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Research Article

Application of the Cole-Hopf Transformation for Finding Exact Solutions to Several Forms of the Seventh-Order KdV Equation

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We use a generalized Cole-Hopf transformation to obtain a condition that allows us to find exact solutions for several forms of the general seventh-order KdV equation (KdV7). A remarkable fact is that this condition is satisfied by three well-known particular cases of the KdV7. We also show some solutions in these cases. In the particular case of the seventh-order Kaup-Kupershmidt KdV equation we obtain other solutions by some ansatzes different from the Cole-Hopf transformation.

1. Introduction

During the last years scientists have seen a great interest in the investigation of nonlinear processes. The reason for this is that they appear in various branches of natural sciences and particularly in almost all branches of physics: fluid dynamics, plasma physics, field theory, nonlinear optics, and condensed matter physics. In this sense, the study of nonlinear partial differential equations NLPDEs and their solutions has great relevance today. Some analytical methods such as Hirota method [1] and scattering inverse method [2] have been used to solve some NLPDEs. However, the use of these analytical methods is not an easy task. Therefore, several computational methods have been implemented to obtain exact solutions for these models. It is clear that the knowledge of closed-form solutions of (NLPDEs) facilitates the testing of numerical solvers, helps physicists to better understand the mechanism that governs the physic models, provides knowledge of the physic problem, provides possible applications, and aids mathematicians in the stability analysis of solutions.

The following are some of the most important computational methods used to obtain exact solutions to NLPDEs: the tanh method [3], the generalized tanh method [4, 5], the extended tanh method [6, 7], the improved tanh-coth method [8, 9], and the Exp-function method [10, 11]. Practically, there is no unified method that could be used to handle all types of nonlinear problems. All previous computational methods are based on the reduction of the original equation to equations in fewer dependent or independent variables. The main idea is to find such variables and, by passing to them, to obtain simpler equations. In particular, finding exact solutions of some partial differential equations in two independent variables may be reduced to the problem of finding solutions of an appropriate ordinary differential equation (or a system of ordinary differential equations). Naturally, the ordinary differential equation thus obtained does not give all solutions of the original partial differential equation, but provides a class of solutions with some specific properties.

The simplest classes of exact solutions to a given partial differential equation are those obtained from a *traveling-wave* transformation.

The general seventh-order Korteweg de Vries equation (KdV7) [12] reads

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0, \tag{1.1}$$

which has been introduced by Pomeau et al. [13] for discussing the structural stability of standard Korteweg de Vries equation (KdV) under a singular perturbation. Some well-known particular cases of (1.1) are the following:

(i) seventh-order Sawada-Kotera-Ito equation [12, 14–16] ($a = 252$, $b = 63$, $c = 378$, $d = 126$, $e = 63$, $f = 42$, $g = 21$):

$$u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0, \tag{1.2}$$

(ii) seventh-order Lax equation [12, 17] ($a = 140$, $b = 70$, $c = 280$, $d = 70$, $e = 70$, $f = 42$, $g = 14$):

$$u_t + 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0, \tag{1.3}$$

(iii) seventh-order Kaup-Kupershmidt equation [12, 18] ($a = 2016$, $b = 630$, $c = 2268$, $d = 504$, $e = 252$, $f = 147$, $g = 42$):

$$u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \tag{1.4}$$

Exact solutions for several forms of (1.1) have been obtained by other authors using the Hirota method [19], the tanh-coth method [19], and He's variational iteration method [20]. However, the principal objective of this work consists on presenting a condition over the coefficients of (1.1) to obtain new exact traveling-wave solutions different from those in [19, 20] by using a Cole-Hopf transformation. We show that some of the previous models (1.2)–(1.4) satisfy this condition, and therefore in these cases we obtain new exact solutions for them.

