APPLICATION OF THE P-VERSION OF THE FINITE-ELEMENT METHOD TO GLOBAL-LOCAL PROBLEMS

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1. INTRODUCTION

The following is a brief survey of some recent developments in finite-element analysis technology which bear upon the three main research areas under consideration in this workshop: (1) analysis methods; (2) software testing and quality assurance; and (3) parallel processing.

The variational principle incorporated in a finite-element computer program, together with a particular set of input data, determines the exact solution corresponding to that input data. Most finite-element analysis computer programs are based on the principle of virtual work. In the following we consider only programs based on the principle of virtual work and denote the exact displacement vector field corresponding to some specific set of input data by $\vec{u}_{EX}$. The exact solution $\vec{u}_{EX}$ is independent of the design of the mesh or the choice of elements. Except for very simple problems, or specially constructed test problems, $\vec{u}_{EX}$ is not known.

We perform a finite-element analysis (or any other numerical analysis) because we wish to make conclusions concerning the response of a physical system to certain imposed conditions, *as if* $\vec{u}_{EX}$ *were known.* We know the finite-element solution only which we denote by $\vec{u}_{FE}$. The solution $\vec{u}_{FE}$ depends not only on the variational principle and the input data but also on the finite-element mesh and the choice of elements. We will assume that the finite-elements are exactly and minimally conforming and therefore the elements are completely characterized by their polynomial degree. We therefore control $\vec{u}_{FE}$ by mesh design and the choice of the polynomial degree of elements.

We wish to compute $\vec{u}_{FE}$ so that $\vec{u}_{FE}$ is close to $\vec{u}_{EX}$ in some sense. For example, if we are interested in determining a stress intensity factor then we wish to have the stress intensity factor computed from $\vec{u}_{FE}$ to be close to the stress intensity factor computed from $\vec{u}_{EX}$ within some prespecified level of tolerance τ. In general, we wish to determine functionals $\Psi_i(\vec{u}_{FE})$ $(i = 1, 2 \ldots, n)$ so that:

$$\left| \frac{\Psi_i(\vec{u}_{EX}) - \Psi_i(\vec{u}_{FE})}{\Psi_i(\vec{u}_{EX})} \right| \leq \tau_i \quad (i = 1, 2, \ldots, n) \tag{1}$$

The question naturally arises: how can we tell whether $\Psi_i(\vec{u}_{FE})$ is close to $\Psi_i(\vec{u}_{EX})$ if we do not know $\vec{u}_{EX}$? The answer is: by performing *extensions.* Both the estimation and control of error are based on extensions.

Extensions are systematic increases in the number of degrees of freedom either by mesh refinement, increase in the polynomial degree of elements or a combination of both. If the extension is by mesh refinement then the process is called *h-extension**. If the extension is by increase in the polynomial degree of elements then the process is called *p-extension***. If the extension is by a combination of *proper mesh refinement and concurrent increase in the polynomial degree of elements* then it is called *h-p extension*. Having performed an extension, we may draw conclusions concerning the overall quality of the approximate solution and the quality of any functional computed from $\vec{u}_{FE}$.

2. OVERALL QUALITY

The overall quality of approximation can be judged in terms of the estimated error in energy norm and errors in equilibrium. Estimation of error in energy norm is outlined in some detail and an example is presented. Procedures for assessment of the quality of approximation in terms of errors in equilibrium are briefly discussed.

2.1. Estimation of error in energy norm.

We know that the strain energy of the error $U(\vec{u}_{EX} - \vec{u}_{FE})$ must decrease monotonically as we systematically refine the mesh or increase the polynomial degree of elements and a well developed, elaborate theoretical basis exists for the estimation of error in energy norm for the h-, p- and h-p extension processes. (See, for example, [1,2,3,4,5].) The error in energy norm is defined as:

$$\|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)} \equiv \sqrt{U(\vec{u}_{EX} - \vec{u}_{FE})} \tag{2}$$

where Ω represents the solution domain and U represents the strain energy. $\|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)}$ is closely related to the root- mean-square of error in stresses [6].

