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THE APPLICATION OF THE MANDELSTAM REPRESENTATION TO PHOTOPRODUCTION OF PIONS FROM NUCLEONS

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James Stutsman Ball
(Thesis)

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ABSTRACT

The Mandelstam representation is applied to the invariant amplitudes for photoproduction. By treating gauge invariance as a subsidiary condition, it is shown that the fixed-momentum-transfer dispersion relations of Chew, Goldberger, Low, and Nambu(CGLN) are probably valid without subtractions for the (-) amplitudes while a three-pion resonances would perhaps require a subtraction in the (+) amplitudes. The two-pion resonance will certainly require a subtraction for the (0) amplitudes, but to a good approximation the contribution of the two-pion intermediate state is found to produce a simple additive correction to the CGLN (0) formula. The strength of this new term is determined by a parameter Λ, which has been introduced elsewhere in treating the photon, three-pion problem. Otherwise, the form of the new term can be expressed in terms of nucleon electromagnetic form factors. Finally, the photoproduction amplitudes are calculated in the threshold region, and an estimate of the size of Λ is made.

I. INTRODUCTION

Recently Chew and Mandelstam have proposed a method for calculating the behavior of systems of strongly interacting particles and have applied it to the problem of pion-pion scattering.[1] This method is based on Mandelstam's generalization of dispersion relations,[2] which provides a means of extending scattering amplitudes into the complex plane for both the energy-and momentum-transfer variables. The Feynman substitution rule then allows one set of invariant amplitudes to represent the three different processes that can be obtained from a four-particle diagram by substitution. The new method has already been applied to the process $\gamma + \pi \to \pi + \pi$,[3] $\pi + \pi \to N + \overline{N}$,[4] and $N + N \to N + N$,[5] in addition to $\pi\pi$ scattering. Our purpose here is to extend the new approach to pion photoproduction from nucleons and in particular to investigate the effect of the pion-pion interaction on photoproduction.

In the case of photoproduction, the invariant amplitudes satisfying the Mandelstam representation will be the scattering amplitudes for the three processes shown in Fig. 1 a, b, and c when the variables are in the appropriate physical region for each process. The Mandelstam singularities appear in the energy variables for each of these processes, $N + \gamma \to N + \pi$, $\gamma + \pi \to \overline{N} + N$, and $\gamma + \overline{N} \to \pi + \overline{N}$. Their location is determined by the masses of intermediate states that have the same quantum numbers as the initial and final states for the reaction in question.

To take full advantage of the Mandelstam representation we must treat simultanously all three processes or channels shown in Fig 1 a, b, and c. Of course, each process will have the same invariant amplitudes, but in order to apply unitarity it is convenient to employ eigenstate amplitudes for the various channels.

In the following section, the invariant amplitudes are defined in terms of invariant-spin and isotopic-spin matrices. The angular-momentum decomposition of the invariant amplitudes in terms of

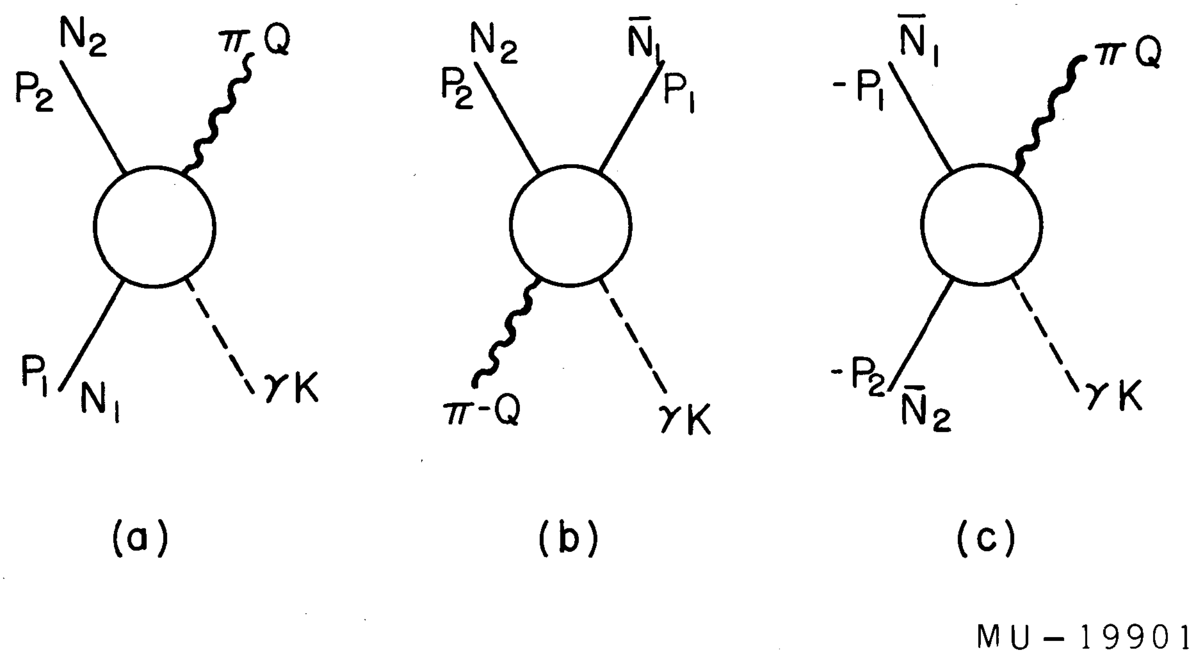

Fig. 1. The three channels of the pion, photon, two-nucleon problem.

multipoles is carried out for photoproduction in Section III, expressing the connection between the invariant amplitudes and the eigenamplitudes for this channel. Section IV deals with the angular-momentum decomposition for $\gamma + \pi \rightarrow N + \overline{N}$. The unitarity condition relates the eigenamplitudes in this case to those of $\pi + \pi \rightarrow N + \overline{N}$ which have been treated earlier[4] in terms of the helicity amplitudes of Jacob and Wick (hereafter referred to as JW).[6] Therefore, it is convenient to employ helicity eigenstates for this channel. The process $\gamma + N \rightarrow \overline{N} + \pi$ (Fig. 1 c) is examined in Section V and found to be related to the photoproduction channel by a simple crossing condition on the invariant amplitudes, making it unnecessary to treat this channel directly. The treatment of the photoproduction channel in Sections II, III, and V closely follows the work of Chew, Goldberger, Low, and Nambu (hereafter CGLN).[7]

In Section VI, the invariant amplitudes are expressed in the Mandelstam form. A general procedure to obtain a complete solution of the photoproduction problem is discussed in Sections VII and VIII. Finally Sections IX and X deal with a low-energy approximation for the photoproduction amplitudes, based on the assumption that pion-pion and pion-nucleon interactions are both dominated by <u>P</u>-wave resonances.

II. CONSTRUCTION OF THE INVARIANT AMPLITUDE

Let P_1 and P_2 denote the initial and final nucleon four-vector momenta and Q and K represent those of the pion and the photon, respectively. Since we will consider all three processes in Fig. 1, it is convenient to define the variables:

$$s = -(P_1 + K)^2 \qquad t = -(Q - K)^2 \qquad \bar{s} = -(P_2 - K)^2 , \tag{2.1}$$

which are the squares of the total energy in the barycentric system for the three processes in Fig. 1. The amplitudes satisfying the Mandelstam representation will have singularities in s, t, and $\bar{s}$, corresponding to the possible intermediate states for each channel. We shall be concerned only with matrix elements linear in the electric charge, as higher-order terms will be smaller at least by a factor of $e^2 = 1/137$. This makes it possible to ignore all intermediate states containing photons. The lowest-mass state that can connect a photon and a nucleon to a pion and a nucleon is, due to baryon conservation, the nucleon itself, producing a pole at $s = M^2$.* The next-higher mass corresponds to the nucleon plus one pion, which is a continuum of states with mass starting at (M + 1) and going to infinity. This produces a branch cut in the s plane running from $(M + 1)^2$ to infinity. The next higher state, two pions and a nucleon, produces a cut from $(M + 2)^2$ to infinity, and similarly for the higher states a branch point will appear at each new threshold. For the $\gamma + \pi \rightarrow N + \bar{N}$ channel, the lowest-mass state is one pion, producing a pole at t = 1. The continuum states start with the two-pion state, which produces a branch cut in the t plane from four to infinity. The $\gamma + \bar{N} \rightarrow \bar{N} + \pi$ channel will produce the same singularities in the

*The units employed throughout are $\hbar = c =$ pion mass = 1 and the metric used is $g_i = 1$ for i = 1, 2, or 3 and $g_0 = -1$.

$\bar{s}$ plane as the photoproduction channel produces in the s plane. It must be remembered that the possible intermediate states also depend on the spin and isotopic-spin quantum numbers, so that the above singularities will not be common to all of the invariant amplitudes.

Conservation of energy-momentum,

$$P_1 + K = P_2 + Q \,, \tag{2.2}$$

and the mass-shell restrictions, $P_1^2 = P_2^2 = - M^2$, $Q^2 = - 1$, and $K^2 = 0$, lead to

$$s + t + \bar{s} = 2M^2 + 1. \tag{2.3}$$

For the photoproduction channel we have

$$s = (E_1 + k)^2 = (E_2 + \omega)^2 \,, \tag{2.4a}$$

$$t = 1 - 2\omega k + 2\, qk \cos\, \theta, \tag{2.4b}$$

and

$$\bar{s} = M^2 - 2\, E_2 k - 2\, qk \cos\, \theta. \tag{2.4c}$$

With reference to the barycentric system, q and k are the magnitudes of the meson and photon momenta, $E_1 = (k^2 + M^2)^{1/2}$ and $E_2 = (q^2 + M^2)^{1/2}$ are the initial and final nucleon energies, $\omega = (q^2 + 1)^{1/2}$ is the meson energy, and $\cos\,\theta = \dfrac{\vec{Q} \cdot \vec{K}}{qk}$ defines the production angle. The quantities E_1, E_2, q, k, and ω depend only on s, being given by the formulas:

$$E_1(s) = \frac{s+M^2}{2\sqrt{s}} \quad , \quad k(s) = \frac{s - M^2}{2\sqrt{s}} \quad , \quad \omega(s) = \frac{s - M^2 + 1}{2\sqrt{s}} \quad ,$$

$$E_2(s) = \frac{s + M^2 - 1}{2\sqrt{s}} \quad , \quad q(s) = \left\{ \frac{[s - (M+1)^2][s - (M-1)^2]}{4s} \right\}^{1/2} .$$

(2.5)

The S matrix for photoproduction may be written

$$S_{fi} = \frac{i}{(2\pi)^2} \quad \frac{M\delta^4(P_1 + K - P_2 - Q)\, \bar{u}(P_2)\, Tu(P_1)}{(4\, E_1\, E_2\, \omega k)^{1/2}} \quad , \tag{2.6}$$

where T is a function of P_1 , P_2 , Q, K, and ϵ , the photon polarization.

The most general form of the matrix, T, allowed by Lorentz invariance is

$$T = \sum_i B_i(s, t, \bar{s})\, N_i(P_1, P_2, K, \epsilon, \gamma) , \tag{2.7}$$

where the N_i's are all of the independent Lorentz-invariant matrices that can be formed containing γ's and ϵ, and the B_i's are functions of s, t, and $\bar{s}$. The gauge invariance requirement on T is considered a subsidiary condition to be applied <u>after</u> the analyticity properties of the spin-independent amplitudes, B_i , have been exhibited.

The first step in the decomposition of T into spin-independent amplitudes, B_i , is the construction of the N_i's , which must be linear and homogenous in ϵ . From Eq. (2.3) we see that there are only three independent momentum four-vectors. If we employ the Dirac equations,

$$(i\,\gamma \cdot P_1 + M)\, u(P_1) = 0 \tag{2.8a}$$

and

$$(i\gamma \cdot P_2 + M)\, u(P_2) = 0 \quad , \qquad (2.8b)$$

and the commutation relation,

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \, , \qquad (2.9)$$

terms in T of the form $\gamma \cdot P_2$ and $\gamma \cdot P_1$ can be commuted to the left and right, respectively, and made to operate directly on the corresponding spinor. Thus they may be replaced by iM, leaving $\gamma \cdot \epsilon$ and $\gamma \cdot K$ as the only γ-matrix invariants.

Further, since we have

$$\gamma \cdot K \gamma \cdot K = K^2 = 0 \, , \qquad (2.10)$$

the N_i's can contain no more than one of each of the factors $\gamma \cdot \epsilon$ and $\gamma \cdot K$. From Eq. (2.2) we obtain

$$K \cdot \epsilon + P_1 \cdot \epsilon = P_2 \cdot \epsilon + Q \cdot \epsilon \, , \qquad (2.11)$$

leaving only three independent scalars of this type. Let $P = \frac{1}{2}(P_1 + P_2)$; then we chose $P \cdot \epsilon$, $K \cdot \epsilon$, and $Q \cdot \epsilon$ as our three scalars containing ϵ.

The above considerations tell us that a given N_i must contain one of $P \cdot \epsilon$, $K \cdot \epsilon$, $Q \cdot \epsilon$, or $\gamma \cdot \epsilon$ and be at most linear in $\gamma \cdot K$. The combinations possible are therefore $\gamma \cdot \epsilon \gamma \cdot K$, $\gamma \cdot K P \cdot \epsilon$, $\gamma \cdot K K \cdot \epsilon$, $\gamma \cdot K Q \cdot \epsilon$, $\gamma \cdot \epsilon$, $P \cdot \epsilon$, $Q \cdot \epsilon$, and $K \cdot \epsilon$. Finally, the fact that the produced meson is a pseudoscalar requires an added factor of γ_5, leading us to define the eight basic forms:

$$N_1 = i\gamma_5 \gamma \cdot \epsilon \gamma \cdot K$$

$$N_2 = 2i\gamma_5 P \cdot \epsilon$$

$$N_3 = 2i\,\gamma_5\, Q \cdot \epsilon$$

$$N_4 = 2i\,\gamma_5\, K \cdot \epsilon$$

$$N_5 = \gamma_5\, \gamma \cdot \epsilon$$

$$N_6 = \gamma_5\, \gamma \cdot K\, P \cdot \epsilon$$

$$N_7 = \gamma_5\, \gamma \cdot K\, K \cdot \epsilon$$

$$N_8 = \gamma_5\, \gamma \cdot K Q \cdot \epsilon\,, \qquad (2.12)$$

where the factors i and 2 are added for convenience in subsequent calculations. We now express T as

$$T = \sum_{i=1}^{8} B_i\,(s,\, t,\, \overline{s})\, N_i\,. \qquad (2.13)$$

Gauge invariance demands that if ϵ is replaced by K in the N_i's, T must vanish. This requirement yields the following relations between the B_i's:

$$B_2\, P \cdot K + B_3\, Q \cdot K = 0 \qquad (2.14a)$$

$$B_5 + B_6\, P \cdot K + B_8\, Q \cdot K = 0\,, \qquad (2.14b)$$

which reduce by two the number of independent amplitudes. As the final T matrix of interest will be gauge invariant, and since we are interested only in the case $K \cdot \epsilon = 0$, it is convenient now to introduce the CGLN matrices:

$$M_i = i\,\gamma_5\, \gamma \cdot \epsilon\, \gamma \cdot K \qquad (2.15)$$

$$M_2 = 2i\gamma_5 (P \cdot \epsilon Q \cdot K - P \cdot K Q \cdot \epsilon) \quad (2.16)$$

$$M_3 = \gamma_5 (\gamma \cdot \epsilon Q \cdot K - \gamma \cdot K Q \cdot \epsilon) \quad (2.17)$$

$$M_4 = 2\gamma_5 (\gamma \cdot \epsilon P \cdot K - \gamma \cdot K P \cdot \epsilon - iM \gamma \cdot \epsilon \gamma \cdot K). \quad (2.18)$$

We now express T as

$$T = \sum_{i=1}^{4} A_i (s, t, \bar{s}) M_i , \quad (2.19)$$

where $A_1 = B_1 - MB_6$, $A_2 = B_2/Q \cdot K = - B_3/ P \cdot K$, $A_3 = - B_8$, and $A_4 = -\frac{1}{2} B_6$.