This paper is organized as follow: In Section 2, we make use of a Cole-Hopf transformation to derive a condition over the coefficients of (1.1) for the existence of traveling-wave solutions. In Section 3 we present new exact solutions for several forms of (1.1) which satisfy the condition given in Section 2. In Section 4, new exact solutions to (1.4) are obtained using other anzatzes. Finally, some conclusions are given.

2. Using the Cole-Hopf Transformation

The main purpose of this section consists on establishing a polynomial equation involving the coefficients a, b, c, d, e, f, and g of (1.1) that allows us to find traveling wave solutions (soliton and periodic solutions) using a special Cole-Hopf transformation. To this end, we seek solutions to (1.1) in the form $u = u(x,t) = u(\xi)$, where

$$\xi = kx + \omega t + \delta, k, \omega, \quad \text{with } k \neq 0,\ \omega \neq 0,\ \delta = \text{const}. \tag{2.1}$$

From (1.1) and (2.1) we obtain the following seventh-order nonlinear ordinary differential equation:

$$bk^3(u'(\xi))^3 + dk^3u^2(\xi)u'''(\xi) + ek^5u''(\xi)u'''(\xi) + u'(\xi)\left(\omega + aku^3(\xi) + ck^3u(\xi)u''(\xi) + fk^5u^{(4)}(\xi)\right) + gk^5u(\xi)u^{(5)}(\xi) + k^7u^{(7)}(\xi) = 0. \tag{2.2}$$

With the aim to find exact solutions to (2.2) we use the following Cole-Hopf transformation [21, 22]:

$$u(\xi) = A\frac{d^2}{d\xi^2}\ln\left(1 + \exp(\xi)\right) + B, \tag{2.3}$$

where $A \neq 0$ and B are arbitrary real constants.

Substituting (2.3) into (2.2), we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of ζ to zero, the following algebraic system is obtained:

$$aB^3k + B^2dk^3 + Bgk^5 + k^7 + \omega = 0, \tag{2.4}$$

$$3aAB^2k + 6aB^3k + AcBk^3 + 2ABdk^3 + Aek^5 + Afk^5 + Agk^5 - 6B^2dk^3 - 54Bgk^5 - 246k^7 + 6\omega = 0, \tag{2.5}$$

$$3aA^2Bk + 12aAB^2k + 15aB^3k + A^2bk^3 + A^2ck^3 + A^2ck^3 - 2AcBk^3 - 16ABdk^3 - 26Afk^5 - 56Agk^5 - 14Aek^5 - 33B^2dk^3 + 135Bgk^5 + 4047k^7 + 15\omega = 0, \tag{2.6}$$

$$aA^3k + 6aA^2Bk + 18aAB^2k + 20aB^3k - 2A^2bk^3 - 4A^2ck^3 - 10A^2dk^3 - 6AcBk^3 - 36ABdk^3 + 66Afk^5 + 246Agk^5 + 42Aek^5 - 52B^2dk^3 + 380Bgk^5 - 11572k^7 + 20\omega = 0. \tag{2.7}$$

From equations (2.4) and (2.5), we obtain

$$\omega = -\left(aB^3k + B^2dk^3 + Bgk^5 + k^7\right), \tag{2.8}$$

$$A = \frac{-6\omega + 246k^7 - 6B^3ak + 54Bgk^5 + 6B^2dk^3}{k^5e + fk^5 + gk^5 + 3B^2ak + Bck^3 + 2Bdk^3}. \tag{2.9}$$

Now, we substitute expression for ω in (2.8) into (2.6) and (2.7) and solve them for b and e to obtain

$$\begin{aligned} b = -\frac{1}{A^2k^2}\Big(&aA^3 + 15aA^2B + 54aAB^2 - A^2ck^2 - 7A^2dk^2 - 12ABck^2 \\ &-84ABdk^2 - 12Afk^4 + 78Agk^4 - 216B^2dk^2 + 720Bgk^4 + 504k^6\Big), \end{aligned} \tag{2.10}$$

$$\begin{aligned} e = -\frac{1}{14Ak^4}\Big(&aA^3 + 12aA^2B + 42aAB^2 - 2A^2ck^2 - 8A^2dk^2 - 10ABck^2 \\ &-68ABdk^2 + 14Afk^4 + 134Agk^4 - 168B^2dk^2 + 600Bgk^4 - 3528k^6\Big). \end{aligned} \tag{2.11}$$