In the case of h- and p-extensions the estimate is of the form:

$$\|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)} \leq \frac{k}{N^{\beta}} \tag{3}$$

where k and β are positive constants, N is the number of degrees of freedom. In the case of h-p extensions the estimate is of the form:

$$\|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)} \leq \frac{k}{\exp(\gamma N^{\theta})} \tag{4}$$

where k, γ and θ are positive constants. These estimators are 'sharp' for large N values hence the 'less than or equal' ($\leq$) can be replaced by 'approximately equal' ($\approx$) in (3), (4) when N is large. Therefore from (3) for large N values we have:

$$\log \|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)} \approx \log k - \beta \log N \tag{5}$$

If we plot $\log \|u_{EX} - u_{FE}\|_{E(\Omega)}$ versus $\log N$ we see a downward sloping straight line. The absolute value of the slope is β, called the *asymptotic rate of convergence*. When β is large then the error decreases rapidly as N is increased. When β is small then the error decreases slowly. Of course, the error also depends on k which

* *h represents the size of elements. h-Extension involves letting* $h_{max} \to 0$.

** *p represents the polynomial degree of elements. p-Extension involves letting* $p_{min} \to \infty$.

is generally not known a priori, but can be estimated from data obtained from properly performed extensions. This will be discussed later. When the estimate is of the form (3) the rate of convergence is said to be *algebraic.*

When the estimate is of the form (4) and we plot $\log \|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)}$ versus $\log N$ then for large N values we see a downward curving line [2,3,4,5,7]. In this case the rate of convergence is *exponential*:

$$\log \|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)} \approx \log k - \gamma (\log e) N^{\theta} \tag{6}$$

where e is the base of the natural logarithm. If we plot $\log \|\vec{u}_{EX} - \vec{u}_{FE}\|_{E(\Omega)}$ versus N^{θ} (not $\log N$ as before) then we see a downward sloping straight line. It is known that under conditions which are generally satisfied in practice $\theta \geq 1/3$ [5].

All error estimation techniques are based on extension. Because in general the exact solution $\vec{u}_{EX}$ is not known, the only information available to us is how the finite-element solution $\vec{u}_{FE}$ behaves when the number of degrees of freedom is increased either through mesh refinement or increase in the polynomial degree of elements. Such information, together with an estimate or hypothesis concerning the magnitude of the error, or its rate of change with respect to N, is essential to all error estimation. Of course, the estimate or hypothesis must be asymptotically correct: as $N \to \infty$ the estimated error must approach zero at the same rate as the true error does. Therefore the quality of error estimators should increase with N.

P-extension makes it convenient and inexpensive to obtain information concerning the rate of change of $U(\vec{u}_{FE})$ with respect to N. In the p-version *hierarchic* basis functions are used. Therefore the stiffness matrices and load vectors corresponding to polynomial degree p are embedded in the stiffness matrices and load vectors of polynomial degree $p+1$. Once a solution is available for polynomial degree p_{max}, all solutions corresponding to $p = 1, 2, \ldots, p_{max} - 1$ can be readily and inexpensively obtained. Specifically, we write:

$$\|\vec{u}_{EX} - \vec{u}_{FE}\|^2_{E(\Omega)} = U(\vec{u}_{EX} - \vec{u}_{FE}) = |U(\vec{u}_{EX}) - U(\vec{u}_{FE})| \approx \frac{k^2}{N^{2\beta}} \tag{7}$$

Let us assume for the moment that $U(\vec{u}_{EX}) > U(\vec{u}_{FE})$. In that case:

$$U(\vec{u}_{EX}) - U(\vec{u}_{FE}) \approx \frac{k^2}{N^{2\beta}} \tag{8}$$

We have three unknowns: $U(\vec{u}_{EX})$, k and β. If we have three values of $U(\vec{u}_{FE})$ and N corresponding to three different values of p, then we have three equations for computing the unknowns. Let us denote these three values by U_p, U_{p-1}, U_{p-2} and N_p, N_{p-1}, N_{p-2} and $U(\vec{u}_{EX})$ by U. Then from (8) we have:

$$\frac{\log \dfrac{U - U_p}{U - U_{p-1}}}{\log \dfrac{U - U_{p-1}}{U - U_{p-2}}} \approx \frac{\log \dfrac{N_{p-1}}{N_p}}{\log \dfrac{N_{p-2}}{N_{p-1}}} \tag{9}$$

