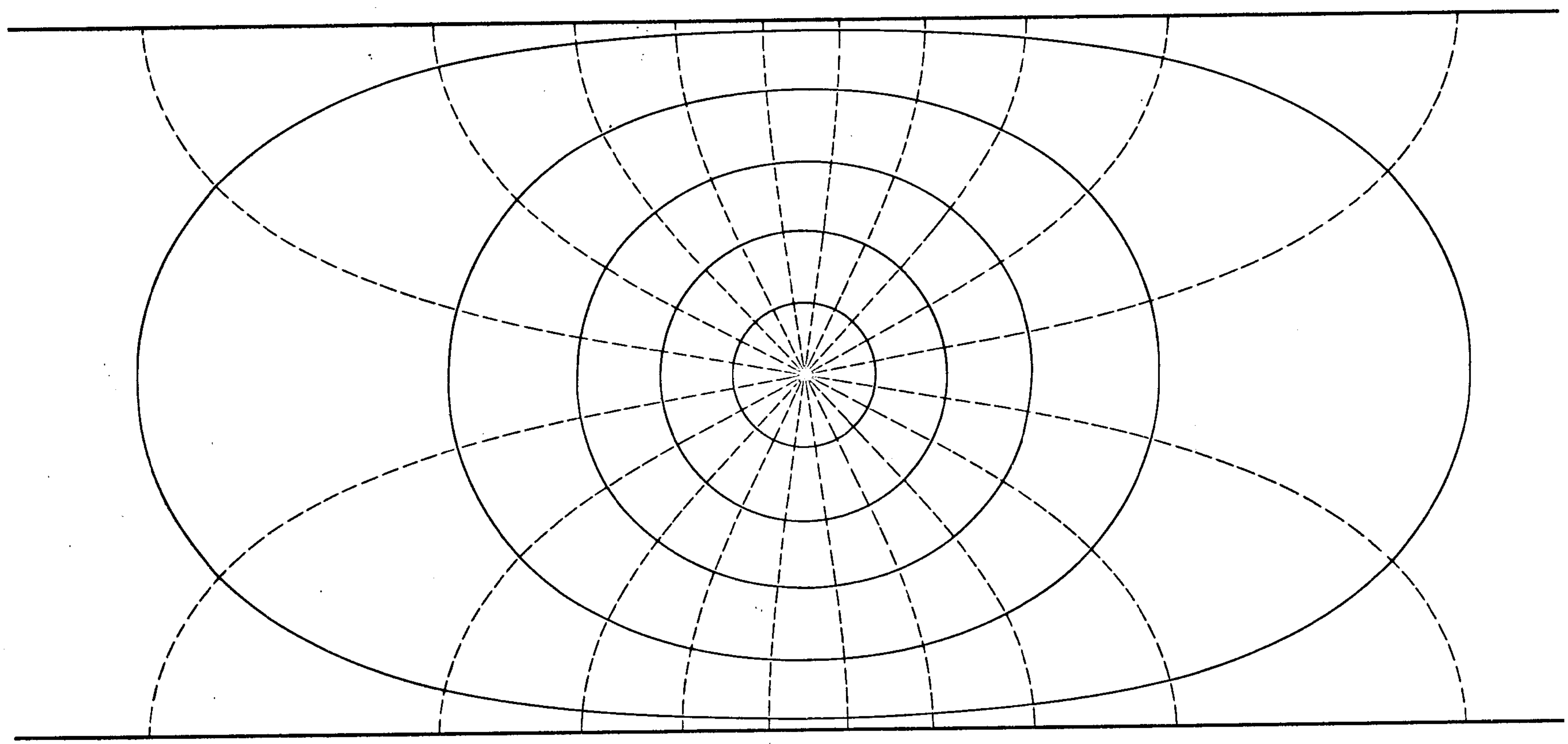


Fig. 3



Exact solution
Asymptotic solution
Bare point charge
φ
0
0.2
0.4
0.6
0.8
1.0
ρ/D
118084

Fig. 4



1180C5

Fig. 5