Substitution of (2.10) and (2.11) into (2.9) gives

$$\frac{A(A+12B)\left(aA^2 - 2Ack^2 - 8Adk^2 + 120gk^4\right)}{aA^3 + 12aA^2B - 2A^2ck^2 - 8A^2dk^2 - 24ABck^2 - 96ABdk^2 + 120Agk^4 - 168B^2dk^2 + 600Bgk^4 - 3528k^6} = 0. \tag{2.12}$$

From this last equation we obtain, in particular,

$$A = -12B, \tag{2.13}$$

or

$$A = \frac{k^2}{a}\left(c + 4d \pm \sqrt{(c+4d)^2 - 120ag}\right). \tag{2.14}$$

These last two expressions are valid for

$$B^2d + 5Bgk^2 + 21k^4 \neq 0. \tag{2.15}$$

Suppose that $A = -12B$. Substituting this expression into (2.10) and (2.11) gives

$$b = -\frac{-216aB^3 - 216B^2dk^2 + 144Bfk^4 - 216Bgk^4 + 504k^6}{144B^2k^2}, \tag{2.16}$$

$$e = \frac{-504aB^3 - 168B^2ck^2 - 504B^2dk^2 - 168Bfk^4 - 1008Bgk^4 - 3528k^6}{168Bk^4}. \tag{2.17}$$

Eliminating B, k, and ω from (2.8), (2.16), and (2.17), we obtain the following polynomial equation in the variables a, b, c, d, e, f and g:

$$28224a^2 + P(b,c,d,e,f,g)a + Q(b,c,d,e,f,g) = 0, \tag{2.18}$$

where

$$\begin{aligned} P(b,c,d,e,f,g) &= 3(56b(e+19f-21g) - 28c(3e-7f+33g) \\ &\quad +(e+3(f+g))((e-5f+15g)(e+3(f+g)) - 336d)), \\ Q(b,c,d,e,f,g) &= (2b+c)(252b^2 - 2b(42c+504d-(e+f+6g)(e-5f+15g)) + 7c^2 \\ &\quad + c(168d-(2f-3g)(e-5f+15g)) \\ &\quad +3d(336d-(e-5f+15g)(e+3(f+g))))). \end{aligned} \tag{2.19}$$

We will call (2.18) the *first discriminant equation*. It is remarkable the fact that (1.2), (1.3), and (1.4) satisfy this equation. All seventh order equations that satisfy the first discriminant equation (2.18) may be solved exactly by means of the Cole-Hopf transformation (2.3).

Now, suppose that $A \neq -12B$. Since $A \neq 0$, then (2.14) holds. Reasoning in a similar way, from (2.14) we obtain the following second polynomial equation in terms of the coefficients of (1.1), which we shall call the *second discriminant equation* for (1.1) associated to Cole-Hopf transformation (2.3):

$$17640a^2 + \widetilde{P}(b,c,d,e,f,g)a + \widetilde{Q}(b,c,d,e,f,g) = 0, \tag{2.20}$$

where

$$\begin{aligned} \widetilde{P}(b,c,d,e,f,g) &= -3\Big(280bg + 14c(3e+5(f+6g)) + 56d(3e+5(f+3g)) \\ &\quad -g(3e+5(f+2g))^2\Big), \\ \widetilde{Q}(b,c,d,e,f,g) &= (b+c+d)\Big(10g^2(b-3d) - g(c+4d)(3e+5f) + 14(c+4d)^2\Big). \end{aligned} \tag{2.21}$$

Direct calculations show that (1.2) and (1.3) satisfy not only the first discriminant equation (2.18), but also the second one. However, (1.4) does not satisfy the second discriminant equation (2.20). All seventh-order equations that satisfy (2.20) can be solved exactly by using the Cole-Hopf transformation (2.3).