Denoting the right hand side of (9) by Q, we have:

$$\frac{U - U_p}{U - U_{p-1}} \approx \left(\frac{U - U_{p-1}}{U - U_{p-2}}\right)^{Q} \tag{10}$$

To obtain an estimate of the exact strain energy U, we need to solve (10). The solution is expected in the neighborhood of U_p. Because convergence of the strain

energy is monotonic, we know that $U \geq U_p$ when $U_p > U_{p-1}$. Conversely, $U \leq U_p$ when $U_p < U_{p-1}$. Eq.(10) would not be different if $U(\vec{u}_{EX}) < U(\vec{u}_{FE})$; therefore, the restriction that $U(\vec{u}_{EX}) > U(\vec{u}_{FE})$ is not essential. Computational experience has shown this estimate to be reliable and generally accurate, with the accuracy of the estimate increasing with the accuracy of U_p.

2.2. Example.

The following test problem is representative of plate and shell intersections and reentrant corner problems in general. An L-shaped plane elastic body of thickness t is loaded by tractions. The tractions are computed from a stress field which satisfies the equilibrium and compatibility equations and the stress free conditions along the reentrant edges. Specifically, the stress field corresponds to the first (symmetric or 'Mode 1') term of the asymptotic expansion of $\vec{u}_{EX}$ about the reentrant corner. (See, for example, [8].) Therefore the exact solution is known. Specifically, the components of $\vec{u}_{EX}$ in the coordinate system shown in Fig. 1 are:

$$u_x = \frac{A}{2G} r^{\lambda} \left[(\kappa - Q(\lambda+1)) \cos\lambda\theta - \lambda \cos(\lambda-2)\theta \right] \tag{11a}$$

$$u_y = \frac{A}{2G} r^{\lambda} \left[(\kappa + Q(\lambda+1)) \sin\lambda\theta + \lambda \sin(\lambda-2)\theta \right] \tag{11b}$$

where A is a generalized stress intensity factor; $\lambda = 0.544483737$; $Q = 0.543075579$; G is the modulus of rigidity and κ depends on Poisson's ratio ν only. For plane strain: $\kappa = 3 - 4\nu$. We assume plane strain conditions and $\nu = 0.3$, therefore in this case $\kappa = 1.8$.

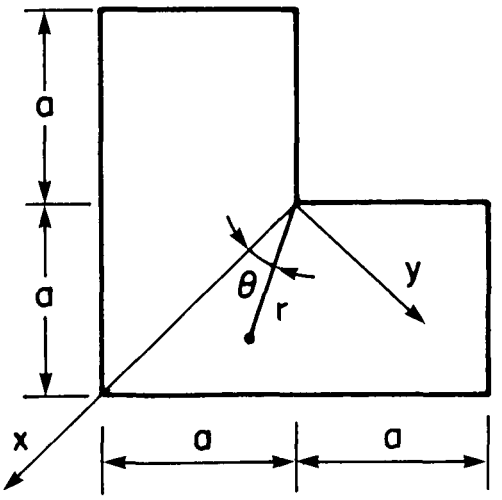

Fig. 1. L-shaped plane elastic body.

The stress tensor components are:

$$\sigma_x = A\lambda r^{\lambda-1} \left[(2 - Q(\lambda+1)) \cos(\lambda-1)\theta - (\lambda-1)\cos(\lambda-3)\theta \right] \tag{12a}$$

$$\sigma_y = A\lambda r^{\lambda-1} \left[(2 + Q(\lambda+1)) \cos(\lambda-1)\theta + (\lambda-1)\cos(\lambda-3)\theta \right] \tag{12b}$$

$$\tau_{xy} = A\lambda r^{\lambda-1} \left[(\lambda-1)\sin(\lambda-3)\theta + Q(\lambda+1)\sin(\lambda-1)\theta \right]. \tag{12c}$$

Because we know the exact displacement and stress fields we can compute the strain energy of the exact solution:

$$U(\vec{u}_{EX}) = 4.15454423\frac{A^2 a^{2\lambda} t}{E} \tag{13}$$

where E is the modulus of elasticity. The relative error in energy norm is defined as follows:

$$(e_r)_E = \sqrt{\frac{|U(\vec{u}_{EX}) - U(\vec{u}_{FE})|}{U(\vec{u}_{EX})}} \tag{14}$$

Using the mesh shown in Fig. 2 finite-element solutions were obtained for p=1 to 8. The computations were performed by a new computer program, called *PROBE* [9]. The number of degrees of freedom, the computed strain energy, the estimated and true relative errors in energy norm, computed from eq. (14), are shown in Table 1.