The isotopic-spin dependence of the A_i's can be removed in a similar manner by constructing all of the possible basic forms containing the isotopic-spin operators τ_1, τ_2, and τ_3 for the nucleon. Watson[8] has shown that the photon may be treated as an isotopic scalar plus the third component of an isotopic vector. If this is done, the basic forms will be charge-independent for each part of the photon interaction. For the vector part, we may treat the photon as a π^o; thus the two charge-independent forms are $\tau_\beta \tau_3$ and $\tau_3 \tau_\beta$, where β is the isotopic-spin index of the final pion. For the scalar part, we form a charge-independent quantity. This is, of course, just τ_β. We thus take as our three basic forms,

$$g_\beta^+ = \frac{1}{2} [\tau_\beta \tau_3 + \tau_3 \tau_\beta] = \delta_{\beta 3} , \quad (2.20)$$

$$g_\beta^- = \frac{1}{2} [\tau_\beta , \tau_3] , \quad (2.21)$$

and

$$g_\beta^o = \tau_\beta , \quad (2.22)$$

and define:

$$A_i = A_i^+ g_\beta^+ + A_i^- g_\beta^- + A_i^0 g_\beta^0 , \qquad (2.23)$$

where the twelve functions $A_i^{(+,-,0)}$ depend only on s, t, and $\bar{s}$.

With the usual representation for the τ's the proton and antiproton states are represented by $X_P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the neutron and antineutron by $X_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The final nucleons and initial antinucleon spinors stand to the left of the g's, while initial nucleons and final antinucleon spinors stand to the right. A physical interpretation for g_β^+, g_β^-, and g_β^0 is provided by considering the isotopic-spin projection operators for $\pi_0 + N \rightarrow N + \pi$:

$$P_{\frac{1}{2}} (\beta , 3) = \frac{1}{3} \delta_{\beta 3} + \frac{1}{3} \frac{[\tau_\beta , \tau_3]}{2} \qquad (2.24)$$

$$P_{\frac{3}{2}} (\beta , 3) = \frac{2}{3} \delta_{\beta 3} - \frac{1}{3} \frac{[\tau_\beta , \tau_3]}{2} . \qquad (2.25)$$

We see that $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ are linear combinations of the g^+ and g^- corresponding to the isotopic vector part of the photon interaction. At the same time, the scalar projection operator g_β^0 contributes only to the $I = \frac{1}{2}$ amplitude.

III. THE ANGULAR-MOMENTUM DECOMPOSITION FOR PHOTOPRODUCTION

The complete amplitude for photoproduction can now be reduced to Pauli spinor by using the explicit form of the Dirac spinors:

$$u_r(P) = \frac{M - i\gamma \cdot P}{[2M(E+M)]^{1/2}} \begin{pmatrix} X_r \\ 0 \end{pmatrix} \qquad (3.1a)$$

and

$$\bar{u} = u^+ \gamma_4 , \qquad (3.1b)$$

where

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$

$$\gamma_0 = i\gamma_4 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

$$\gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 ,$$

$$\chi_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

Here χ_1 and χ_{-1} correspond to nucleon spin up and down, respectively, in the rest system of the particle.

The differential cross-section for meson production in the barycentric system is expressed as

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| \chi_f \mathcal{F} \chi_i \right|^2 , \qquad (3.2)$$

where we define

$$\chi_f \mathcal{F} \chi_i = \bar{u}_f T u_i , \tag{3.3}$$

and $\mathcal{F}$ is given by

$$\mathcal{F} = i\vec{\sigma} \cdot \vec{\epsilon} \mathcal{F}_1 + \frac{\sigma \cdot \vec{Q} \sigma \cdot \vec{K} \vec{\epsilon}}{qK} \mathcal{F}_2 + i \frac{\vec{\sigma} \cdot \vec{K} \vec{Q} \cdot \vec{\epsilon}}{qk} \mathcal{F}_3 + i \frac{\vec{\sigma} \cdot \vec{Q} \vec{Q} \cdot \epsilon}{q^2} \mathcal{F}_4 . \tag{3.4}$$

The $\mathcal{F}_i$'s are functions of A_1, A_2, A_3, and A_4. We will define F_1, F_2, F_3, and F_4 as follows:

$$F_1 = 4\pi \frac{2W}{W - M} \frac{\mathcal{F}_1}{[(E_2 + M)(E_1 + M)]^{1/2}}$$

$$= A_1 + (W - M) A_4 - \frac{t - 1}{2(W - M)} (A_3 - A_4) \tag{3.5}$$

$$F_2 = 4\pi \frac{2W}{W - M} \left(\frac{E_2 + M}{E_1 + M} \right)^{1/2} \frac{\mathcal{F}_2}{q}$$

$$= - A_1 + (W + M) A_4 - \frac{t - 1}{2(W + M)} (A_3 - A_4) \tag{3.6}$$

$$F_3 = 4\pi \frac{2W}{W - M} \frac{\mathcal{F}_3}{[(E_2 + M)(E_1 + M)]^{1/2} q}$$

$$= (W - M) A_2 + (A_3 - A_4) \qquad (3.7)$$

$$F_4 = 4\pi \frac{2W}{W - M} \left(\frac{E_2 + M}{E_1 + M}\right)^{1/2} \frac{\mathcal{F}_4}{q^2}$$

$$= -(W + M) A_2 + (A_3 - A_4) . \qquad (3.8)$$

It is well known that, for photoproduction, the unitarity condition requires the phase of an amplitude leading to an outgoing pion-nucleon state of definite angular-momentum, isotopic spin, and parity to be the same as the phase of the pion-nucleon scattering amplitude leading to the same final state.[9] To use the unitarity condition we must decompose the $\mathcal{F}$'s into definite parity eigenamplitudes.

This angular-momentum decomposition has been carried out by C G L N; they obtain

$$\mathcal{F}_1 = \sum_{\ell=0}^{\infty} [\ell M_{\ell+} + E_{\ell+}] P'_{\ell+1}(x)$$

$$+ [(\ell + 1)M_{\ell-} + E_{\ell-}] P'_{\ell-1}(x)$$

$$(3.9)$$

$$\mathcal{F}_2 = \sum_{\ell=1}^{\infty} [(\ell + 1) M_{\ell+} + \ell M_{\ell-}] P'_{\ell}(x)$$

$$(3.10)$$

$$\mathcal{F}_3 = \sum_{\ell=1}^{\infty} [E_{\ell+} - M_{\ell+}] P''_{\ell+1}(x) + [E_{\ell-} + M_{\ell-}] P''_{\ell-1}(x) \tag{3.11}$$

$$\mathcal{F}_4 = \sum_{\ell=1}^{\infty} [M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}] P''_{\ell}(x) , \tag{3.12}$$

where $x = \cos\theta$. The energy-dependent amplitudes $M_{\ell\pm}$ and $E_{\ell\pm}$ refer to transitions initiated by magnetic and electric radiation, respectively, leading to final states of orbital angular momentum ℓ and total angular momentum $\ell \pm \frac{1}{2}$.

These expressions can be inverted by using the following orthogonality relations,

$$\int_{-1}^{1} dx\, P'_j(x) \left[P_{\ell-1}(x) - P_{\ell+1}(x) \right] = 2\,\delta_{j\ell} \tag{3.13}$$

and

$$\int_{-1}^{1} dx\, P''_j(x) \left\{ \frac{P_{\ell-2}(x) - P_{\ell}(x)}{2\ell - 1} - \frac{P_{\ell}(x) - P_{\ell+2}(x)}{2\ell+3} \right\} = 2\,\delta_{\ell j} . \tag{3.14}$$

This yields

$$M_{\ell+} = \frac{1}{2(\ell+1)} \int_{-1}^{1} dx \left[\mathcal{F}_1 P_{\ell}(x) - \mathcal{F}_2 P_{\ell+1}(x) - \mathcal{F}_3 \frac{P_{\ell-1}(x) - P_{\ell+1}(x)}{2\ell+1} \right] , \tag{3.15}$$

for $\ell > 0$,

$$E_{\ell +} = \frac{1}{2(\ell + 1)} \int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) - \mathcal{F}_2 P_{\ell+1}(x) + \mathcal{F}_3 \ell \frac{P_{\ell-1}(x) - P_{\ell+1}(x)}{2\ell+1} + \mathcal{F}_4 (\ell+1) \frac{P_\ell(x) - P_{\ell+2}(x)}{2\ell+3} \right], \tag{3.16}$$

$$M_{\ell -} = \frac{1}{2\ell} \int_{-1}^{1} dx \left[-\mathcal{F}_1 P_\ell(x) + \mathcal{F}_2 P_{\ell-1}(x) + \mathcal{F}_3 \frac{P_{\ell-1}(x) - P_{\ell+1}(x)}{2\ell+1} \right], \tag{3.17}$$

and $\ell > 0$, and

$$E_{\ell -} = \frac{1}{2\ell} \int_{-1}^{1} dx \left[\mathcal{F}_1 P_\ell(x) - \mathcal{F}_2 P_{\ell-1}(x) - \mathcal{F}_3 (\ell+1) \frac{P_{\ell-1}(x) - P_{\ell+1}(x)}{2\ell+1} - \ell \mathcal{F}_4 \frac{P_{\ell-2}(x) - P_\ell(x)}{2\ell-1} \right] \tag{3.18}$$

for $\ell > 1$.

Superscripts (+, -, 0) may be added to each quantity in Eqs. (3.2) to (3.12) and (3.15) to (3.18) to designate its isotopic-spin dependence.

IV. THE ANGULAR-MOMENTUM DECOMPOSITION OF THE SCATTERING AMPLITUDE FOR THE PROCESS $\pi + \gamma \rightarrow N + \overline{N}$

In a discussion of the kinematics of the process $\pi + \gamma \rightarrow N + \overline{N}$ depicted in Fig. 1b, it is useful to introduce the four-vectors P'_1 and Q' representing the energy-momentum of the antinucleon and pion. We can write

$$P'_1 = P_1 \tag{4.1a}$$

and

$$Q' = - Q. \tag{4.1b}$$

Then Eq. (2.3) becomes

$$Q' + K = P_2 + P'_1 . \tag{4.2}$$

Again, expressing s, t, and $\bar{s}$ in terms of P'_1 and Q', we have

$$s = -(K - P')^2 = M^2 - 2Ek' - 2pk' \cos \theta m \tag{4.3a}$$

$$t = -(Q' + K)^2 = (2E)^2 , \tag{4.3b}$$

and

$$\bar{s} = -(P_2 - K)^2 = M^2 - 2Ek' + 2pk' \cos \theta' , \tag{4.3c}$$

where p and k' are the magnitudes of the nucleon and photon momenta, E is the nucleon energy, and $\cos \theta' = \vec{P}_2 \cdot \vec{K} / Pk'$, all in the barycentric system. We can write E, p, and k' as

$$k' = \frac{t - 1}{2\sqrt{t}} , \tag{4.4a}$$

$$p = \frac{1}{2} \quad (t - 4M^2)^{1/2} , \qquad (4.4b)$$

and

$$E = \frac{1}{2} \quad (t)^{1/2} . \qquad (3.4c)$$

The S matrix for $\pi + \gamma \rightarrow N + \overline{N}$ is

$$S_{fi} = \frac{i}{(2\pi)^2} \quad \frac{M\delta^4 (P_2 + P'_1 - K - Q') \, \bar{u} (P_2) \, T (-P'_1 , P_2, -Q', K) \, v (P'_1)}{(4 E_1 E_2 \omega k)^{1/2}} , \qquad (4.5)$$

where E_1, E_2, ω, k, are the energies of the four particles in whatever reference system is employed.

The complete amplitude can again be reduced to Pauli spinor from by employing Eq. (3.1) and

$$v_r (P) = \frac{M + i \gamma \cdot P}{[2M (E + M)]^{1/2}} \begin{pmatrix} 0 \\ y_r \end{pmatrix} , \qquad (4.6)$$

where $y_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $y_{-1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ refer to an antinucleon with spin up and down, respectively, in the rest system of the particle.

The differential cross section for $\gamma + \pi \rightarrow N + \overline{N}$ in the barycentric system is

$$\frac{d\sigma}{d\Omega} = \frac{p}{k} \mid \chi_N G y_{\overline{N}} \mid^2 , \qquad (4.7)$$

where

$$G = \frac{\vec{P}_2 \cdot \vec{\epsilon}}{p} G_1 + \frac{i \vec{\sigma} \cdot \vec{P}_2 \times \vec{\epsilon}}{p} G_2 + \frac{i\sigma \cdot \vec{P}_2 \vec{P}_2 \cdot \vec{K} \times \vec{\epsilon}}{p^2 k'} G_3$$

$$+ i \frac{\vec{\sigma} \cdot \vec{K} \times \vec{\epsilon}}{k'} G_4 . \qquad (4.8)$$

The G's are the following functions of A_1, A_2, A_3, and A_4:

$$G_1 = \frac{k'p}{16\pi E} [A_1 + t A_2] \quad (4.9)$$

$$G_2 = - \frac{k' p}{4\pi} A_3 \quad (4.10)$$

$$G_3 = \frac{(M - E) k'}{8 \pi E} [A_1 + \sqrt{t} A_4] \quad (4.11)$$

$$G_4 = \frac{k'}{16 \pi E} [2 M A_1 - t A_4] . \quad (4.12)$$

In making the angular-momentum decomposition of the G's, it is useful to introduce the helicity amplitudes treated by JW. The differential cross section in the barycentric system expressed in terms of the helicity amplitudes is

$$\frac{d\sigma}{d\Omega} = \frac{p}{k'} \left| f^{\lambda_\gamma}_{\lambda_N \lambda_{\overline{N}}} \right|^2 \quad (4.14)$$

where λ_γ is the photon helicity, while λ_N and $\lambda_{\overline{N}}$ are the helicities of the nucleon and antinucleon, respectively. For a given λ_γ, we may express $f^{\lambda_\gamma}_{\lambda_N \lambda_{\overline{N}}}$ in Pauli spinor form by:

$$f^{\lambda_\gamma}_{\lambda_N \lambda_{\overline{N}}} = \chi_{\lambda_N} f^{\lambda_\gamma} y_{\lambda_{\overline{N}}} \quad (4.15)$$

where

$$f^{\lambda_\gamma} = \begin{pmatrix} -f^{\lambda_\gamma}_{\frac{1}{2}, \frac{1}{2}} & f^{\lambda_\gamma}_{\frac{1}{2}, \frac{1}{2}} \\ -f^{\lambda_\gamma}_{-\frac{1}{2}, \frac{1}{2}} & f^{\lambda_\gamma}_{-\frac{1}{2}, -\frac{1}{2}} \end{pmatrix}$$

where $+\frac{1}{2}$ helicity for the antinucleon is represented by y_{-1} and $-\frac{1}{2}$ helicity is y_{+1}.

The helicity and the z component of spin states are identical for the $\bar{N} + N$ system provided the z axis is picked to lie in the direction of the final nucleon momentum. In this case the scattering amplitudes defined in Eq. (4.8) and (4.16) are identical,

$$f^{\lambda_\gamma} = G^{\lambda_\gamma} e^{i\phi} , \qquad (4.17)$$

up to an arbitrary phase. This phase represents both the arbitrary phase between nucleon and antinucleon states and the phase of the photon state. Because the results of Frazer and Fulco[4] will be needed to use the unitarity condition for this channel, we must pick the antinucleon phase consistent with their choice. The photon phase will be adjusted later to make the A_i's real when the eigenamplitudes for the $\pi + \gamma \rightarrow N + \bar{N}$ channel are real.