3. Solutions to the KdV7

Suppose that $A = -12B$. Then (1.1) satisfies (2.18). In this case, we obtain the following solution to (1.1):

$$u_1(x,t) = -\frac{A}{12}\left(1 - 3\operatorname{sech}^2\left(\frac{1}{2}\xi\right)\right),$$
$$\xi = \xi(x,t) = kx + \omega t + \delta, \quad \omega = -\frac{k}{1728}\left(-aA^3 + 12A^2dk^2 - 144Agk^4 + 1728k^6\right),$$
$$b = \frac{1}{8A^2k^2}\left(-aA^3 + 12A^2dk^2 + 96Afk^4 - 144Agk^4 - 4032k^6\right),$$
$$e = -\frac{1}{48Ak^4}\left(aA^3 - 4A^2ck^2 - 12A^2dk^2 + 48Afk^4 + 288Agk^4 - 12096k^6\right). \tag{3.1}$$

So that, from u_1 we can obtain explicit solutions to (1.2), (1.3), and (1.4). They are

(i) Sawada-Kotera-Ito equation (1.2):

$$A = 2k^2, \qquad \omega = \frac{1}{6}k^7, \tag{3.2}$$

$$u(x,t) = -\frac{1}{6}k^2 + \frac{1}{2}k^2\operatorname{sech}^2\left(\frac{1}{2}kx + \frac{1}{12}k^7t + \delta\right), \tag{3.3}$$

(ii) Lax equation (1.3):

$$A = 2k^2, \qquad \omega = \frac{1}{27}k^7,$$
$$u(x,t) = -\frac{1}{6}k^2 + \frac{1}{2}k^2\operatorname{sech}^2\left(\frac{1}{2}kx + \frac{1}{54}k^7t + \delta\right), \tag{3.4}$$

(iii) Kaup-Kupershmidt equation (1.4):

$$A = \frac{1}{2}k^2, \qquad \omega = \frac{1}{48}k^7, \tag{3.5}$$

$$u(x,t) = -\frac{1}{24}k^2 + \frac{1}{8}k^2\operatorname{sech}^2\left(\frac{1}{2}kx + \frac{1}{96}k^7t + \delta\right). \tag{3.6}$$

Figure 1 shows the graph of (3.6) for $k = 1.9$, $\delta = -1$, $-3 \le x \le 3$ and $0 \le t \le 3$.

It may be verified that condition (2.15) holds when A is given by either (3.2) or (3.5) since $k \neq 0$.

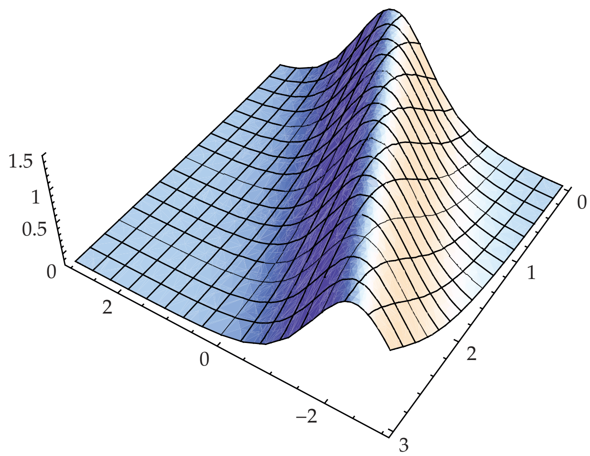

Figure 1: Graph of solution given by (3.6).