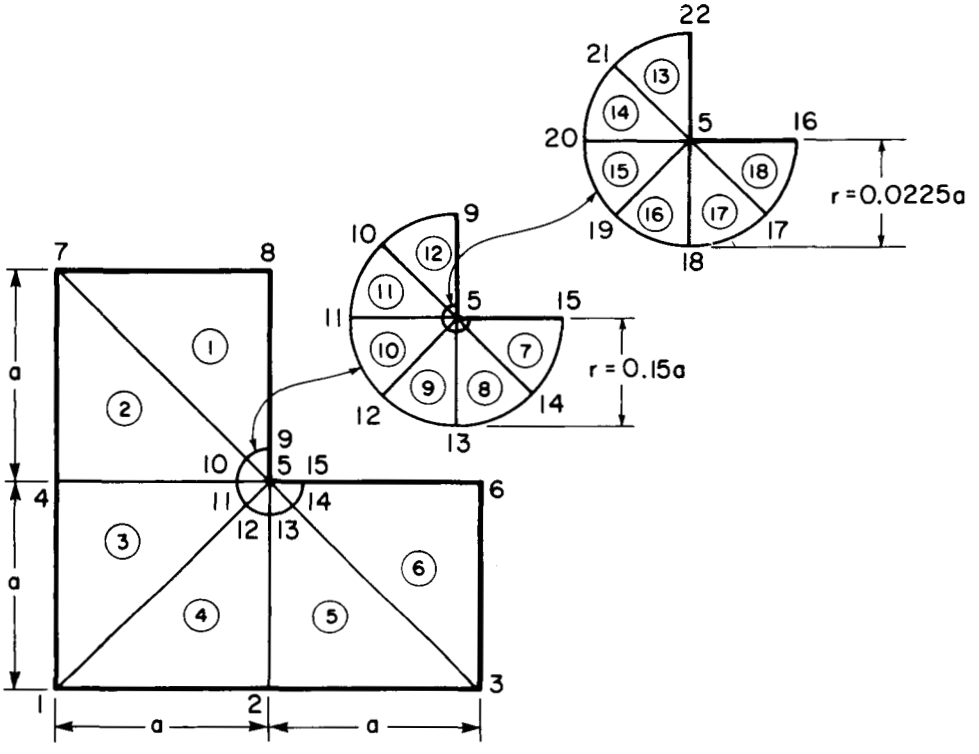

Fig. 2. Mesh design.

The results presented in Table 1 are typical of the quality of the error estimate we can obtain by means of the procedure described in Section 2.1. When the mesh is strongly graded toward the point of singularity then the convergence path (the $\log(e_r)_E$ versus $\log N$ curve) looks like an inverted S [3,4,5,10]. For low N values the rate of convergence is nearly exponential and the downward slope increases with N. In this segment the estimated error is conservative. Near the inflection

Table 1. Estimated and true relative error in energy norm.

p	N	$\frac{U(\vec{u}_{FE})E}{A^2a^{2\lambda}t}$	2β	$Est.'d$ $(e_r)_E$	$True$ $(e_r)_E$
1	41	3.8860880	–	–	25.42
2	119	4.1248326	–	–	8.46
3	209	4.1481150	1.93	5.34	3.93
4	335	4.1526504	2.76	2.02	2.14
5	497	4.1536354	3.05	1.01	1.48
6	695	4.1539746	2.45	0.80	1.17
7	929	4.1541390	1.83	0.75	0.99
8	1199	4.1542378	1.39	0.75	0.86
∞	∞	4.1545442	1.09	–	0

point, (i.e. where the curvature of the convergence path changes from negative to positive) the estimate is the least accurate and not conservative, nevertheless as we see in this example, it remains close. The estimate then becomes progressively more accurate as the asymptotic range of the p-extension is entered. In this case the correct asymptotic rate of convergence is $\beta = \lambda = 0.5445$. At $p = 8$ the computed value of β is approximately 0.7 with β decreasing.