Picking a definite helicity for the photon, $\lambda_\gamma = +1$, and taking $\vec{K}$ to lie in the x-z plane, we can express G^{λ_γ} in terms of the G_i's by substituting

$$\epsilon = \epsilon^{+1} = -\frac{1}{\sqrt{2}} (\vec{e}_x \cos\theta' + \sin\theta' \, \vec{e}_z + i\vec{e}_y) , \qquad (4.18a)$$

and

$$\frac{\vec{K}}{k'} = \cos\theta' \, \vec{e}_z - \sin\theta' \, \vec{e}_x \qquad \frac{\vec{P}_2}{p} = \vec{e}_z \qquad (4.18b)$$

into Eq. (4.8), where ϵ^{+1} is the usual polarization vector for a photon of helicity $+1$ traveling in the $\vec{K}$ direction, and where $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are unit vectors defining the coordinate system. We then find

$$G^1 = -\frac{1}{\sqrt{2}} \begin{pmatrix} (G_1 + G_3 + G_4)\sin\theta' & (G_2 + G_4)(1+\cos\theta') \\ (G_2 - G_4)(1-\cos\theta') & -(-G_1 + G_3 + G_4)\sin\theta' \end{pmatrix}. \tag{4.19}$$

The expansions for the helicity amplitudes, as given by JW for the particular cases of interest, are

$$f_{\frac{1}{2}\,\frac{1}{2}}^{\;1} = \frac{1}{(k'p)^{1/2}} \sum_J (J+\tfrac{1}{2})\, T_J^{++1}\, d_{1\,0}^{\,J}(\theta'), \tag{4.20}$$

$$f_{\frac{1}{2}-\frac{1}{2}}^{\;1} = \frac{1}{(k'p)^{1/2}} \sum_J (J+\tfrac{1}{2})\, T_J^{+-1}\, d_{1\,1}^{\,J}(\theta'), \tag{4.21}$$

$$f_{-\frac{1}{2}\,\frac{1}{2}} = \frac{1}{(p\,k')^{1/2}} \sum_J (J+\tfrac{1}{2})\, T_J^{-+1}\, d_{1\,-1}^{\,J}(\theta'), \tag{4.22}$$

and

$$f_{-\frac{1}{2}\,-\frac{1}{2}}^{\;1} = \frac{1}{(pk')^{1/2}} \sum_J (J+\tfrac{1}{2})\, T_J^{--1}\, d_{1\,0}^{\,J}(\theta'), \tag{4.23}$$

where the energy-dependent amplitudes $T_J^{(\pm)(\pm)1}$ are the T-matrix elements for transitions initiated by a photon of helicity $+1$ producing a nucleon of helicity $\pm\frac{1}{2}$ and an antinucleon of helicity $\pm\frac{1}{2}$ with total angular momentum J. The first superscript on $T_J^{(\pm)(\pm)1}$ represents the nucleon helicity and the second that of the antinucleon. A direct comparison between Eqs. (4.16) and (4.19) gives the G's in terms of the f's, and this yields an expansion for the G's in terms of eigenamplitudes:

$$G_1 = -\Sigma\, (J + \tfrac{1}{2})\, \beta^-_J \; P'_J\,(x') \tag{4.24}$$

$$G_2 = -\Sigma \left\{ \alpha^-_J \left[\frac{J\, P''_{J+1}(x') + (J+1)\, P''_{J-1}}{2} \right] + (J + \tfrac{1}{2})\, \alpha^+_J \; P''_J\,(x') \right\} \tag{4.25}$$

$$G_3 = +\Sigma \left\{ \alpha^+_J \left[\frac{J\, P''_{J+1}(x') + (J+1)\, P''_{J-1}(x')}{2} \right] - (J + \tfrac{1}{2})\, \alpha^-_J \; P''_J(x') - (J + \tfrac{1}{2})\, \beta^+_J\, P'_J\,(x) \right\} \tag{4.26}$$

$$G_4 = -\Sigma \left\{ \frac{\alpha^+_J}{2} \left(J\, P''_{J+1}(x') + (J+1)\, P''_{J-1}\right) - (J + \tfrac{1}{2})\, \alpha^-_J\, P''_J\,(x') \right\} \tag{4.27}$$

where

$$\alpha^+_J = \frac{T_J^{+,-,1} + T_J^{-,+,1}}{J\,(J+1)\,\sqrt{2pk'}} \quad , \tag{4.28a}$$

$$\alpha^-_J = \frac{T_J^{+,-,1} - T_J^{-,+,1}}{J\,(J+1)\sqrt{2\,pk'}} \quad , \tag{4.28b}$$

$$\beta^{\pm}_J = \frac{T_J^{+,+,1} \pm T_J^{-,-,1}}{\sqrt{J\,(J+1)}\;\sqrt{2\,pk'}} \quad , \tag{4.28c}$$

$$\beta_J^- = \frac{T_J^{+,+,1} - T_J^{-,-,1}}{\sqrt{J(J+1)}\,\sqrt{2pk'}} , \qquad (4.28d)$$

and the photon phase has been adjusted to make the A_i's real when the T_J's are real.

The energy-dependent amplitudes α_J^+ and β_J^+ represent transitions initiated by magnetic radiation leading to triplet nucleon-antinucleon final states of parity $(-1)^J$ and total angular momentum J. Electric transitions leading to a triplet final state of parity $(-1)^{J+1}$ are represented by α_J^-, while β_J^- represents an electric transition to a single final state of angular momentum J.

Again one may add superscripts to each term of Eqs. (4.7) to (4.12) and (4.24) to (4.27) to denote the term's isotopic spin character. Some physical meaning of g_β^0, g_β^+, and g_β^- for this channel is obtained by noticing that the isotopic-spin projection operators for the process $\pi + \pi^0 \rightarrow N + \overline{N}$ given by Frazer and Fulco,[4]

$$P_0(\beta, 3) = \frac{1}{\sqrt{6}} \delta_{\beta 3} , \qquad (4.29)$$

and

$$P_1(\beta, 3) = \frac{1}{4} \left[\tau_\beta , \tau_3 \right] , \qquad (4.30)$$

are just proportional to g^+ and g^- while g^0 leads only to the $I = 1$ amplitude.

V. THE CROSSING CONDITION

A form of crossing symmetry for the amplitudes $A_i^{(\pm, 0)}$ can be obtained by requiring that the amplitude for photoproduction of a pion from a nucleon of momentum P_1 be equal to the amplitude for production of the charge-conjugate pion from an antinucleon of momentum P_1. Thus we obtain

$$\bar{u}(P_2)\, T(P_2, P_1, \tau_\beta)\, u(P_1) = \bar{v}(P_1)\, T(-P_1, -P_2, \tau_\beta^*)\, v(P_2) \quad . \tag{5.1}$$

by applying the substitution rule to obtain the matrix elements for antinucleon photoproduction and using the fact that τ_β^* represents production of a pion that is charge conjugate to the pion represented by τ_β.

To express Eq. (5.1) in terms of the A_i's, we employ

$$v = C^* u^* \tag{5.2a}$$

and

$$\bar{u} = u^+\beta = v^T C^T \beta \, . \tag{5.2b}$$

Where A^T means the transpose of A and, for $\mu = 0, 1, 2,$ and 3, we have

$$G\gamma_\mu C^{-1} = \gamma^*_\mu \, , \tag{5.3a}$$

$$\beta\gamma_\mu \beta = -\gamma^+_\mu \, , \tag{5.3b}$$

and

$$C\gamma_5 C^{-1} = -\gamma^*_5 \, , \; \beta\gamma_5\beta = -\gamma^+_5 \, , \tag{5.3c}$$

where

$$\beta = i\gamma_0$$

and

$$CC^+ = 1.$$

Expressing the v's in terms of the u's in Eq. (5.1) we obtain:

$$u^T(P_1)\, C^T\beta\, T\,(-P_1,\ -P_2,\ \tau_\beta{}^*)\, C^*\beta\beta u^*(P_2) = \bar{u}\,(P_2)\, T\,(P_2, P_1, J_\beta) u\,(P_1). \tag{5.4}$$

It then follows that

$$T(P_2,\ P_1,\ \tau_\beta) = [C^T\beta\, T\,(-P_1, -P_2,\ \tau_\beta{}^*)\, C^*\beta]^T = -\,\beta C^{-1}[T(-P_1, -P_2, \tau_\beta{}^*)]^T C\beta. \tag{5.5}$$

The exchange of $-P_1$ and P_2 in T changes s into $\bar{s}$ and P into $-P$. The operator βC^{-1}. replaces $\gamma_\mu{}^T$ by $-\gamma_\mu$ and replaces $\gamma_5{}^T$ by $+\gamma_5$. The net effect of the combined matrix operations on the right-hand side of Eq. (5.5) is to interchange the order of γ matrix factors and to change the sign of γ_μ. Equation (5.5) implies the following relation for each of the independent spin amplitudes:

$$A_1\,(s, \bar{s})\, M_1 = -i\,\gamma\,.\,K\,\gamma\,.\,\epsilon\,\gamma_5\, A_1\,(\bar{s},\ s) = A_1\,(\bar{s},\ s)\, M_1 \tag{5.6}$$

$$A_2(s, \bar{s})\, M_2 = -2i\,\gamma_5\,[\,P\cdot k\,Q\cdot\epsilon - P\cdot\epsilon Q\cdot K]\ A_2(\bar{s}, s) = M_2 A_2(\bar{s}, s) \tag{5.7}$$

$$A_3(s, \bar{s})\, M_3 = (\gamma\,.\,\epsilon\, Q\cdot K - \gamma\cdot K - \gamma\,.\,\epsilon)\,\gamma_5\, A_3\,(\bar{s},\ s)$$

$$A_3(s, \bar{s})\, M_3 = -M_3\, A_3\,(\bar{s}, s) \tag{5.8}$$

$$A_4(s,\bar{s})\, M_4 = -2\,[\gamma \cdot \epsilon\, P \cdot K - \gamma \cdot K P \cdot \epsilon \mp i\, M \gamma \cdot K \gamma \cdot \epsilon]\, \gamma_5\, A_4\,(\bar{s}, s)$$

$$= A_4\,(\bar{s},\, s)\, M_4\,. \qquad (5.9)$$

The crossing condition in isotopic spin space can be obtained by examining Eqs. (5.6) to (5.9) and writing the isotopic spin operators explicitly. If 1 and 2 denote states with momentum P_1 and P_2 respectively, then Eq. (5.6) becomes

$$\left\langle 2 \,\middle|\, A_1^{\,+}(s,\bar{s})\, g_\beta^{\,+} + A_1^{\,-}(s,s)\, g_\beta^{\,-} + A_1^{\,0}\,(s,\bar{s})\, g_\beta^{\,0} \,\middle|\, 1 \right\rangle$$

$$= \left\langle 1 \,\middle|\, A_1^{\,+}\,(s,s)\,(g_\beta^{\,+})^* + A_1^{\,-}(s,s)\,(g_\beta^{\,-})^* + A_1^{\,0}(s,s)(g_\beta^{\,0})^* \,\middle|\, 2 \right\rangle \,.$$

$$(5.10)$$

Taking the transpose and noting that $g_\beta^{\,+}$, and $g_\beta^{\,0}$ are hermitian, while $g_\beta^{\,-}$ is antihermitian, we obtain

$$A_1^{\,+}\,(s,\bar{s}) = A_1^{\,+}\,(\bar{s}, s)\,, \qquad (5.11a)$$

$$A_1^{\,0}\,(s,\bar{s}) = A_1^{\,0}\,(\bar{s}, s), \qquad (5.11b)$$

and

$$A_1^{\,-}\,(s,\bar{s}) = -A_1^{\,-}(\bar{s}, s)\,. \qquad (5.11c)$$

The procedure is identical for A_2, A_3, and A_4, yielding $A_1^{(+,0)}$, $A_2^{(+,0)}$, $A_3^{(-)}$, and $A_4^{(+,0)}$ **to be s**ymmetric under exchange of s and $\bar{s}$, while $A_1^{\,-}$, $A_2^{\,-}$ $A_3^{(+,0)}$, and $A_4^{\,-}$ are antisymmetric.

VI. THE MANDELSTAM REPRESENTATION

It is now assumed that the invariant functions $B_i^{(\pm, 0)}$ satisfy the spectral representation proposed by Mandelstam and that they possess no kinematical singularities that might be introduced by a poor choice of the invariant spin matrices. The motivation for the last assumption is that the N_i chosen are essentially unique as the simplest forms allowed by invariance considerations.

We may now express the B's as

$$B_i = \frac{R_s^{\,i}}{s - M^2} + \frac{R_t^{\,i}}{t - t} + \frac{R_{\bar{s}}^{\,i}}{\bar{s} - M^2}$$

$$+ \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \, \frac{\rho_s^{\,i}(s')}{s' - s} + \frac{1}{\pi}\int_{4}^{\infty} dt' \, \frac{\rho_t^{\,i}(t')}{t' - t}$$

$$+ \frac{1}{\pi}\int_{(M+1)^2}^{\infty} d\bar{s}' \, \frac{\rho_{\bar{s}}^{\,i}(\bar{s}')}{\bar{s}' - \bar{s}} + \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_{4}^{\infty} dt' \, \frac{b_{12}^{\,i}(s', t')}{(s' - s)(t' - t)}$$

$$+ \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_{(M+1)^2}^{\infty} d\bar{s}' \, \frac{b_{13}^{\,i}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})}$$

$$+ \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} d\bar{s}' \int_{4}^{\infty} dt' \, \frac{b_{23}^{\,i}(\bar{s}', t')}{(\bar{s}' - \bar{s})(t' - t)} \, .$$

(6.1)

The one-dimensional spectral functions in Eq. (6.1) represent reducible Feynman diagrams which have the same form as the diagrams that produce the poles. Thus an amplitude that has a pole in a particular variable will in general also have a one-dimensional spectral function in that variable. The possibility of an over-all subtraction constant (independent of s, t, and $\bar{s}$) or of polynomials multiplying the one-dimensional spectral function is removed by unitarity requirements on the asymptotic behavior of B_i.

Since A_1, A_3, and A_4 contain no kinematic factors in their relation to the B's, they may be written:

$$A_i^{(+,-,0)} = \Gamma_i^{(+,-,0)}\left(\frac{1}{s-M^2} \pm \frac{1}{\bar{s}-M^2}\right)$$

$$+ \frac{1}{\pi}\int \rho^i(s')^{(+,-,0)}\left[\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}}\right]$$

$$+ \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_4^{\infty} dt' \frac{a_{i_{12}}^{(+,-,0)}(s',t')}{(s'-s)(t'-t)}$$

$$+ \frac{1}{\pi^2}\int_{(M+1)^2}^{\infty} ds' \int_{(M+1)^2}^{\infty} d\bar{s}' \frac{a_{i_{13}}^{(+,-,0)}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})} \qquad (6.2)$$

$$+ \frac{1}{\pi^2} \int_{(M+1)^2}^{\infty} d\bar{s}' \int_4^{\infty} dt' \frac{a_{i_{23}}^{(+,-,0)}(\bar{s}', t')}{(\bar{s}' - \bar{s})(t' - t)}$$

for $i = 1, 3$, and 4.

Where the spectral functions $a_{i_{12}}$, $a_{i_{13}}$, and $a_{i_{23}}$ are real functions and the Γ 's have been given by CGLN:

$$\Gamma_1^{(+,-,0)} = \frac{e_r g_r}{2} \tag{6.3a}$$

$$\Gamma_3^{(+,-)} = \Gamma_4^{(+,-)} = -\frac{1}{2} g_r (\mu'_{p_r} - \mu_{n_r}) \tag{6.3b}$$

$$\Gamma_3^{0} = \Gamma_4^{0} = -\frac{1}{2} g_r (\mu'_{p_r} + \mu_{n_r}), \tag{6.3c}$$

where μ'_{p_r} and μ_{n_r} are the rationalized anomalous static nucleon moments and e_r and g_r are the rationalized and renormalized electronic charge and pion-nucleon coupling constant, respectively. These have the values

$$\mu'_{p_r} = 1.78 \frac{e_r}{2M},$$

$$\mu_{n_r} = -1.91 \frac{e_r}{2M},$$

$$\frac{e_r}{4\pi} = \frac{1}{137} ,$$

and

$$\frac{g_r^{\;2}}{4\pi} \simeq 14 .$$

The upper sign in Eq. (6.2) is to be used with the even A_1's$^{(+,-,0)}$, while the lower goes with the odd functions.

The gauge condition, Eq. (2.14a), is now applied to B_2 and B_3. Expressing Eq. (2.14a) in terms of the s, $\bar{s}$, and t variables, we obtain

$$(s - \bar{s}) B_2 = 2 (t - 1) B_3 . \tag{6.4}$$

We see that if B_3 is to remain finite as s approaches infinity then $R_t^{\;2}$ and $\rho_t^{\;2}$ (t) must be zero. If Eq. (6.4) is evaluated at $t = 1$, we obtain

$$(s - M^2) B_2 (s, 1) = R_t^{\;3} , \tag{6.5}$$

where $R_t^{\;3}$ is a known constant. We can take advantage of Eq. (6.5) by making a subtraction at $t = 1$, obtaining a spectral function for

$$\frac{B_2}{t-1} = -\frac{1}{2} A_2 .$$

The resulting spectral representation for A_2 is

$$A_2^{(\pm, 0)} = -\frac{e_r g_r}{t-1} \left(\frac{1}{s - M^2} \pm \frac{1}{\bar{s} - M^2} \right)$$

$$+ \frac{1}{\pi^2} \int_{(M+1)^2}^{\infty} ds' \int_4^{\infty} dt' \frac{a_{2_{13}}^{(+,-,0)}(s', t')}{(s' - s)(t' - t)}$$

$$+ \frac{1}{\pi^2} \int_{(M+1)^2}^{\infty} ds' \int_{(M+1)^2}^{\infty} d\bar{s}' \frac{a_{2_{13}}^{(+,-,0)}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})}$$

$$+ \frac{1}{\pi^2} \int_{(M+1)^2}^{\infty} d\bar{s}' \int_4^{\infty} dt' \frac{a_{2_{23}}^{(+,-,0)}(s', t')}{(\bar{s}' - \bar{s})(t' - t)} \quad . \tag{6.6}$$

It should be noted that this subtraction procedure has removed all one-dimensional spectral functions.