Now, let us assume that $A \neq -12B$. In this case, A is given by (2.14). Then (1.1) satisfies (2.20). We obtain the following solution to (1.1):

$$u_2(x,t) = \frac{k^2}{4a}\left(c + 4d \pm \sqrt{(c+4d)^2 - 120ag}\right)\operatorname{sech}^2\left(\frac{1}{2}\xi\right) + B,$$
$$\xi = \xi(x,t) = kx + \omega t + \delta, \quad \omega = -\left(aB^3k + B^2dk^3 + Bgk^5 + k^7\right),$$
$$b = \frac{1}{2A^2k^2}\left(6k^2\left(d\left(A^2 + 12AB + 72B^2\right) + 2Ak^2(2f - 3g) - 168k^4\right) - aA\left(A^2 + 18AB + 108B^2\right)\right),$$
$$e = -\frac{1}{2Ak^4}\left(-aAB(A+6B) + 4Bdk^2(A+6B) - 2Ak^4(f+g) + 504k^6\right). \tag{3.7}$$

So that, from u_2 we can obtain explicit solutions to (1.2) and (1.3). The first is

(i) Sawada-Kotera-Ito equation (1.2):

$$A = 2k^2, \quad \omega = -252B^3k - 126B^2k^3 - 21Bk^5 - k^7,$$
$$u(x,t) = \frac{1}{2}k^2\operatorname{sech}^2\left(\frac{1}{2}(kx + \omega t + \delta)\right) + B, \tag{3.8}$$

where

$$B \neq -\frac{k^2}{3}, \quad B \neq -\frac{k^2}{2}. \tag{3.9}$$

Restriction (1.2) guarantees condition (2.15). However, when $B = -k^2/2$, we obtain following solution to (1.2):

$$A = 2k^2, \qquad \omega = \frac{19}{2}k^7,$$
$$u(x,t) = -\frac{1}{2}k^2 \tanh^2\left(\frac{1}{2}\left(kx + \frac{19}{2}k^7 t + \delta\right)\right), \tag{3.10}$$

and when $B = -k^2/3$, another solution is given by

$$A = 2k^2, \qquad \omega = \frac{4}{3}k^7,$$
$$u(x,t) = -\frac{k^2}{3} + \frac{1}{2}k^2 \operatorname{sech}^2\left(\frac{1}{2}\left(kx + \frac{4}{3}k^7 t + \delta\right)\right), \tag{3.11}$$

(ii) The second solution is Lax equation (1.3)

$$A = 2k^2, \qquad \omega = -140B^3k - 70B^2k^3 - 14Bk^5 - k^7,$$
$$u(x,t) = \frac{1}{2}k^2 \operatorname{sech}^2\left(\frac{1}{2}(kx + \omega t + \delta)\right) + B, \tag{3.12}$$

for any real number B.

4. Other Exact Solutions to the Seventh-Order Kaup-Kupershmidt Equation

As we remarked in the previous section, the seventh-order Kaup-Kupershmidt equation (1.4) does not satisfy the second discriminant equation (2.20). However, we can use other methods to obtain exact solutions different from those we already obtained, for instance,

(i) an exp rational ansatz:

$$u(\xi) = a + \frac{b}{1 + p\exp(\xi) + q\exp(-\xi)}, \quad \xi = kx + \omega t + \delta, \tag{4.1}$$

(ii) the tanh-coth ansatz [23]:

$$u(\xi) = p + a\tanh(\xi) + b\coth(\xi) + c\tanh^2(\xi) + d\coth^2(\xi), \quad \xi = kx + \omega t + \delta, \tag{4.2}$$

where a, b, c, d, k, p, q, k, ω, and δ are constants.