2.3. Equilibrium tests.

Smallness of error in energy norm is a necessary but not sufficient condition for ensuring that the overall quality of the finite-element solution is good. It is possible to produce examples where the estimated error in energy norm is small (under 1 percent) yet the error in overall equilibrium is large (well over 10 percent).

Although we do not know $\vec{u}_{EX}$ we know that $\vec{u}_{EX}$ satisfies the equations of equilibrium and the law of action and reaction. We can, therefore, assess the quality of the finite-element solution by examining to what degree $\vec{u}_{FE}$ satisfies equilibrium and the law of action and reaction. Specifically, we can perform: (1) overall equilibrium tests; (2) element by element equilibrium tests and (3) action-reaction tests.

In the *overall equilibrium test* we 'cut' the structure from its supports and integrate the tractions, computed from the $\vec{u}_{FE}$, to obtain the reactions. In this way a free body diagram is produced. The error in equilibrium must be small in relation to the magnitude of applied forces. For example, we can define:

$$F = \sqrt{\sum_{i=1}^{N} r_i^2} \tag{15}$$

where r_i are the (global) load vector components. The term F is a suitable measure of the magnitude of the applied load.

In the *element-by-element equilibrium test* individual elements (or any group of elements) are separated from the model and tested for equilibrium. Specifically, we denote the element domain by Ω_e and its boundary by $\partial\Omega_e$. We compute:

$$q_i^{(e)} = \int_{\Omega_e} (\sigma_{ij,j} + X_i)\, t\, dx_1 dx_2 + \int_{\partial\Omega_e} \sigma_{ij} n_j\, t\, ds \quad (i,j = 1,2) \tag{16}$$

where σ_{ij} are the stress components computed from the finite element solution; X_i represents the applied body force components and n_j represents the unit normal to $\partial\Omega_e$. The summation convention is used. We should have $|q_i^{(e)}| \equiv \sqrt{q_i^{(e)} q_i^{(e)}}$ as well as the absolute values of each component of both expressions on the right hand side of (16) small in relation to the magnitude of the applied loads. If $|q_i^{(e)}|$ is small but the absolute values of the integral expressions are not small then there is a local error but, according to Saint-Venant's principle, the effect of the local error will not be substantial at some distance from the element in question. If $|q_i^{(e)}|$ is large, even after p-extension was performed, then element e, and possibly its neighbors, should be subdivided. Thus the element by element equilibrium test provides information about the *quality of mesh design*. In many cases minor local refinement (for example, dividing one element into two elements) can have a highly beneficial effect on the overall quality of approximation when p-extension is used.

In the *action-reaction test* we compute the stress resultants along interelement boundaries and external element boundaries where tractions are applied. Along interelement boundaries the stress resultants computed for neighboring elements should have nearly the same absolute value and opposite sense. Along external boundaries the resultants of the applied tractions and the tractions computed from the finite element solution should be nearly the same.

Examples of equilibrium tests are presented in [11].

3. LOCAL QUALITY

Having ascertained that the overall solution quality is acceptable, we are ready to compute the quantities which are of principal interest, i.e. $\Psi_i(\vec{u}_{FE})$. Smallness of error in energy and equilibrium does not guarantee that all functionals computed from $\vec{u}_{FE}$ are accurate. It is advisable to perform convergence tests on at least the more difficult functionals. We demonstrate the procedure by computing the direction and magnitude of the principal stresses at a point close to the reentrant corner. We selected the point $r = 0.025a$; $\theta = 30°$ in the coordinate system shown in Fig. 1. The stress components computed from the exact solution are:

$$\sigma_x = 3.67198Aa^{\lambda-1} \quad \sigma_y = 7.68696Aa^{\lambda-1} \quad \tau_{xy} = 0.698375Aa^{\lambda-1} \tag{17}$$

Therefore the principal stresses σ_1 and σ_2 and the direction of the first principal stress σ_1 from the positive x-axis, denoted by θ_1, are:

$$\sigma_1 = 7.804Aa^{\lambda-1} \quad \sigma_2 = 3.554Aa^{\lambda-1} \quad \theta_1 = 80.4° \tag{18}$$

In general functionals, other than the strain energy, do not converge monotonically, nevertheless the fact that convergence has occurred should be obvious. Here σ_1 and σ_2 happen to converge monotonically but θ_1 does not. We see that the state of stress is known with sufficient accuracy for engineering purposes at p=4 (335 degrees of freedom, see Table 1). Extension beyond p=4 merely confirms that convergence has occurred to within the range of precision normally expected in engineering computations and thereby establishes reliability of the data.