The crossing condition now requires

$$a_{i_{12}}^{(\pm,0)}(s', t') = \pm a_{i_{23}}^{(\pm,0)}(s', t') \tag{6.7}$$

and

$$a_{i_{13}}^{(\pm,0)}(s', \bar{s}') = \pm a_{i_{13}}^{(\pm,0)}(\bar{s}', s') \tag{6.8}$$

for i = 1, 2, 3, 4.

The upper sign in Eqs. (6.6) and (6.7), and (6.8) is again to be used with the even A_i's $^{(+,-,0)}$ while the lower goes with the odd functions.

Mandelstam has given a general method for determining the regions in which the spectral functions $a_{i_{12}}^{(+,-,0)}$, $a_{i_{13}}^{(+,-,0)}$ and $a_{i_{23}}^{(+,-,0)}$ are nonzero.[2] These boundaries result from considering the lowest-mass intermediate states possible in any pair of the variables s, t, and $\bar{s}$.

Examining first the t spectrum, we must consider the processes $\gamma + \pi \rightarrow n\,\pi$. Conservation of G parity requires n to be odd for the istopic-vector part of the photon, while n must be even for the isotopic-scalar part. Thus the (+, -) spectra will have intermediate states containing odd numbers of pions, starting with the three-pion state, while the 0 spectrum will contain only intermediate states with even numbers of pions, starting with the two pion state. The s and $\bar{s}$ spectra both start with the pion and nucleon intermediate state; with no differences for the various isotopic combinations.

Following Mandelstam's method, we find that the region in which the functions $a_{i_{12}}^{(+,-)}$ are nonzero is bounded by the following curve (see Fig. 2),

$$[s - (M+1)^2]\;[s - (M-1)^2]\;[t - 9] - 8\,(3s - M^2 + 1) = 0. \tag{6.9}$$

The spectral functions $a_{i_{12}}^{0}$ are bounded by the two curves (see Fig. 3),

$$[s-(M+2)^2]\,[s-(M-2)^2]\,t(t-4) - 2t\,(9s+31M^2-28) - (4M^2-1) = 0 \tag{6.10a}$$

and

$$[s-(M+1)^2]\,[s-(M-1)^2]\,t\,(t-16) - 8t\,(9s+M^2-1) - 16\,(M^2-1) = 0. \tag{6.10b}$$

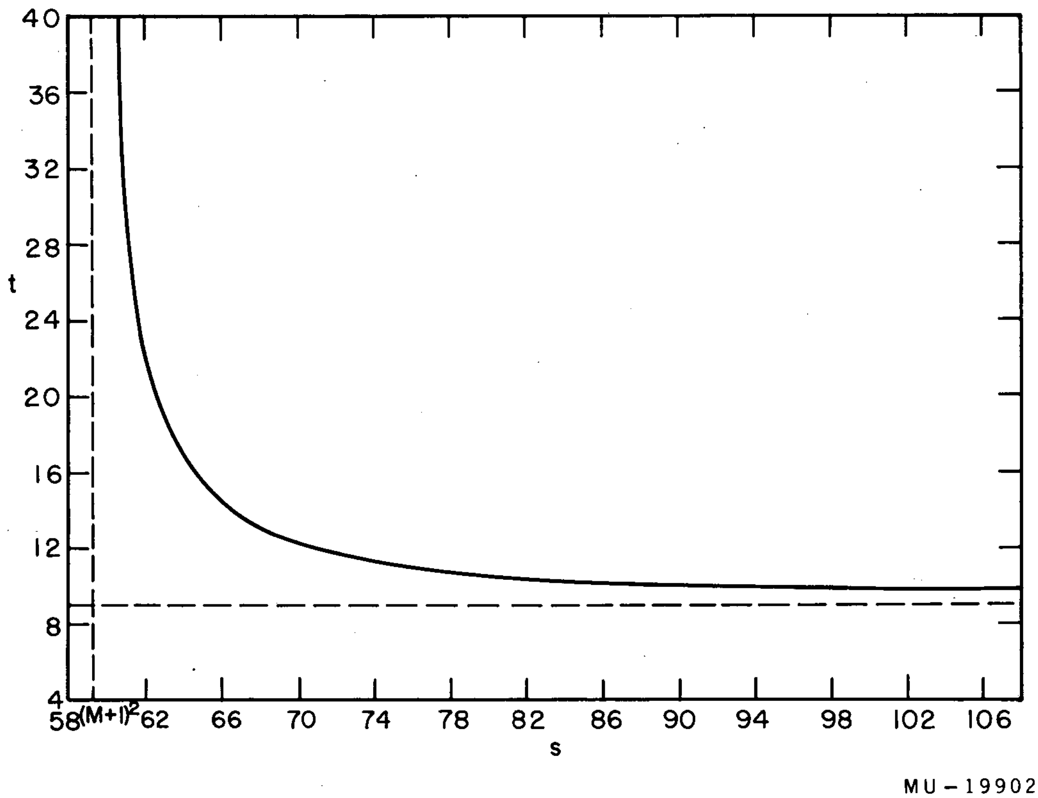

Fig. 2. The boundary curve for $a_{i_{12}}^{(+,-)}$. The dashed lines are the asymptotes of the curve.

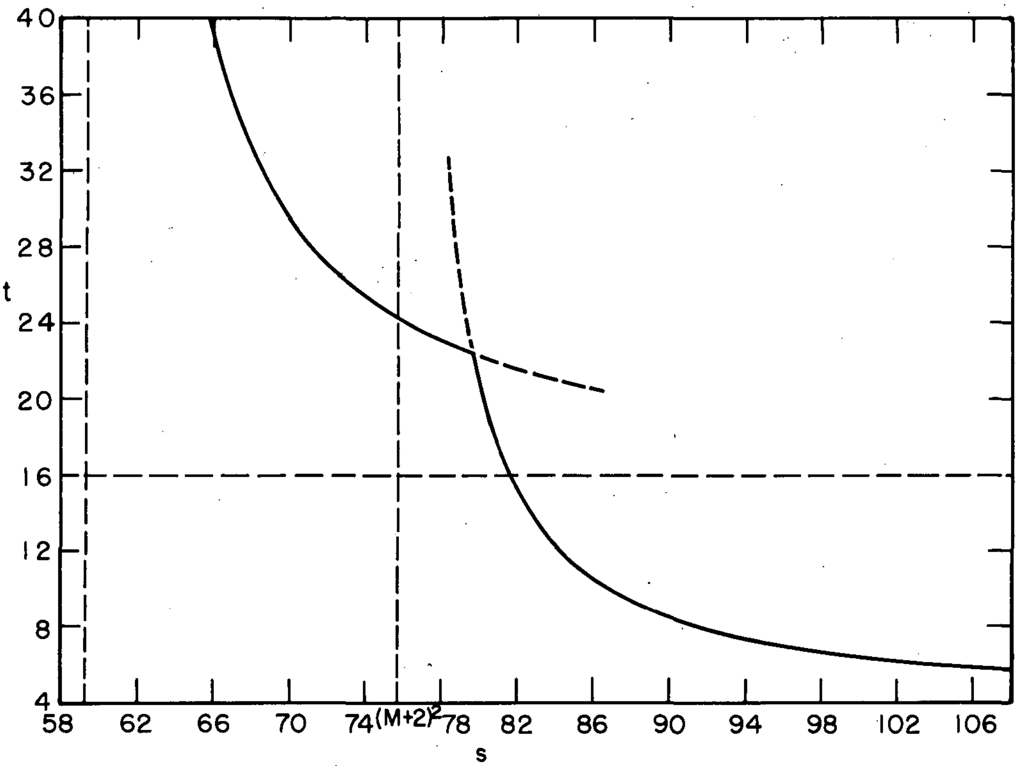

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Fig. 3. The boundary curve for $a_{i_{12}}^{(0)}$. The dashed ~~lin~~ lines are the asymptotes of the curve.

The curves bounding $a_{i_{23}}^{(+,-)}$ and $a_{i_{23}}^{0}$ can be obtained from Eqs. (6.9), (6.10a), and (6.10b) by changing s to $\overline{s}$. The spectral functions $a_{i_{13}}^{(+,-,0)}$ are bounded by the curve (see Fig. 4) ,

$$[\overline{s} - (M+1)^2]\,[\overline{s} - (M-1)^2]\,[s - (M+1)^2]\,[s - (M-1)^2]$$

$$-(4M^2 - 1)\,[\,2\,s\,\overline{s} - 2\,(M^2 - 1)\,(s + \overline{s}) + 2M^4 - 1] = 0. \qquad (6.11)$$

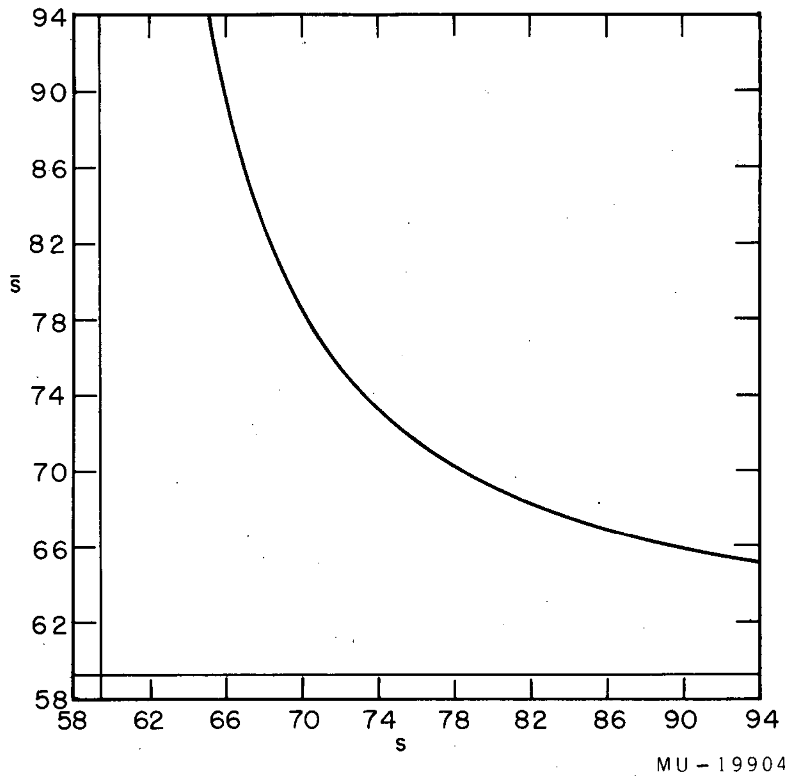

Fig. 4. The boundary curve for $a_{i_{13}}$. The boundary lines are the asymptotes of the curve.

VII. ONE-DIMENSIONAL DISPERSION RELATIONS

It is now possible to obtain one-dimensional dispersion relations with either s, t, or $\bar{s}$ held fixed. We define the following functions:

$$\mathrm{Im}_{I}\, A_i\,(s,\, t) = \rho_i\,(s) + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} d\bar{s}' \,\frac{a_{i_{13}}\,(s,\, \bar{s}')}{\bar{s}' + t + s - 2M^2 - 1}$$

$$+ \frac{1}{\pi}\int_{4}^{\infty} dt' \,\frac{a_{i_{12}}\,(s,\, t')}{t' - t} \qquad (7.1a)$$

$$\mathrm{Im}_{II}\, A_i\,(s,t) = \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \,\frac{a_{i_{12}}\,(s',\, t)}{s' - s}$$

$$\mp \frac{1}{\pi}\int \frac{d\bar{s}'\, a_{i_{23}}\,(\bar{s}',\, t)}{\bar{s}' + s + t - 2M^2 - 1} \qquad (7.1b)$$

$$\mathrm{Im}_{III}\, A_i\,(s,\bar{s}) = \pm\, \rho_i\,(\bar{s}) + \frac{1}{\pi}\int_{(M+1)^2} ds' \,\frac{a_{i_{13}}\,(s',\, \bar{s})}{s' - s}$$

$$+ \frac{1}{\pi}\int_{4}^{\infty} dt' \,\frac{a_{i_{23}}\,(\bar{s},\, t')}{t' + \bar{s} + s - 2M^2 - 1}\,. \qquad (7.1c)$$

It can be seen from Eqs. (6.2) and (6.6) that $Im_i A_i$ is the imaginary part of A_i when the variables are in the physical region for process I, $\gamma + N \rightarrow \pi + N$, and represents the analytic continuation of $Im\, A_i$ outside of this region. Functions $Im_{II} A_i$ and $Im_{III} A_i$ have the same meaning for process II, $\gamma + \pi \rightarrow N + \overline{N}$, and process III, $\gamma + N \rightarrow \pi + \overline{N}$, respectively.

The dispersion relations with one variable held fixed are now obtained from Eqs. (6.2), (6.6), and (7.1):

<u>Fixed s</u>:
$$A_i(s,t,\bar{s}) = \text{Poles} + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \frac{\rho_i(s')}{s'-s}$$
$$+ \frac{1}{\pi}\int_{4}^{\infty} dt' \frac{Im_{II}A_i(s,t')}{t'-t} + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} d\bar{s}' \frac{Im_{III}A_i(s,\bar{s}')}{\bar{s}'-\bar{s}} \tag{7.2}$$

<u>Fixed t</u>:
$$A_i(s,t,\bar{s}) = \text{Poles} + \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds'\, Im_I A_i(s',t)\left(\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}}\right) \tag{7.3}$$

Fixed $\bar{s}$:
$$A_i(s,t,\bar{s}) = \text{Poles} \pm \frac{1}{\pi}\int_{(M+1)^2}^{\infty} d\bar{s}' \frac{\rho_i(\bar{s}')}{\bar{s}'-\bar{s}}$$
$$\pm \frac{1}{\pi}\int_{4}^{\infty} dt' \frac{Im_{II}A_i(\bar{s},t')}{t'-t} \pm \frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds' \frac{Im_{III}A_i(\bar{s},s')}{s'-s}. \tag{7.4}$$

The crossing symmetry has been employed to produce Eqs. (7.3) and (7.4). These are, of course, simply different ways of representing the same functions.

In previous work on pion-nucleon scattering and on photoproduction, only the fixed momentum-transfer dispersion relation, Eq. (7.3), has been employed. It is noteworthy that, according to the above considerations, this is the only one of the three for which a subtraction is not required by elementary perturbation-theory arguments. In practice however, a strongly interacting intermediate state connecting $(\pi\gamma)$ to $N\overline{N}$ may necessitate a subtraction. To illustrate this point, we consider for example a resonant 2π intermediate state. This can be represented schematically by the diagram in Fig. 5. As the π-π interaction becomes stronger, the life time of the 2π state becomes longer, and more pion pairs are exchanged between the resonating pions. This can be represented in Fig. 5 by adding more pairs to ladder of pions representing the intermediate state. If we now look at the singularities produced in s by this diagram, we see that as more pairs are added, the contribution of this diagram comes from higher values of s'. Finally as the interaction becomes strong enough to produce a bound state, the contribution to the s spectrum moves to infinity, requiring a subtraction. Another way to understand this effect is to recall that if there were a 2π bound state, we should certainly have to add a new pole in t together with the associated ρ_t. Thus if one wants to treat only the lower intermediate states in the s spectrum, approximating $\mathrm{Im}_I A_i$ in Eq. (7.3) by the first few terms of a polynomial expansion, a resonance in an intermediate state of the t spectrum may necessitate a subtraction. The approximation of replacing $\mathrm{Im}_I A_i$ in Eq. (7.3) by its polynomial expansion in the physical region, will be discussed in a later section.

The fixed s-dispersion relation, Eq. (7.2), is useful for the photoproduction channel in that the $\cos\theta$ dependence of A_i is given explicitly. This allows the use of projection operators to obtain an integral representation for the eigenamplitudes.