4.1. Solutions by the exp *Ansatz*

From (1.4) and (4.1) we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Solving it, we get the solution

$$u(x,t) = -\frac{1}{24}k^2 + \frac{qk^2}{\exp(kx + (1/48)k^7t + \delta) + 4q^2\exp(-(kx + (1/48)k^7t + \delta)) + 4q}. \tag{4.3}$$

From (4.3) the following solutions are obtained:

$$u(x,t) = -\frac{1}{24}k^2 + \frac{1}{8}k^2\operatorname{sech}^2\left(\frac{1}{2}\left(kx + \frac{1}{48}k^7t + \delta\right)\right), \quad q = \frac{1}{2}. \tag{4.4}$$

Observe that solutions given by (3.6) and (4.4) coincide. Consider

$$\begin{gathered} u(x,t) = -\frac{1}{24}k^2 - \frac{1}{8}k^2\operatorname{csch}^2\left(\frac{1}{2}\left(kx + \frac{1}{48}k^7t + \delta\right)\right), \quad q = -\frac{1}{2}, \\ u(x,t) = \frac{1}{24}k^2 - \frac{1}{4}k^2\frac{1}{1 - \sin(kx - (1/48)k^7t + \delta)}, \quad q = \frac{\sqrt{-1}}{2},\ k \longrightarrow k\sqrt{-1},\ \delta \longrightarrow \delta\sqrt{-1}, \\ u(x,t) = \frac{1}{24}k^2 - \frac{1}{4}k^2\frac{1}{1 + \sin(kx - (1/48)k^7t + \delta)}, \quad q = -\frac{\sqrt{-1}}{2},\ k \longrightarrow k\sqrt{-1},\ \delta \longrightarrow \delta\sqrt{-1}. \end{gathered} \tag{4.5}$$

4.2. Solutions by the tanh - coth *Ansatz*

We change the tanh and coth functions to their exponential form and then we substitute (4.2) into (1.4). We obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of ζ to zero results in an algebraic system in the variables a, b, c, d, p, k, and ω. Solving it with the aid of a computer, we obtain the following solutions of (1.4).

(i) $a = 0,\ b = 0,\ c = 0,\ d = -(1/2)k^2,\ p = (1/3)k^2,\ \omega = (4/3)k^7$:

$$\begin{gathered} u(x,t) = \frac{1}{3}k^2 - \frac{1}{2}k^2\coth^2\left(k\left(x + \frac{4}{3}k^6t\right)\right). \\ u(x,t) = -\frac{1}{3}k^2 - \frac{1}{2}k^2\cot^2\left(k\left(x - \frac{4}{3}k^6t\right)\right). \end{gathered} \tag{4.6}$$

(ii) $a = 0,\ b = 0,\ c = -(1/2)k^2,\ d = 0,\ p = (1/3)k^2,\ \omega = (4/3)k^7$:

$$u(x,t) = \frac{1}{3}k^2 - \frac{1}{2}k^2\tanh^2\left(k\left(x + \frac{4}{3}k^6t\right)\right). \tag{4.7}$$

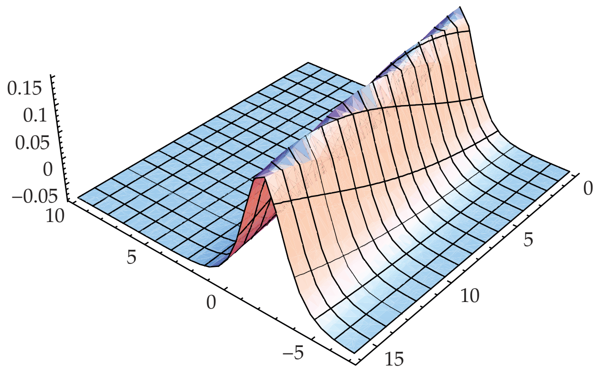

Figure 2: Graph of solution (4.7).