This test problem demonstrates that accurate stress data can be obtained in the very close proximity of stress singularities. Other examples and additional discussion of this point are presented in [10,12].

Table 2. Principal stresses at $r = 0.025a$; $\theta = 30°$.

p	$\frac{\sigma_1}{Aa^{\lambda-1}}$	$\frac{\sigma_2}{Aa^{\lambda-1}}$	θ_1 (degrees)
1	7.096	2.854	92.1
2	7.441	3.047	84.1
3	7.532	3.294	81.6
4	7.731	3.483	80.9
5	7.754	3.525	80.6
6	7.773	3.545	80.5
7	7.786	3.551	80.5
8	7.791	3.553	80.4
∞	7.804	3.554	80.4

It is possible also to compute various functionals from $\vec{u}_{FE}$ using advanced methods of extraction [13,14,15]. For example, we may wish to determine the value of the generalized stress intensity factor A [see eq.'s (11a,b), (12a,b,c)]. Such procedures based on [13,14,15] have been implemented in *PROBE* [9].

4. CONCLUSIONS AND RECOMMENDATIONS

(1) Extensions are essential for both the estimation and control of error in finite-element computations.

(2) We are in a much better position today than we were, even just one year ago, from the point of view of understanding how an advanced finite element software system should be designed so that (a) the solution is obtained at very nearly the theoretically optimal efficiency and (b) the user is provided with the capability to estimate and control the quality of engineering data computed from the finite-element solution at a small marginal cost. This is because now we understand the interplay between mesh design and the polynomial degree of elements.

(3) P-extension, coupled with properly graded meshes, is the most efficient method for controlling error in finite-element computations.

(4) The proper mesh design is such that points of singularity (and areas where the solution changes rapidly over short distances) are isolated by one or more layers of small elements, with the elements graded in geometric progression toward the points of singularity. In this way both the global and local behavior of the solution can be represented without compromising the accuracy of either.

(5) Implementation of advanced extraction methods for the the computation of certain engineering data, such as stress intensity factors, will further increase the efficiency and reliability of computations.

(6) The p-version is well suited for implementation on parallel processors because the data are organized in relatively few, large units. This logical organization reduces the overhead associated with parallel processing.

(7) The substantial increase of efficiency in finite-element computations through the use of h-p extension and the availability of parallel and vector processing technology make it possible and desirable to model plate and shell problems

using hierarchic sequences of plate and shell theories in conjunction with fully three-dimensional representation. The various plate and shell theories are nothing more than specializations of the three-dimensional theory of elasticity through restrictions imposed on the variation of the displacement field in the direction of the normal. Such restrictions generally do not hold near supports, stiffeners, cut-outs, plate and shell intersections, etc. which are the areas where cracking and delaminations originiate and therefore of the greatest concern to analysts and designers. These areas can be properly modeled by three-dimensional representation only. The use of hierarchic extensions toward higher order plate and shell theories will permit us to assess and control the quality of approximation in relation to three-dimensional theory.

(8) Although linear theory is properly the first and most generally used approach to structural modeling, it should be possible to ascertain by a posteriori analysis whether engineering conclusions drawn from a numerical model would be different if geometric and material nonlinearities were considered. We can view linear theory as the simplest of a hierarchic system of theories. Much the same way as we estimate error by the use of extension processes within the framework of linear theory, we should be able to estimate error by extension within the hierarchic system of theories. This important area has not received much attention in the past. Because it bears on the reliability of computed data, and the engineering conclusions based on them, it deserves serious consideration.

(9) In some areas our ability to compute data is already greater than the material scientists' ability to tell us what data should be computed. For example, it is not fully understood what parameters govern crack initiation. The reason, at least in part, is that the conventional finite-element method tends to yield 'fuzzy' data in areas where stresses change substantially over short distances. Proper use of h-p extension, coupled with advanced extraction methods, permits us to compute any stress field parameter with arbitrary precision. This removes an uncertainty from the phenomenological characterization of material response to various stress fields. Of course, such characterization can be developed only through joint experimental-analytical investigations.

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