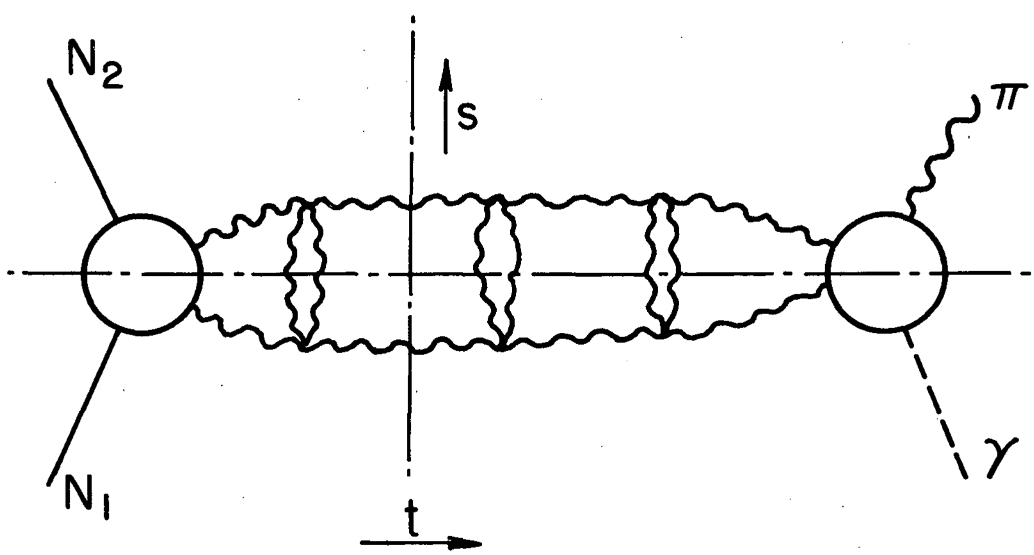

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Fig. 5. A typical diagram representing a two-pion state connecting $\gamma + \pi$ to $N\bar{N}$.

VIII. THE APPROACH THROUGH EIGENAMPLITUDES

Before we use the basic machinery developed in the preceding sections, it is useful to examine the type of information provided by the unitarity conditions for photoproduction and $\pi + \gamma \to N + \overline{N}$. For photoproduction below the threshold for production of two pions, unitarity gives the phase of each eigenamplitude in terms of the corresponding pion-nucleon phase. Of course, the pion-nucleon phase must be supplied as starting information.

The unitarity condition for a particular angular-momentum state of the process $\pi + \gamma \to N + \overline{N}$ is

$$2\,\mathrm{Im}\left\langle \overline{N}N \,\middle|\, \pi\gamma \right\rangle_J = \sum_n \left\langle \overline{N}N \,\middle|\, n \right\rangle_J \left\langle n \,\middle|\, \pi\gamma \right\rangle_J , \qquad (8.1)$$

where the sum n runs over all physical intermediate states having the same quantum numbers as the $N\overline{N}$ and $\pi\gamma$ states. Mandelstam has shown that Eq. (8.1) is valid in the nonphysical region $4M^2 > t > 4$.[10] We see that only for $t > 4M^2$ does the right of Eq. (8.1) contain $\left\langle \overline{N}N \,\middle|\, \pi\gamma \right\rangle$. Thus, for $4M^2 > t > 4$ the imaginary part of the A_i's will be a function that must be supplied by solutions of other scattering problems. At present, the only information available is for $\gamma + \pi \to \pi + \pi$ and $\pi + \pi \to N + \overline{N}$, therefore we are restricted to treating only the 2π intermediate state. It should be noted, however, that in the treatment of $\gamma + \pi \to \pi + \pi$ by H. S. Wong, it has been necessary to introduce a new coupling constant, which if large enough would make the 2π intermediate state an important singularity.[3] A further enhancement of this state would arise from the pion-pion p-wave resonance proposed by Chew and Mandelstam.

In order to make use of all the information obtained from the unitarity conditions, it is convenient to develop dispersions relations for the individual eigenamplitudes. The first step toward this end is to insert Eqs. (3.5) to (3.8) into Eqs. (3.13) to (3.18), obtaining projection operators to be applied to the A_i's. These projection operators are then applied to the fixed s representation of the A_i's, Eq. (7.2), yielding an integral representation for each eigenamplitude. By examining these expressions we can obtain the analyticity properties of the eigenamplitudes.

In general, the eigenamplitudes will have the singularities present in the A_i's plus kinematical singularities arising from the relation between the $\mathcal{F}$'s and the A_i's, and from expressing $\bar{s}$ and t as functions of s and cos θ. It is possible, however, to construct a function from each eigenamplitude that is free from kinematic singularities in the $\sqrt{s}$ or w plane.

A simple reflection property in W for the eigenamplitudes can be obtained by noting that $\mathcal{F}_1(-W) = -\mathcal{F}_2(W)$ and $\mathcal{F}_3(-W) = \mathcal{F}_4(W)$, and using these relations in Eqs. (3.15) to (3.18). The resulting relation for the eigenamplitudes is

$$M_{\ell+}(-W) = \frac{1}{\ell+1}\left[(\ell+2)\, M_{(\ell+1)-}(W) + E_{(\ell+1)-}(W)\right] \tag{8.2a}$$

and

$$E_{\ell+}(-W) = \frac{1}{\ell+1}\left[M_{(\ell+1)-} - \ell E_{(\ell+1)-}(W)\right]. \tag{8.2b}$$

If Eqs. (8.2a) and (8.2b) are used together with the analyticity properties of the eigenamplitudes, the following dispersion relation results:

$$F_\ell^{\,i}(W) = G_\ell^{\,i}(W) + \frac{1}{\pi}\int_{-\infty}^{-(M+1)} dW' \, \frac{\mathrm{Im}F_\ell^{\,i}(W')}{W' - W}$$

$$+ \frac{1}{\pi}\int_{M+1}^{\infty} dW' \, \frac{\mathrm{Im}F_\ell^{\,i}(W')}{W' - W}\,, \tag{8.3}$$

where the eigenamplitudes $F_\ell^{\ 1}(W)$ and $F_\ell^{\ 2}(W)$ differ from $M_{\ell +}$ and $E_{\ell +}$, respectively, by factors that remove the kinematical singularities. The function $G_\ell^{\ i}(W)$ contains all the singularities arising from the t and $\bar{s}$ spectra and may be obtained from Eq. (7.2) if $Im_{II}\, A_i$ and $Im_{III}\, A_i$ are replaced by their appropriate polynomial expansions in the physical regions for channels II and III, respectively. When the projection operation is applied to Eq. (7.2) after the above replacement has been made, $G_\ell^{\ i}(W)$ will now result.

It is now possible to write a solution to Eq. (8.3) imposing the unitarity requirement on $F_\ell^{\ i}$ in the physical region for photoproduction. The reflection law in the W plane for the pion-nucleon eigenamplitudes has been given by MacDowell to be[11] $f_{\ell +}(W) = f_{(\ell+1)-}(-W)$, where $f_{\ell\pm} = \sin\delta_{\ell\pm}\, e^{i\delta_{\ell\pm}}/q$. Thus it is seen that the function $f_{\ell +}(W)$ that has the same phase as $F_\ell(W)$ for $W > 0$ will also have the correct phase for $W<0$. We will now assume that the pion-nucleon problem has been solved by use of the N/D technique employed by Chew and Mandelstam in the pion-pion problem.

The function N_ℓ will be a real analytic function of W for $W > M + 1$ and $W < -(M+1)$, while D_ℓ will have two branch cuts running from $M + 1$ to ∞ and from $-(M+1)$ to $-\infty$ and will be analytic elsewhere. The function N_ℓ / D_ℓ will be the eigenamplitude for pion-nucleon scattering that is free of kinematical singularities. The relation of N_ℓ / D_ℓ to the pion-nucleon phase shifts will be

$$\frac{N_\ell}{D_\ell} = R_\ell(W) \begin{cases} \operatorname{Sin}\delta_{\ell +}\, e^{i\delta_{\ell +}} & [\text{ for } W > M+1] \\ \operatorname{Sin}\delta_{(\ell+1)-}\, e^{i\delta_{(\ell+1)-}} & [\text{ for } W < -(M+1)\ ]. \end{cases} \tag{8.4}$$

where $R_\ell(W)$ is the factor needed to remove the kinematical singularities. The phase requirements on F_ℓ will now be satisfied by a solution to Eq. (8.3) of the type employed by Chew and Low:[12]

$$F_\ell^{\,i}(W) = G_\ell^{\,i}(W) + \frac{1}{D_\ell(W)}\left[\frac{1}{\pi}\int_{-\infty}^{-(M+1)} dW' \, \frac{G_\ell^{\,i}(W')\,N_\ell(W')}{(W'-W)\,R_\ell(W')} + \frac{1}{\pi}\int_{M+1}^{\infty} dW' \, \frac{G_\ell^{\,i}(W')\,N_\ell(W')}{(W'-W)\,R_\ell(W')}\right] . \quad (8.5)$$

This approach to the photoproduction problem is based on the assumption, discussed by Chew, that nearby singularities in the complex plane dominate the behavior of amplitudes in the low-energy region.[13] The solution given by Eq. (8.5) includes the lowest singularities and must be considered the first step in a series of approximations that will include successively higher singularities.

The main obstacle preventing the use of Eq. (8.5) at present is the lack of information about the pion-nucleon problem. The only pion-nucleon eigenamplitude that has been studied by the N/D method so far is the resonant $I = 3/2$, $J = 3/2$, p-wave.[14] If only the 3-3 amplitude and the pole terms are to be treated, Eq. (8.5) represents little improvement over the fixed - t dispersion-relation approach which has been used by CGLN. Since they included the 3-3 amplitude, we must conclude that, with the pion-nucleon information presently available, no significant improvement can be made in the treatment of the s or $\bar{s}$ spectra. However, the 2π branch cut in the t spectrum can now be included, which will be the first modification to the pole terms for the (0) amplitudes.

IX. PHOTOPRODUCTION NEAR THRESHOLD

The threshold region provides us with a situation in which the contribution of the 2π intermediate state is maximized, first by virtue of a small denominator in Eq. (7.2) and second because the measurable cross sections will not be dominated by the 3-3 resonance of the pion-nucleon system.

The approximation now employed is to assume that the $I = 1/2$ phases for pion-nucleon scattering are negligible, meaning that the s and $\bar{s}$ branch cuts may be ignored. The amplitude $F_\ell^{\,i}$ will then just be given by its projection from the poles and from the t spectrum. For this reason we can do the sum over the F_ℓ's, undoing the projection and allowing us to work directly with $A_i^{(0)}$. The resulting functions $A_i^{(0)}$ will just be given by Eq. (7.2) in which the polynomial expansion is used for $\mathrm{Im}_{II} A_i^{0}$, with $\mathrm{Im}_{III} A_i^{0} = 0$.

We will now employ unitarity for process II to obtain $\mathrm{Im}_{II} A_i^{0}$. The unitarity condition resulting from the 2π intermediate state is

$$2\mathrm{Im}\, T_J^{\lambda_N \lambda_{\bar{N}} \lambda_\gamma}(t) = \tau_J^{\lambda_N \lambda_{\bar{N}}}(t)\, t_J^{\lambda\gamma}(t), \qquad (9.1)$$

where $\tau_J^{\lambda_N \lambda_{\bar{N}}}(t)$ is the helicity eigenamplitudes for $\pi + \pi \to N + \bar{N}$

and $t_J^{\lambda\gamma}(t)$ is the helicity eigenamplitude for $\pi + \gamma \to \pi + \pi$. This relation is exact for $4 < t < 16$ and will be assumed to be approximately true for larger t.

As the process $\pi + \gamma \to \pi + \pi$ contains only odd angular-momentum states, we will neglect F-wave and higher states, keeping only the p-wave state. We are concerned only with photon helicity $\lambda_\gamma = +1$, because Eqs. (4.24) to (4.27) are expressed for this photon state. Equation (9.1) becomes

$$\mathrm{Im}\, T_1^{++1} = \frac{1}{2}\, \tau_1^{++}\, t_1^{*1} = \mathrm{Im}\, T_1^{--1} \qquad (9.2)$$

and

$$\text{Im } T_1^{+-1} = \frac{1}{2} \tau_1^{+-} t_1^{*1} = \text{Im } T_1^{-+1} , \qquad (9.3)$$

where we donote + 1/2 helicity as + and - 1/2 as -. The relation between T_1^{++1} and T_1^{--1}, and between T_1^{+-1} and T_1^{-+1} arises from the fact that only the odd-parity part of the $J = 1$ nucleon-antinucleon system contributes.

Thus we have

$$\text{Im } \alpha_1^{+} = \frac{\text{Im } T_1^{+-1}}{(2 pk')^{1/2}} \qquad (9.4a)$$

$$\text{Im } \beta_1^{+} = \frac{\text{Im } T_1^{++1}}{(Pk')^{1/2}} , \qquad (9.4b)$$

and

$$\text{Im } \alpha_1^{-} = \text{Im } \beta_1^{-} = 0. \qquad (9.4c)$$

The amplitudes τ_1^{++} and τ_1^{+-} have been treated by Frazer and Fulco.[4] They defined

$$T_{+}^{1} = \left(\frac{2q'}{p}\right)^{1/2} \tau_1^{++} \qquad (9.5a)$$

$$T_{-}^{1} = \left(\frac{2q'}{p}\right)^{1/2} \tau_1^{+-} \qquad (9.5b)$$

where q' and P are the magnitude of the initial meson and final nucleon momenta in the barycentric system. If we now define

$$\frac{t_1^{1}}{k'} = \left(\frac{2q'}{k'}\right)^{1/2} M^1 , \qquad (9.6)$$

where q' and k' are the magnitudes of initial photon and final meson momenta in the barycentric system for the process $\gamma + \pi \rightarrow \pi + \pi$,

$$\text{Im}\ \alpha_1^{\ +} = \frac{1}{2\sqrt{2}}\ T_-^{\ 1}\ M^{1*} \qquad (9.7a)$$

and

$$\text{Im}\ \beta_1^{\ +} = \frac{1}{2}\ T_+^{\ 1}\ M^{1*}\ . \qquad (9.7b)$$

From Eqs. (4.24) to (4.27) with Eqs. (9.7a) and (9.7b) it follows that:

$$\text{Im}\ G_1 = 0 \qquad (9.8)$$

$$\text{Im}\ G_2 = 0 \qquad (9.9)$$

$$\text{Im}\ G_3 = \frac{3}{4}\ M_+^{\ 1*}\left(\frac{T_-^{\ 1}}{\sqrt{2}} - T_+^{\ 1}\right) \qquad (9.10)$$

$$\text{Im}\ G_4 = -\frac{3}{4}\sqrt{2}\ M^{1*}\ T_-^{\ 1}\ . \qquad (9.11)$$

Solving Eqs. (4.9) to (4.12) for A_1, A_2, A_3, and A_4, we obtain

$$A_1 = \frac{8\pi E}{p^2 k'}\ [E\ G_3 + (E - M)\ G_4] \qquad (9.12)$$

$$A_2 = \frac{2\pi}{k' P E}\ \left[2G_1 - \frac{E}{P}\ G_3 - \frac{E-M}{P}\ G_4\right] \qquad (9.13)$$

$$A_3 = - \frac{4\pi}{pk'} G_2 \quad (9.14)$$

$$A_4 = \frac{4\pi}{p^2 k'} [M G_3 + (M - E) G_4] . \quad (9.15)$$

These equations together with Eqs. (9.8) to (9.11) yield

$$\text{Im } A_1 = - \frac{6\pi E}{p^2 k'} \left[E T_+^1 - \frac{M T_-^1}{\sqrt{2}} \right] M^{1*} , \quad (9.16)$$

$$\text{Im } A_2 = + \frac{3\pi}{2p^2 k' E} \left[E T_+^1 - \frac{M T_-^1}{\sqrt{2}} \right] M^{1*} , \quad (9.17)$$

$$\text{Im } A_3 = 0, \quad (9.18)$$

and

$$\text{Im } A_4 = - \frac{3\pi}{p^2 k'} \left[M T_+^1 - E \frac{T_-^1}{\sqrt{2}} \right] M^{1*} . \quad (9.19)$$

It is now convenient to introduce the notation of Frazer and Fulco (hereafter denoted FF):[15]

$$g_1^V(t) = + \frac{e F_\pi^*(t) (t - 4)^{1/2}}{4p^2} \left[\frac{E T_-^1(t)}{\sqrt{2}} - M T_+^1 (t) \right] \quad (9.20)$$

$$g_2^V(t) = \frac{e F_\pi^*(t)(t-4)^{1/2}}{8 p^2 E} \left[E T_+^1 - \frac{M}{\sqrt{2}} T_-^1 \right] , \tag{9.21}$$

where $F_\pi(t)$ is the pion form factor. If we now use the fact that the P-wave $\gamma + \pi \to \pi + \pi$ amplitude will have the phase of $\pi - \pi$ scattering, then $M^{1^*}(t)$ may be written

$$M^{1^*}(t) = e \frac{k'(t)(t-4)^{1/2}}{12\pi} h(t) F_\pi^*(t) , \tag{9.22}$$

where $h(t)$ is a real function and the factors $k'(t)$, $(t-4)^{1/2}$, e, and 12π are included for convenience in subsequent calculations.