Figure 2 shows the graph of $u_3(x,t)$ for $k = -0.68$, $-7 \le x \le 10$, and $0 \le t \le 16$.

$$u(x,t) = -\frac{1}{3}k^2 - \frac{1}{2}k^2\tan^2\left(k\left(x - \frac{4}{3}k^6 t\right)\right). \tag{4.8}$$

Figure 3 shows the graph of $u_4(x,t)$ for $k = 1.18$, $-3 \le x \le 2$, and $0 \le t \le 2$

(iii) $a = 0,\ b = 0,\ c = -(1/2)k^2,\ d = -(1/2)k^2,\ p = (1/3)k^2,\ \omega = (256/3)k^7$:

$$\begin{aligned} u(x,t) &= \frac{k^2}{3} - \frac{k^2}{2}\coth^2\left(k\left(x + \frac{256}{3}k^6 t\right)\right) - \frac{k^2}{2}\tanh^2\left(k\left(x + \frac{256}{3}k^6 t\right)\right), \\ u(x,t) &= -\frac{k^2}{3} - \frac{k^2}{2}\cot^2\left(k\left(x - \frac{256}{3}k^6 t\right)\right) - \frac{k^2}{2}\tan^2\left(k\left(x - \frac{256}{3}k^6 t\right)\right). \end{aligned} \tag{4.9}$$

4.3. Other Exact Solutions

We may employ other methods to find exact solutions to the seventh-order Kaup-Kupershmidt equation. Thus, a solution to (1.4) of the form

$$u(\xi) = A\frac{d^2}{d\xi^2}\log(s + \exp(\xi) + \exp(2\xi)) + B, \quad \xi = kx + \omega t + \delta \tag{4.10}$$

is

$$\begin{gathered} u(\xi) = \frac{-k^2s^2(16s-1) + 2k^2s(8s-5)e^{\xi} + k^2(160s^2-62s+1)e^{2\xi} + 2k^2(8s-5)e^{3\xi} - k^2(16s-1)e^{4\xi}}{24s^2(4s-1) + 48s(4s-1)e^{\xi} + 24(2s+1)(4s-1)e^{2\xi} + 48(4s-1)e^{3\xi} + 24(4s-1)e^{4\xi}}, \\ \xi = \xi(x,t) = kx + \omega t + \delta, \quad \omega = \frac{4096s^3 + 960s^2 + 264s - 1}{48(4s-1)^3}k^7, \quad A = \frac{1}{2}k^2, \quad B = -\frac{k^2(16s-1)}{24(4s-1)}. \end{gathered} \tag{4.11}$$

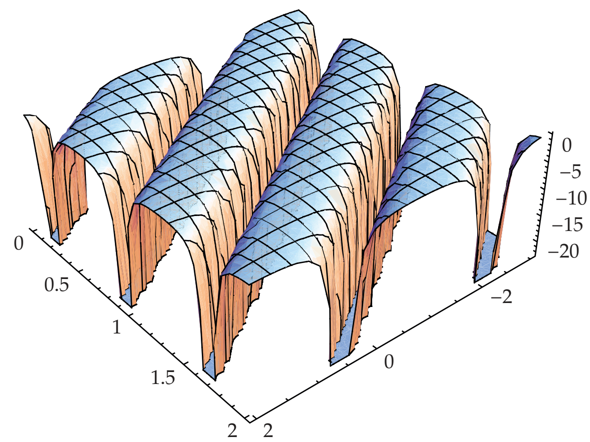

Figure 3: Graph of solution (4.8).

In particular, taking $s = \pm 1$, we get

$$u(x,t) = \frac{k^2\left(4\cosh\left(kx + (197/48)k^7 t + \delta\right) - 10\cosh\left(2\left(kx + (197/48)k^7 t + \delta\right)\right) + 33\right)}{24\left(2\cosh(kx + (197/48)k^7 t + \delta) + 1\right)^2}, \quad (4.12)$$

$$u(x,t) = \frac{k^2\left(52\sinh\left(kx + (3401/6000)k^7 t + \delta\right) - 34\cosh\left(2\left(kx + (3401/6000)k^7 t + \delta\right)\right) - 223\right)}{120\left(2\sinh(kx + (3401/6000)k^7 t + \delta) + 1\right)^2}. \quad (4.13)$$

Following solution results from (4.12) by replacing k with $k\sqrt{-1}$ and δ with $\delta\sqrt{-1}$:

$$u(x,t) = \frac{k^2\left(10\cos\left(2kx - (197/24)k^7 t + 2\delta\right) - 4\cos\left(kx - (197/48)k^7 t + \delta\right) - 33\right)}{24\left(2\cos(kx - 197k^7 t/48 + \delta) + 1\right)^2}. \quad (4.14)$$

The choice $s = 1/16$ gives

$$u(x,t) = \frac{8k^2\left(15\sinh\left(kx - k^7 t + \delta\right) + 17\cosh\left(kx - k^7 t + \delta\right) + 4\right)}{\left(15\sinh(kx - k^7 t + \delta) + 17\cosh(kx - k^7 t + \delta) + 16\right)^2}. \quad (4.15)$$

Finally, by using the ansatze

$$u(\xi) = \frac{p + q\cosh(2\xi)}{(m + \sinh(\xi))^2}, \qquad u(\xi) = \frac{r + s\cosh(2\xi)}{(m + \cosh(\xi))^2}, \quad (4.16)$$

we obtain

$$u(x,t) = -\frac{5k^2\left(11 + 5\cos\left(2kx - 19k^7 t + 2\delta\right)\right)}{8\left(\sqrt{10} \pm 5\sin\left(kx - (19/2)k^7 t + \delta\right)\right)^2}, \tag{4.17}$$

$$u(x,t) = \frac{5k^2\left(11 - 5\cosh\left(2kx + 19k^7 t + 2\delta\right)\right)}{8\left(\sqrt{10} \pm 5\cosh\left(kx + (19/2)k^7 t + \delta\right)\right)^2}. \tag{4.18}$$

The correctness of the solutions given in this work may be checked with the aid of either *Mathematica 7.0* or *Maple 12*. For example, to see that function defined by (4.12) is a solution to the seventh-order Kaup-Kupershmidt (1.4), we may use *Mathematica 7.0* as follows:

```
In [1]:= kk7 = ∂t # + 2016 #^3∂x # + 630 (∂x #)^3 + 2268 # ∂x # ∂x,x # + 504
        #^2∂x,x,x # + 252 ∂x,x # ∂x,x,x # + 147 ∂x # ∂x,x,x,x # + 42 # ∂x,x,x,x,x # +
        ∂x,x,x,x,x,x,x # &;

     (*This defines the differential operator associated with the
seventh-order Kaup-Kupershmidt equation*)
```

$$\mathtt{sol} = \frac{k^2\left(33 + 4\cosh\left[k\,x + \frac{197\,k^2\,t}{48} + \delta\right] - 10\cosh\left[2\left(k\,x + \frac{197\,k^7\,t}{48} + \delta\right)\right]\right)}{24\left(1 + 2\cosh\left[k\,x + \frac{197\,k^2\,t}{48} + \delta\right]\right)^2} \tag{4.19}$$

```
(*This defines the solution*)

In [3]:= Simplify [kk7 [TrigToExp [sol]]]

     (*Here we apply the differential operator to the solution and then we
make use of the Simplify command*)

Out [3]:= 0
```

5. Conclusions

In this work we have obtained two conditions associated to Cole-Hopf transformation (2.3) for the existence of exact solutions to the general KdV7. Two cases have been analyzed. The first one is given by (2.13) and the other by (2.14). The first case leads to first discriminant equation (2.18) which is satisfied by Sawada-Kotera-Ito equation (1.2), Lax equation (1.3), and Kaup-Kupershmidt equation (1.4). The second case gives the second discriminant equation (2.20), which is fulfilled only by (1.2) and (1.3). However, when both conditions (2.13) and (2.14) hold, we may obtain solutions different from those that result from the two cases we mentioned. We did not consider this last case. Other works that related the problem of finding exact solutions of nonlinear PDE's may be found in [24, 25].

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