Expressing the $\text{Im}\, A_i^0$'s in this notation, we obtain

$$\text{Im}\, A_1^0(t) = -t\, h(t)\, g_2^V(t) , \tag{9.23}$$

$$\text{Im}\, A_2^0(t) = +h(t)\, g_2^V(t) , \tag{9.24}$$

and

$$\text{Im}\, A_4^0(t) = +h(t)\, g_1^V(t) . \tag{9.25}$$

The final expressions for the A_i^0's are then:

$$A_1^0 = \frac{e_r g_r}{2} \left[\frac{1}{s - M^2} + \frac{1}{\bar{s} - M^2} \right] + \frac{1}{\pi} \int_4^\infty dt' \frac{t'\, h(t')\, g_2^V(t')}{t' - t} \tag{9.26}$$

$$A_2^0 = + \frac{e_r q_r}{(s - M^2)(\bar{s} - M^2)} - \frac{1}{\pi}\int_4^\infty dt' \frac{h(t')\, g_2^V(t')}{t' - t} \tag{9.27}$$

$$A_3^0 = - \frac{1}{2} g_r (\mu'_{P_r} + \mu_{N_r}) \left[\frac{1}{s - M^2} - \frac{1}{\bar{s} - M^2} \right] \tag{9.28}$$

$$A_4^0 = - \frac{1}{2} g_r (\mu'_{P_r} + \mu_{N_r}) \left[\frac{1}{s - M^2} + \frac{1}{\bar{s} - M^2} \right]$$

$$- \frac{1}{\pi} \int_4^\infty dt' \frac{h(t')\, g_1^V(t')}{t' - t} , \tag{9.29}$$

where the factor (-1) arises from the sign of $i\epsilon$ that appears in the t denominators when s is given a small imaginary part and $\bar{s}$ is held fixed.

It is now possible to take advantage of the appearance of the functions g_2^V and g_1^V in these integrals. From FF we observe that

$$G_1^V(t) = \frac{1}{\pi} \int_4^\infty dt' \frac{g_1^V(t')}{t' - t} \tag{9.30a}$$

and

$$G_2^V(t) = \frac{1}{\pi} \int_4^\infty dt' \frac{g_2^V(t')}{t' - t} . \tag{9.30b}$$

The function h(t) has been shown by H. S. Wong to be well represented by the form[3]

$$h(t) = \frac{3\Lambda}{3\sqrt{2}F_\pi(1)e}\left(\frac{1}{t} + \frac{\beta}{t+a}\right)\left(\frac{1+a}{1+a+\beta}\right), \qquad (9.31)$$

where, if FF's solution of the F_π is used, we have $\beta = -65$, $a = 5$, $F_\pi(1) = 1.08$, and Λ is the arbitrary constant previously mentioned. Knowning the form of h(t), we can by forming subtracted forms of Eqs. (9.30a) and (9.30b) express the integrals in Eqs. (9.26) to (9.29) directly in terms of $G_1^V(t)$ and $G_2^V(t)$.

The resulting expressions for A_1^0, A_2^0, A_3^0, and A_4^0 are

$$A_1^0 = \frac{e_r g_r}{2}\left[\frac{1}{s-M^2} + \frac{1}{\bar{s}-M^2}\right]$$

$$+ \lambda'\left[(1+\beta)G_2^V(t) - \frac{\beta a}{t+a}\left(G_2^V(t) - G_2^V(-a)\right)\right], \qquad (9.32)$$

$$A_2^0 = \frac{e_r g_r}{(s-M^2)(\bar{s}-M^2)} - \lambda'\left[\frac{G_2^V(t) - G_2^V(0)}{t}\right.$$

$$\left. + \beta\,\frac{G_2^V(t) - G_2^V(-a)}{t+a}\right], \qquad (9.33)$$

$$A_3^{\;0} = -\frac{1}{2} g_r \left(\mu'_{P_r} + \mu_{N_r}\right) \left[\frac{1}{s - M^2} - \frac{1}{\bar{s} - M^2} \right] , \tag{9.34}$$

and

$$A_4^{\;0} = -\frac{1}{2} g_r \left(\mu'_{P_r} + \mu_{N_r}\right) \left[\frac{1}{s - M^2} + \frac{1}{\bar{s} - M^2} \right]$$

$$- \lambda' \left[\frac{G_1^{\;V}(t) - G_1^{\;V}(0)}{t} + \beta \, \frac{G_1^{\;V}(t) - G_1^{\;V}(-a)}{t + a} \right] , \tag{9.35}$$

where

$$\lambda' = \frac{3}{3\sqrt{2}} \frac{\Lambda}{e} \left(\frac{1 + a}{1 + a + \beta} \right) \frac{1}{F_\pi(1)} .$$

In order to compare the above results with experimental data, we must know the (+) and (-) amplitudes in the threshold region. These amplitudes have been given by CGLN, but because of a $1/M$ expansion used within their dispersion integrals, they were forced to introduce undetermined correction terms $N^{(+)}$ and $N^{(-)}$ into each of these amplitudes. Since the effect of our new Λ-dependent terms will be fairly sensitive to the values of N^+ and N^-, we will recalculate the (+, -) amplitudes avoiding any expansions in $1/M$. The fixed - t dispersion relation without subtractions as given in Eq. (7.3) will be used. It should be noted that a strongly interacting 3π state as has been proposed by Chew[16] would necessitate the use of a subtracted form of Eq. (7.3), but only for the + amplitude, as this 3π state would have $I = 0$. However, the (+) amplitude does not contribute to charged-pion photoproduction, which will prove to be most sensitive to the value of Λ.

A polynomial expansion in $\cos\theta$ for $\mathrm{Im}_I A_i^{(+,-)}$ in Eq. (7.2) will now converge for $\cos\theta$ within an ellipse with foci at +1 and -1 and semi-major axis given by the value of $\cos\theta$ at the nearest singularity in t. Thus for a maximum value of t allowed for convergence, there is also a minimum corresponding to the negative limit of the ellipse. The relation between these limits is

$$t_{max} + t_{min} = 2 - 4\omega k. \tag{9.36}$$

If we now express t_{max} as given by the boundary of $a_{i_{12}}^{(+,-)}$, we obtain

$$t_{min} = -4\omega(s)\,k(s) - 7 - \frac{8(3s - M^2 + 1)}{[s - (M+1)^2][s - (M-1)^2]} . \tag{9.37}$$

Since $\mathrm{Im}\, A_i^{(+,-)}$ will be used in the integral in Eq. (8.1), the region of convergence is determined by the maximum value of t_{min} and is found to be $9 > t > -19.3$. For comparison, we state the result rigorously proved by Oehme and Taylor that the polynomial expansion converges, at least for t, in the range $0 > t > t_0 \approx -12$.[17] It can be seen from Fig. 6 that the smallest value of t corresponding to physical $\cos\theta$ will be larger than -10 in the energy region we are considering; therefore the expansion for $\mathrm{Im}\, A_i^{(+,-)}$ should converge rapidly, making it plausible to neglect the high-angular-momentum contributions to $\mathrm{Im}\, A_i^{(+,-)}(s', t)$. A general feature of pion-nucleon scattering below 400 Mev is that the only large phase shift is in the $J = 3/2$, $I = 3/2$, state. Thus a reasonable first approximation is obtained by including in $\mathrm{Im}\, A_i^{(+,-)}(s't)$ only the parts containing this large pion-nucleon phase.

We must now calculate the absorptive parts of the A_i's for photoproduction ($s > (M+1)^2$) including only the 33 part. Since, in

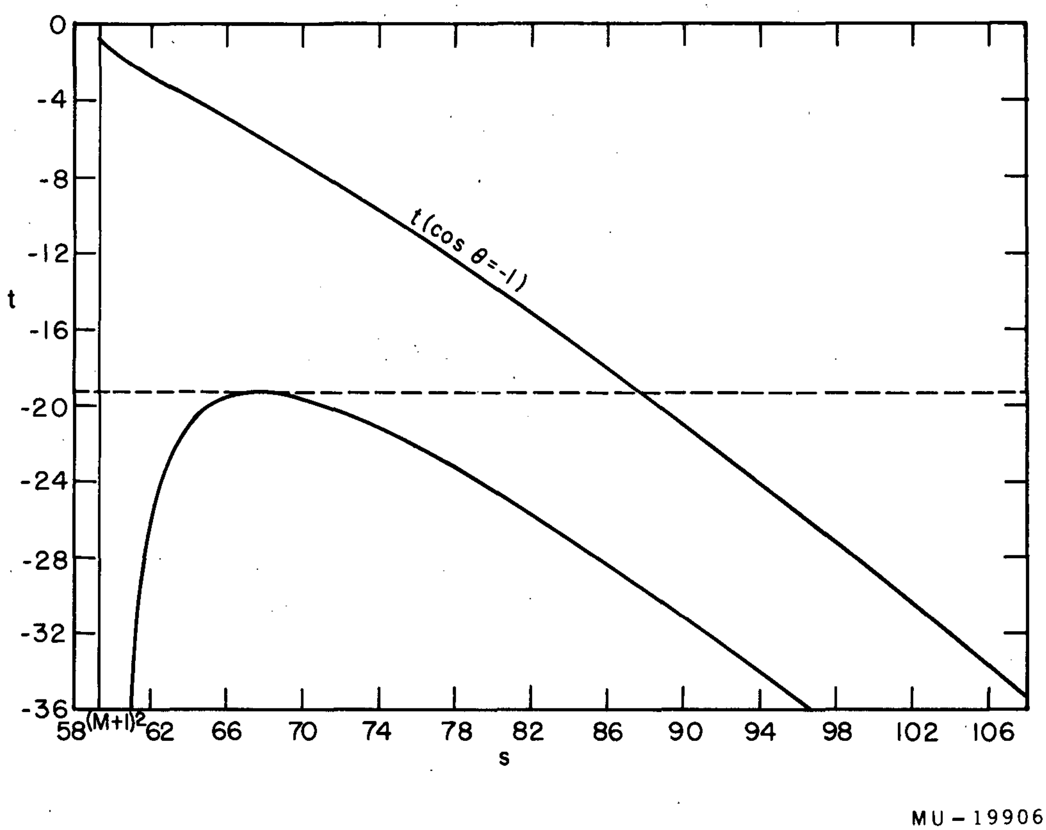

Fig. 6. Plot of t_{min} and t for cos $\theta = 0$ as a function of s.

the previous treatment by CGLN, the M_{1+} amplitude was found to be much more important than the E_{1+}, the imaginary part of E_{1+} is neglected in the following expressions.

From Eqs. (3.5) to (3.12) we see that

$$\text{Im } F_1 = \frac{4\pi 12\ sx}{(s-M^2)\,[(\sqrt{s}+M)^2-1]^{1/2}}\ \text{Im } M_{1+}(s)\,, \tag{9.38}$$

$$\text{Im } F_2 = \frac{4\pi\ 2\,[(\sqrt{s}+M)^2]^{1/2}}{q(s)\,K(s)}\ \text{Im } M_{1+}(s)\,, \tag{9.39}$$

$$\text{Im } F_3 = \frac{-\,4\pi\ 6\sqrt{s}}{q(s)\,K(s)\,[\sqrt{s}+M)^2-1]^{1/2}}\ \text{Im } M_{1+}(s)\,, \tag{9.40}$$

and

$$\text{Im } F_4 = 0. \tag{9.41}$$

Equations (3.5) to (3.8) can be solved, yielding

$$A_1 = \frac{1}{2\sqrt{s}}\left[(\sqrt{s}+M)\,F_1 - (\sqrt{s}-M)\,F_2 + M(2M^2-s-s) \times\left\{\frac{F_3}{\sqrt{s}-M} + \frac{F_4}{\sqrt{s}+M}\right\}\right] \tag{9.42}$$

$$A_2 = \frac{1}{2\sqrt{s}}\left[F_3 - F_4\right] \tag{9.43}$$

$$A_3 = \frac{1}{2\sqrt{s}} \left[F_1 + F_2 + \frac{s - \bar{s}}{2} \left\{ \frac{F_3}{\sqrt{s} - M} + \frac{F_4}{\sqrt{s} + M} \right\} \right] \tag{9.44}$$

$$A_4 = \frac{1}{2\sqrt{s}} \left[F_1 + F_2 + \frac{2M^2 - s - \bar{s}}{2} \left\{ \frac{F_3}{\sqrt{s} - M} + \frac{F_4}{\sqrt{s} + M} \right\} \right] . \tag{9.45}$$

Taking the imaginary parts of A_1, A_2, A_3, and A_4, employing Eqs. (9.38) to (9.41), and expressing x in terms of s and t,

$$x = \cos\theta = \frac{t - 1 + 2\omega(s)\,k(s)}{2q(s)\,k(s)} ,$$

we find

$$\mathrm{Im}\, A_1(s,t) = C(s) \left[\omega(s)\,(\sqrt{s} + M) + t + 1 \right] \mathrm{Im}\, M_{1+}(s) , \tag{9.46}$$

$$\mathrm{Im}\, A_2(s,t) = -3\, C(s)\, \mathrm{Im}\, M_{1+}(s) , \tag{9.47}$$

$$\mathrm{Im}\, A_3(s,t) = C(s) \left[\frac{3}{2} \left(\frac{t - 1}{\sqrt{s} + M} \right) + \omega(s) - (W + M) \right] \mathrm{Im}\, M_{1+}(s) , \tag{9.48}$$

and

$$\mathrm{Im}\, A_4(s,t) = C(s) \left[\frac{3}{2} \left(\frac{t - 1}{\sqrt{s} + M} + \omega(s) + 2(W + M) \right] \mathrm{Im}\, M_{1+}(s) , \tag{9.49}$$

where $C(s) = \dfrac{4\pi}{q(s)\,k(s)} \left[(\sqrt{s} + M)^2 - 1 \right]^{-1/2}$.

Now, from Eqs. (2.23) and (2.24), we see that

$$M_{1+}^{+} = \frac{1}{3} M_{1+}^{1/2} + \frac{2}{3} M_{1+}^{3/2} \tag{9.50}$$

and

$$M_{1+}^{-} = \frac{1}{3} M_{1+}^{1/2} - \frac{1}{3} M_{1+}^{3/2} . \tag{9.51}$$

Since the $I = \frac{1}{2}$ phase is small, we can write

$$\operatorname{Im} M_{1+}^{+} \cong -2 \operatorname{Im} M_{1+}^{-} \cong \frac{2}{3} \operatorname{Im} M_{1+}^{3/2} . \tag{9.52}$$

The final expressions for the A_i's are

$$A_1^{\pm} = \frac{e_r g_r}{2} \quad \frac{1}{s - M^2} \pm \frac{1}{\bar{s} - M^2}$$

$$\pm \binom{2}{1} \frac{1}{3\pi} \int_{(M+1)^2}^{\infty} ds' \, C(s') \left[\omega(s') \left(\sqrt{s'} + M\right) + t + 1\right] \operatorname{Im} M_{1+}^{3/2}(s) \tag{9.53}$$

$$\times \left\{ \frac{1}{s' - s} \pm \frac{1}{s' - \bar{s}} \right\} ,$$

$$A_2^{\pm} = - \frac{e_r g_r}{t - 1} \left(\frac{1}{s - M^2} \pm \frac{1}{\bar{s} - M^2} \right)$$

$$\mp \binom{2}{1} \frac{1}{\pi} \int_{(M+1)^2}^{\infty} ds' \, C(s') \operatorname{Im} M_{1+}^{3/2} (s') \left\{ \frac{1}{s' - s} + \frac{1}{s' - \bar{s}} \right\} , \tag{9.54}$$

$$A_3^{\pm} = -\frac{1}{2}\, g_r\, (\mu'_{P_r} - \mu_{N_r})\, \left(\frac{1}{s - M^2} \mp \frac{1}{\bar{s} - M^2} \right)$$

$$\pm \binom{2}{1} \frac{1}{3\pi} \int_{(M+1)^2}^{\infty} ds'\; C(s') \left[\frac{3}{2} \left(\frac{t - 1}{\sqrt{s'} + M} \right) + \omega(s') - (/s' + M) \right]$$

$$\times \; \mathrm{Im}\, M_{1+}^{3/2}(s') \left\{ \frac{1}{s' - s} \mp \frac{1}{s' - \bar{s}} \right\} , \qquad (9.55)$$

and

$$A_4^{\pm} = -\frac{1}{2}\, g_r\, (\mu'_{P_r} - \mu_{N_r})\, \left(\frac{1}{s - M^2} \pm \frac{1}{\bar{s} - M^2} \right)$$

$$\pm \binom{2}{1} \frac{1}{3\pi} \int_{(M+1)^2}^{\infty} ds'\; C(s') \left[\frac{3}{2} \left(\frac{t - 1}{\sqrt{s'} + M} \right. + \omega(s') + 2\, (\sqrt{s'} + M) \right]$$

$$\times \; \mathrm{Im}\, M_{1+}^{3/2}(s') \left\{ \frac{1}{s' - s} \pm \frac{1}{s' - s} \right\} . \qquad (9.56)$$

These integrals can be carried out numerically if the CGLN expression for $M_{1+}^{3/2}$ is used with an effective range formula to represent the 33 amplitude for pion-nucleon scattering. It should be noted that expressions (9.53) to (9.56) are identical to those of CGLN except that no $1/M$ expansion has been made and only M_{1+} has been kept in the imaginary parts of the amplitudes.

X. EVALUATION OF THE PHOTOPRODUCTION AMPLITUDES

Evaluation of the integrals in Eq. (9.53) to Eq. (9.56) is accomplished by using the CGLN solution for $M_{1+}^{3/2}$:

$$\frac{M_{1+}^{3/2}(s)}{q(s)\,k(s)} = \frac{\mu_p - \mu_n}{2f}\,\frac{f_{33}}{q^2(s)} \tag{10.1}$$

A relativistic effective-range formula suggested by Chew and Wong,[18]

$$\mathrm{Im}\, f_{33} = \frac{q^5}{q^6 + \Gamma (s - s_r)^2 (s - M^2)^2} \tag{10.2}$$

is used to represent the 33 amplitude, where Γ and s_r are parameters which have been adjusted to fit the Chiu and Lomon[19] δ_{33} at 150 and 220 Mev and to the low-energy behavior of δ_{33} as given by Barnes <u>et al.</u>[20] The resulting parameters are $\Gamma = 3.5\times 10^{-4}$ and $s_r = 76.6$. In performing the integrations, we expanded the denominators in powers of $\cos\theta$, keeping only the first two terms because the expansion converges quite rapidly since $\cos\theta$ is always multiplied by the nucleon velocity.

The Re $M_{1+}(s)$ produced by the integrals in Eqs. (9.53) to (9.56) must be considered an iterative solution for $M_{1+}(s)$. As there seems to be no guarantee that such a procedure will converge, we projected this contribution from the $\mathcal{F}$'s by means of Eq. (3.15) and replaced it by the value given by Eq. (10.1). It was noted, however, that this correction was not large, indicating that the solution given by Eq. (10.1) is reasonably good.

To form the scattering amplitude for any of the charge states of interest, we must know the matrix element of g_β^+, g_β^-, and g_β^0 for each of these states. These matrix elements as evaluated by CGLN are given in Table I.

Table I

Matrix elements of $g^{(\pm, 0)}$
for the four possible charge configurations.

	$\gamma + p \rightarrow \pi^0 + p$	$\gamma + n \rightarrow \pi^0 + n$	$\gamma + p \rightarrow \pi^+ + n$	$\gamma + n \rightarrow \pi + p$
g^+	1	1	0	0
g^-	0	0	$\sqrt{2}$	$-\sqrt{2}$
g^0	1	-1	$\sqrt{2}$	$\sqrt{2}$

The scattering amplitudes for $\gamma + p \rightarrow \pi^0 + p$, denoted $\mathcal{F}^{\pi^0}$, and $\gamma + p \rightarrow \pi^+ + n$, denoted $\mathcal{F}^{\pi^+}$, are formed as

$$\mathcal{F}^{\pi^0} = \mathcal{F}^+ + \mathcal{F}^0 \tag{10.3a}$$

and

$$\frac{\mathcal{F}^{\pi^0}}{\sqrt{2}} = \mathcal{F}^- + \mathcal{F}^0 . \tag{10.3b}$$

In Table II the calculated values of $\mathcal{F}^{\pi^0}$ for $\Lambda = 0$ are given, again with only the first two terms in $x = \cos\,\theta$ retained.* Only the pole terms for the (0) amplitude have been included. We define $\omega^* = \sqrt{s} - M$. The photon laboratory energy can be obtained from

$$K_L = \frac{s - M^2}{2M} = \omega^* + \frac{\omega^{*2}}{2M} . \tag{10.4}$$

In the case of charged-pion production, an expansion of the meson-current pole term is not possible. We separate these terms as follows:

$$\mathcal{F}^{\pi+} = \mathcal{F}' \; \frac{\mathcal{F}^R}{1 - vx} \tag{10.5}$$

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*The values $M = 6.7$ and $f^2 = 0.08$ have been employed throughout.

Table II

Values of the scattering amplitudes, $\mathcal{F}$, for photoproduction of π^0 with $\Lambda = 0$.

ω^*	$\mathcal{F}_1 \times 10^3$	$\mathcal{F}_2 \times 10^3$	$\mathcal{F}_3 \times 10^3$	$\mathcal{F}_4 \times 10^3$
1.00	-2.780	0	0	0
1.05	-2.682 + 6.865x	3.424 - 0.064x	-7.104 + 0.123x	-0.360 + 0.016x
1.10	-2.556 + 10.159x	5.199 - 0.128x	-10.563 + 0.250x	-0.704 + 0.045x
1.15	-2.398 + 13.046x	6.844 - 0.193x	-13.631 + 0.380x	-1.033 + 0.081x
1.20	-2.204 + 15.828x	8.504 - 0.257x	-16.619 + 0.513x	-1.350 + 0.124x
1.25	-1.969 + 18.632x	10.243 - 0.321x	-19.659 + 0.648x	-1.654 + 0.170x
1.30	-1.690 + 21.531x	12.099 - 0.384x	-22.831 + 0.787x	-1.949 + 0.223x
1.40	-0.986 + 27.796x	16.263 - 0.503x	-29.765 + 1.073x	-2.510 + 0.334x
1.50	-0.078 + 34.784x	21.077 - 0.609x	-37.602 + 1.368x	-3.040 + 0.458x

where v is the velocity of the final meson.

It is now possible to expand $\mathcal{F}'^{\pi+}$, and the resulting expressions for $\Lambda = 0$ are given in Table III. Values of $\mathcal{F}_3^R$, $\mathcal{F}_4^R$, v, Im M_{1+}^{+}, and Im M_{1+}^{-} are given in Table IV, and $\mathcal{F}_1^R$ and $\mathcal{F}_2^R$ are zero.

The imaginary part of any of the $\mathcal{F}$'s may be obtained with the aid of Eqs. (3.9) to (3.12). The pole terms for $\mathcal{F}^0$ are given in Table V.

The differential cross section for unpolarized photons and nucleons is

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} |M|^2 = \frac{q}{k} \Big\{ |\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 + \frac{1}{2} |\mathcal{F}_3|^2 + \frac{1}{2} |\mathcal{F}_4|^2 + \mathrm{Re}\,\mathcal{F}_1^* \mathcal{F}_4 + \mathrm{Re}\,\mathcal{F}_2^* \mathcal{F}_3$$
$$+ \cos\theta \left[\mathrm{Re}\,\mathcal{F}_3^* \mathcal{F}_4 - 2\,\mathrm{Re}\,\mathcal{F}_1^* \mathcal{F}_2 \right]$$
$$- \cos^2\theta \left[\frac{1}{2} |\mathcal{F}_3|^2 + \frac{1}{2} |\mathcal{F}_4|^2 + \mathrm{Re}\,\mathcal{F}_1^* \mathcal{F}_4 + \mathrm{Re}\,\mathcal{F}_2^* \mathcal{F}_3 \right]$$
$$- \cos^3\theta \ \mathrm{Re}\,\mathcal{F}_3^* \mathcal{F}_4 \Big\} . \qquad (10.6)$$

The differential cross section for $\gamma + P \rightarrow P + \pi^0$ in the threshold region may be expressed as

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left[A + B \cos + C \cos^2\theta + D \cos^3\theta \right] . \qquad (10.7)$$

Table III

Scattering amplitudes, $\mathcal{F}$, excluding the meson-current term for photoproduction of π^+ with $\Lambda = 0$.

ω^*	$\frac{1}{\sqrt{2}} \mathcal{F}'_1 \times 10^3$	$\frac{1}{\sqrt{2}} \mathcal{F}'_2 \times 10^3$	$\frac{1}{\sqrt{2}} \mathcal{F}'_3 \times 10^3$	$\frac{1}{\sqrt{2}} \mathcal{F}'_4 \times 10^3$
1.00	19.679	0	0	0
1.05	19.403 - 2.824x	-2.773 + 0.036x	2.899 - 0.083x	0.049 - 0.002x
1.10	19.116 - 4.219x	-4.082 + 0.072x	4.353 - 0.168x	0.099 - 0.006x
1.15	18.818 - 5.471x	-5.217 + 0.108x	5.670 - 0.254x	0.151 - 0.011x
1.20	18.507 - 6.702x	-6.300 + 0.142x	6.979 - 0.343x	0.205 - 0.017x
1.25	18.180 - 7.964x	-7.383 + 0.175x	8.334 - 0.433x	0.262 - 0.024x
1.30	17.836 - 9.288x	-8.496 + 0.207x	9.767 - 0.525x	0.319 - 0.032x
1.40	17.090 - 12.203x	-10.886 + 0.264x	12.959 - 0.713x	0.439 - 0.052x
1.50	16.260 - 15.512x	-13.539 + 0.310x	16.628 - 0.907x	0.566 - 0.076x

Table IV

Values of $\mathcal{F}^R$, the meson velocity, Im $M_{1+}{}^{+}$, and Im $M_{1+}{}^{-}$.

ω^*	$\frac{1}{\sqrt{2}}\mathcal{F}_3{}^R \times 10^3$	$\frac{1}{\sqrt{2}}\mathcal{F}_4{}^R \times 10^3$	Meson velocity V	Im $M_{1+}{}^{+} \times 10^3$	Im $M_{1+}{}^{-} \times 10^3$
1.00	0	0	0	0	0
1.05	5.947	-1.818	0.2854	0.012	-0.006
1.10	8.098	-3.382	0.3911	0.053	-0.027
1.15	9.565	-4.735	0.4649	0.130	-0.065
1.20	10.667	-5.910	0.5217	0.252	-0.126
1.25	11.532	-6.936	0.5676	0.433	-0.217
1.30	12.232	-7.834	0.6058	0.691	-0.346
1.40	13.284	-9.318	0.6661	1.537	-0.769
1.50	14.023	-10.475	0.7118	3.090	-1.545

Table V

The pole terms for the (0) amplitude.

ω^*	$\mathcal{F}_1^0$	$\mathcal{F}_2^0$	$\mathcal{F}_3^0$	$\mathcal{F}_4^0$
1.00	-1.290			
1.05	-1.330 + 0.439x	0.030 - 0.010x	-0.403 + 0.018x	-0.162 + 0.007x
1.10	-1.371 + 0.623x	0.043 - 0.020x	-0.570 + 0.036x	-0.316 + 0.020x
1.15	-1.410 + 0.766x	0.055 - 0.030x	-0.699 + 0.055x	-0.462 + 0.036x
1.20	-1.448 + 0.887x	0.066 - 0.040x	-0.808 + 0.073x	-0.602 + 0.055x
1.25	-1.485 + 0.995x	0.076 - 0.051x	-0.903 + 0.092x	-0.735 + 0.075x
1.30	-1.521 + 1.093x	0.086 - 0.062x	-0.990 + 0.112x	-0.863 + 0.097x
1.40	-1.590 + 1.269x	0.106 - 0.084x	-1.143 +0.151x	-1.106 + 0.146x
1.50	-1.655 + 1.425x	0.124 - 0.107x	-1.277 + 0.191x	-1.332 + 0.199x

In Figs. 7 to 10 the values of A, b, C and D calculated from the $\mathcal{F}$s in Table II are given together with experimental data.[21] The fact that D has been set equal to zero in the analysis of the experimental data, while the calculated value of D is comparable to B, makes a quantitative comparison between our values of B and C and those from experiment unreliable.

In Fig. 11, $|M|^2$ at 90 deg for $\gamma + P \rightarrow N + \pi^+$ as calculated from Tables III and IV is given, together with experimental data.[22] Also included in Fig. 11 are the results of a theoretical calculation by C. S. Robinson based on the results of CGLN with $N^+ = N^- = 0$.[23]

In obtaining these cross sections a correction has been made for the mass difference between π^+ and π^0 by using as a unit the mass of the pion in question.* This means that, in effect, the value of the nucleon mass used in the calculation for the π^0 amplitude was too small, being 6.7 instead of 6.9. Since all energies are expressed relative to the nucleon mass, no serious error will be introduced by this procedure.

It is now possible to estimate how large a Λ would be allowed on the basis of present experimental information. First we will take G_1^V and G_2^V to be linear functions of t for $0 > t > -5$, and will use

$$\frac{G_1^{V'}(0)}{G_1^V(0)} \simeq 0.08 \simeq \frac{G_2^{V'}(0)}{G_2^V(0)} = a \qquad (10.8)$$

as given by FF. We may express $G_1^V(t)$ and $G_2^V(t)$

$$G_1^V(t) = G_1^V(0)[1 + at] = \frac{e}{2}[1 + at] \qquad (10.9)$$

---

*The conversion factors used are $\mu_{\pi_0} = 135$ Mev, $\mu_{\pi_+} = - 140$ Mev, $(\frac{1}{\mu_{\pi_0}})^2 = 18.66$ mb, and $(\frac{1}{\mu_{\pi_-}})^2 = 19.96$ mb.

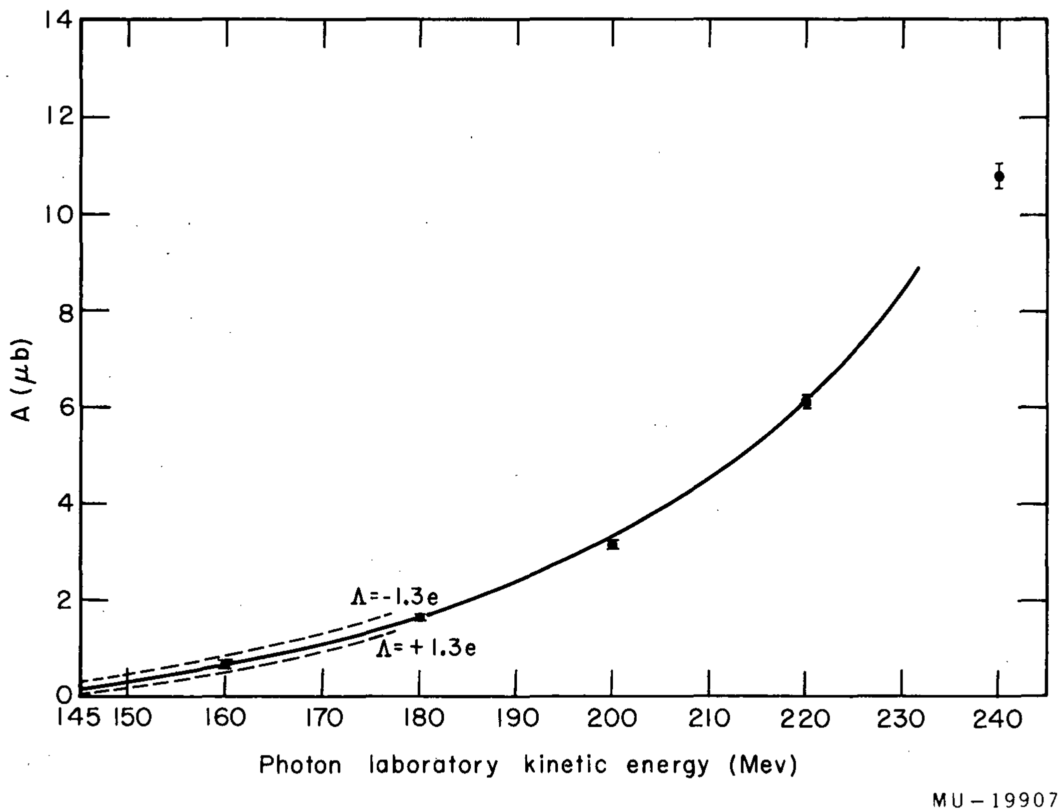

Fig. 7. The coefficient A for π^0 photoproduction. Solid line: prediction with $\Lambda = 0$; dashed lines: predictions with $\Lambda = 1.8e$ and $\Lambda = -1.8e$. The experimental points are those of Goldansky et al.[21]

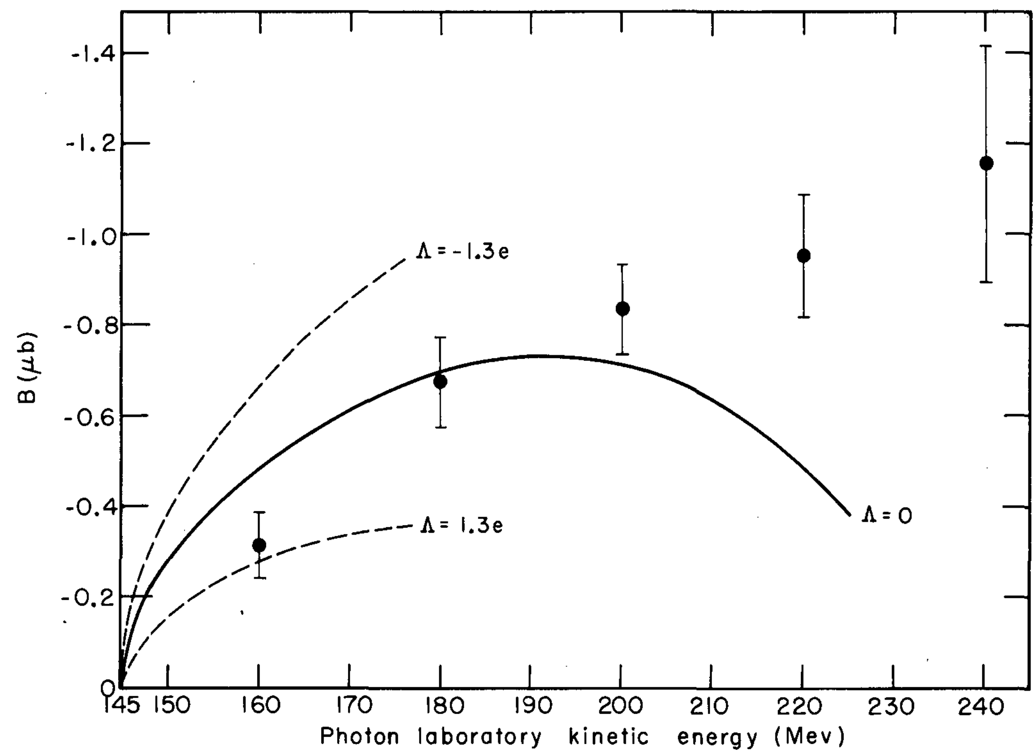

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Fig. 8. The coefficient B for π^0 photoproduction. Solid line: prediction with $\Lambda = 0$; dashed lines: predictions with $\Lambda = 1.8e$ and $\Lambda = -1.8e$. The experimental points are those of Goldansky et al.[21]

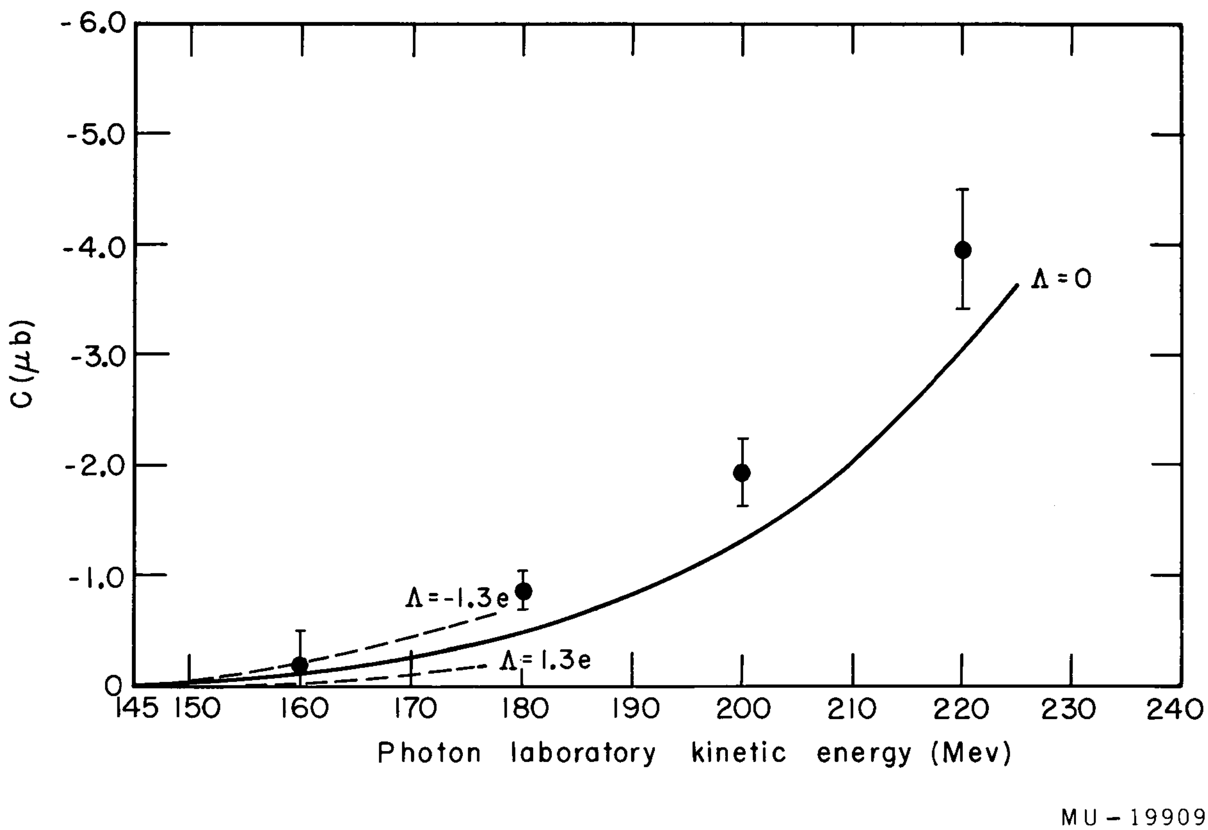

Fig. 9. The coefficient C for π^0 photoproduction. Solid line: prediction with $\Lambda = 0$; dashed lines: predictions with $\Lambda = 1.8e$ and $\Lambda = -1.8e$. The experimental points are those of Goldansky et al.[21]

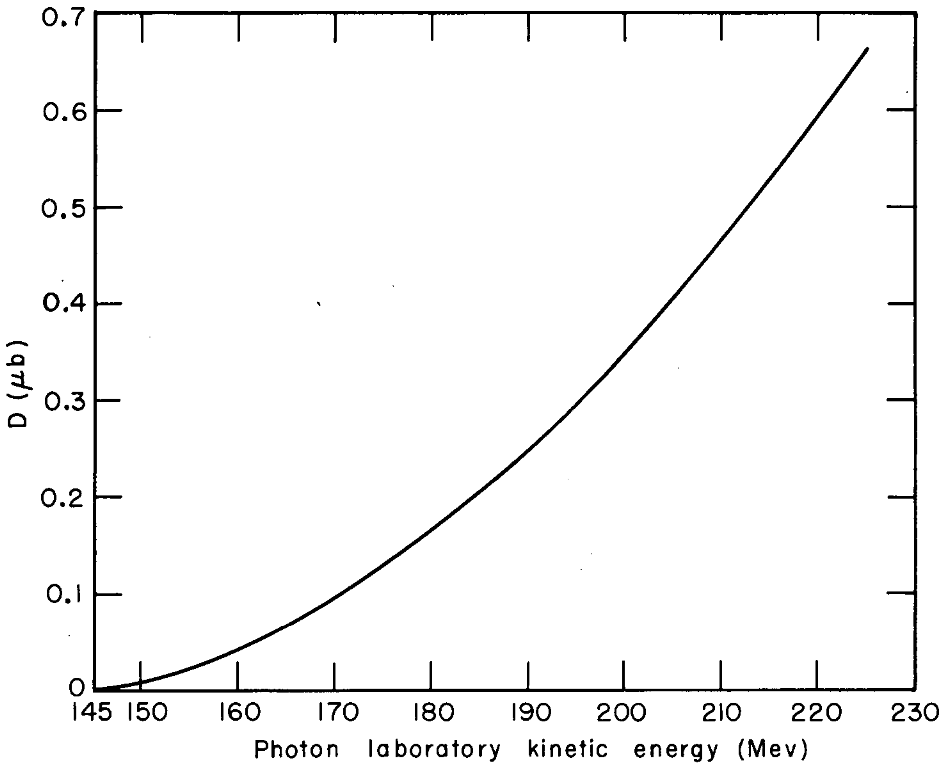

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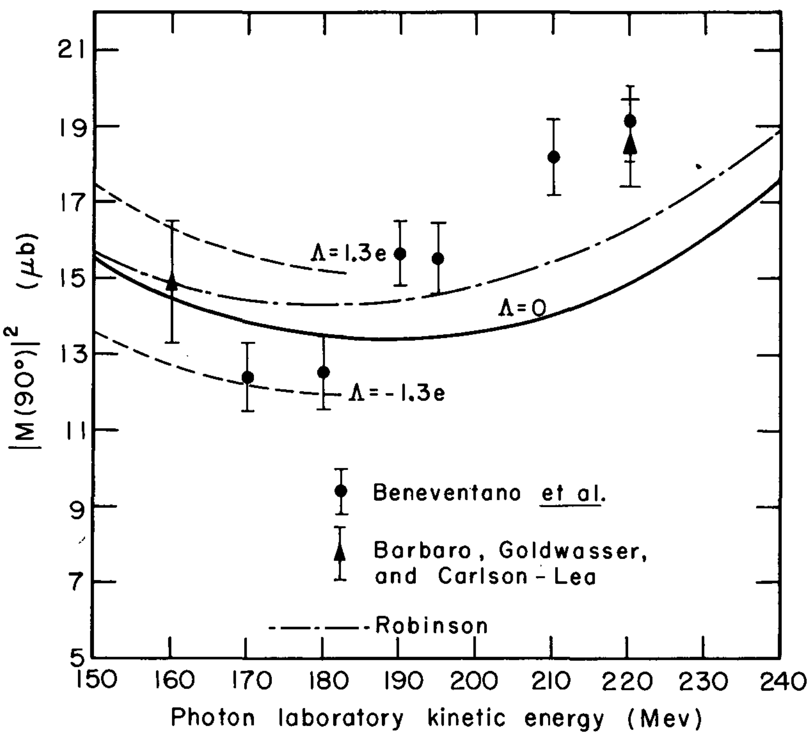
Fig. 10. The coefficient D for π^0 photoproduction as predicted for $\Lambda = 0$.



MU-19911

Fig. 11. The matrix element squared at θ = 90 deg for π^+ photoproduction. Solid line: prediction for Λ = 0; dashed lines: predictions for Λ = 1.8 e and Λ = - 1.8e; dot-dash line: prediction of CGLN as calculated by Robinson.[23] The experimental points are those of Barbaro et al. and Beneventano et al.[22]

and

$$G_2^V(t) = G_2^V(0)[1 + a t] = \frac{(\mu'_p - \mu_N)}{2} [1 + a t] . \tag{10.10}$$

The quantities most sensitive to Λ are the threshold values of $(k/q)(d\sigma/d\Omega$ for π^+ and $d\sigma(\pi^-)/d\sigma(\pi^+)$. This can be seen by noticing that the correction to A_1 is several times the correction to the other amplitudes causing a large correction to $\mathcal{F}_1$. Also, since $\mathcal{F}_1$ is larger for charged-pion production, $|\mathcal{F}_1|^2$ will be sensitive to small changes in $\mathcal{F}_1$.

At threshold, the correction to be added to $\mathcal{F}_1^0$ is

$$\Delta \mathcal{F}_1^0 = 6.8 \times 10^{-4} \Lambda/e , \tag{10.11}$$

which produces a fractional change of $1 + (0.074 \Lambda/e)$ in $|M|^2$ for π^+ production. Since $\mathcal{F}^0$ enters with opposite signs into π^- production, $R = d\sigma(\pi^-)/d\sigma(\pi^+)$ at threshold will be even more sensitive to Λ. Including the correction given by Eq. (10.11) in R, we obtain

$$R = 1.28 \left[\frac{1 - (0.031 \Lambda/e)}{1 + (0.037 \Lambda/e)}\right]^2 \simeq 1.28 [1 - (0.14 \Lambda/e)] \tag{10.12}$$

The quantities A, B, and C for π^0 production and $|M|^2$ at $\theta = 90$ deg for π^+ production have been calculated for $\Lambda = \pm 1.8e$ (see Figs. 7 through 11).

XI. CONCLUSIONS

The results we have obtained in this work may be summarized by saying that after a more careful analysis of photoproduction based on the Mandelstam representation, the work of CGLN survives almost unchanged from a practical point of view. The only modification is an additive term to correct the (0) amplitude in terms of the parameter Λ. However, it should be remembered that a change in the treatment of CGLN may also be required in the (+) amplitude if there is a resonant three-pion intermediate state in the t spectrum.

The evaluation of the dispersion integrals in their relativistic form for the (+) and (-) amplitudes did not produce any significant change from CGLN. The values $N^+ = -0.062$ and $N^- = 4.5 \times 10^{-3}$ were obtained, indicating that the often used procedure of setting $N^+ = N^- = 0$ does not cause an important error.

To investigate what limit the various experimental data place on the size of Λ, consider first the coefficients A, B, and C giving the angular distribution for π^0 production. The value of A, which is the most accurately determined by experiment, proves to be quite insensitive to the values of Λ. The calculated values of A for $|\Lambda| < 1.8$ e are all in good agreement with the experimental data. While the coefficients B and C are more sensitive to Λ, the difficulty encountered in comparing the theoretical values of these coefficients with those from experiment make these data a poor test for Λ. A further uncertainty in the theoretical values of A, B, and C arises from the possibility of a strongly interacting three-pion intermediate state, which would require a subtraction in the (+) amplitude.

The threshold π^+ data provides a better test of the magnitude of Λ. As can be seen from Fig. 11, with $f^2 = 0.08$ the experimental data constrain Λ to lie between 1.3e and -1.3e. Changing f^2 by ± 0.01, which is perhaps the maximum allowed by other considerations, can be compensated in the threshold π^+ cross section by giving Λ a value $\mp$ 1.75 e. (This compensating effect will not remain at higher energies, as the energy dependence of the Λ term is different from that of the other terms).

The value of $R = d\sigma(\pi^-) / d\sigma(\pi^+)$ at threshold given by formula (10.12) provides a measure of Λ which is insensitive to small variations in f^2. However, the corrections and extrapolation necessary to obtain R cast some doubt as to its exact value. The range $-1.8\ e < \Lambda < 1.8\ e$ corresponds to $1.0 < R < 1.6$, which is roughly the current uncertainty in the $-/+$ ratio.

An estimate of Λ based on the π^0 lifetime has been made by H. S. Wong.[3] His results are $|\Lambda| \gtrsim e$; however, the possibility of a resonant three-pion state produces some uncertainty in this estimate.

As the theoretical understanding of pion-nucleon scattering improves, an approach to photoproduction through multipole amplitudes as outlined in Section VIII should be carried through. Such a procedure could extend the description of photoproduction to the region in which phases other than the 3-3 become important. It could also improve the crude CGLN formula for the magnetic-dipole amplitude which has been accepted here as the basis for many of our calculations.

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