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Application of the quark-confining string to the ψ spectroscopy*

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We report the results of the nonrelativistic application of the quark-confining string model to the study of ψ spectroscopy. This string model is defined by a relativistic-invariant, gauge-invariant, and reparametrization-invariant action describing quarks interacting with color SU(3) gauge fields. The model has no gluonic degrees of freedom, but has instead string degrees of freedom. Quark masses and the quark-gluon coupling constant are the only parameters of the model. In the Schrödinger limit and in the absence of light quarks, the longitudinal modes of the quark-antiquark pair and the rotational modes of the string for a meson reduce to the charmonium model with a linear potential. String vibrations, which are absent in the charmonium model, provide additional levels. They start coming in at around 4.0 GeV; the density of states increases as an exponential function of the mass. The two lowest vibrational levels in the e^+e^- channel have energies at 4.0 GeV and 4.4 GeV. Relativistic corrections are estimated to be small for the low-lying states so that the Schrödinger approximation is justified. We consider this application to ψ spectroscopy as a test of the model.

I. INTRODUCTION

The discovery[1] of the ψ family of particles marks an important turning point in the development of particle physics. Recent development[2] strongly suggests the existence of the charmed quark proposed by Glashow and collaborators,[3] and the ψ particles are charm-anticharm quark bound states. The states of the ψ spectroscopy are very narrow; their masses are accurately measured and a lot of their properties are known. Their spectroscopy therefore provides an ideal testing ground for any comprehensive model of hadron dynamics.

The ψ states appear to be most naturally described as nonrelativistic bound states of a charmed quark and its antiquark, as first suggested by Appelquist and Politzer.[4] It is clear in this treatment that the dominant nonrelativistic potential between the quarks is linear. In the nonrelativistic approximation, the bound system may be described by the Schrödinger equation[5]

$$\left[2M-\frac{1}{M}\frac{\partial^2}{\partial r^2}+kr+\frac{l(l+1)}{Mr^2}\right][r\Phi(r)]=E[r\Phi(r)] \; , \tag{1.1}$$

where E is the mass of the state, M is the charmed-quark mass, k is the strength of the linear potential, and Φ is the radial part of the wave function for a state of orbital angular momentum l. This we refer to as the charmonium model.

Equation (1.1) describes a set of radial and orbital excitations of ψ states with no spin-orbit or hyperfine splittings. Such splittings presumably arise as relativistic corrections in some relativistic theory to which Eq. (1.1) is an approximation. This simple picture does not seem to account for the rich spectrum of states observed between 3.9 and 4.5 GeV in the e^+e^- channel; it predicts only one state (~4.15 GeV) in this region.

There are many other terms one may add to the above equation (e.g., additive constant, Coulomb, square well, etc.). However, their introduction requires the introduction of new parameters, in addition to the two, namely k and M, already present in Eq. (1.1). Furthermore, the contribution of any additional terms is probably of the same order of magnitude as the relativistic correction terms (e.g., the spin-orbit splitting). Hence the only way to decide if the above linear potential term needs modifications or not is to evaluate the relativistic corrections, among which are the fine and the hyperfine structures.

In the charmonium model, per se, we are unable to deduce relativistic corrections unambiguously, or to understand the origin of the linear potential. The only reliable way to investigate this problem is to write down a relativistic invariant model which in the nonrelativistic limit gives Eq. (1.1) for a quark-antiquark bound state. One can then pick up all the relativistic corrections and evaluate them perturbatively.

A more fundamental issue is the origin of the linear potential. To describe the potential in relativistic language, fields must be introduced. It is well known that gauge fields in one space and one time dimensions have a linearly rising Coulomb potential. However, we live in four-dimensional Minkowski space.

It has been suggested that the local field theory of quarks interacting with color Yang-Mills gauge fields, namely, quantum chromodynamics (QCD), may give stringlike solutions, with a linear potential in the nonrelativistic approximation. Mesons with heavy quarks then obey Eq. (1.1) in the leading approximation. Unfortunately, to demonstrate stringlike hadrons as solutions of QCD is extremely difficult.[6] Equally difficult is to calculate hadron properties from QCD in a reliable way. An easier task will be to start from a precisely defined, relativistic-invariant field-theoretic model where quark confinement is explicit, and where the linear potential arises straightforwardly. To such a model we address ourselves.

In this paper, we consider the dynamics of nonrelativistic heavy-quark bound states in the "quark-confining string" (QCS) model recently proposed by one of us (S.-H.T.).[7] This is a classically Lorentz-invariant, gauge-invariant, and reparametrization-invariant model of quarks and color SU(3) gauge fields interacting in a two-dimensional world sheet (i.e., string). The color dynamics of this model is analogous to that of two-dimensional QCD; in particular, all physical states are color singlets; there are no independent gauge-field degrees of freedom, and a linear potential arises naturally from the color Coulomb force. There are additional string degrees of freedom describing the embedding of the string in Minkowski space. The quarks are Dirac fields in Minkowski space.

In this work we investigate only the application of the QCS model to the ψ spectroscopy in the nonrelativistic limit.[8] We ignore all the light quarks and consider only the charmed quark. This we refer to as the charm string. We shall assume the charmed quark to be heavy and the quark-gluon coupling e small enough so that a Schrödinger approximation is valid for the charm-anticharm quark bound system. The relativistic effects are to be introduced as corrections. We discover that even in this nonrelativistic limit, there are new features of the string model that are absent in the charmonium model.

Pictorially, the ψ meson is composed of a quark and an antiquark linked by the appropriate color electric flux line (see Fig. 1). This electric flux defines the string. When the string is straight (note that straightness is meaningful only in the nonrelativistic limit), the electric field is essentially a constant. The string and the quarks can rotate as a unit. Ignoring its vibrational motion, we obtain from the QCS model the Schrödinger equation (1.1) where $k=2e^2/3$. However, the string can also vibrate. These vibrational modes provide new states beyond the charmonium picture. In the e^+e^- channel, the two lowest vibrational levels lie at 4.0 and 4.4 GeV.

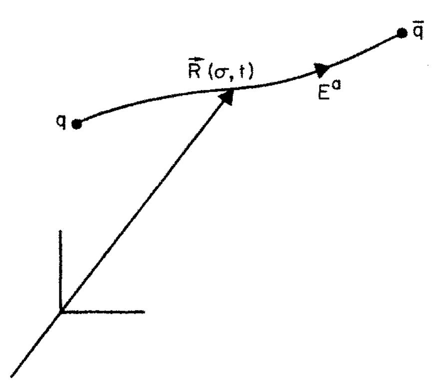

FIG. 1. The physical picture of a meson charm string.

As the QCS model is a relativistic model, all relativistic corrections are, in principle, determined. For the lowest P state, the $\Delta(E_{J=2}-E_{J=0})$ splitting is of the order of 0.14 GeV. This indicates that the Schrödinger treatment of the ψ spectroscopy is a valid approximation. However, a complete evaluation of the spin-orbit coupling, hyperfine splittings, and other relativistic effects is nontrivial and is beyond the scope of this paper. Throughout we emphasize the physics of the string picture instead of mathematical rigor. This work is organized as follows. The QCS model is reviewed in Sec. II. We also discuss our choice of gauge and parameters. In Sec. III and Sec. IV we consider the charm string in the absence of vibrational modes. In the nonrelativistic limit, we show how the quantization of angular momentum emerges from the rotation of the string. Then the charmonium equation (1.1) results. It is amusing to note that quantum mechanically, the ground state of the charm string has a spherically symmetric wave function. In Sec. V we study the lowest vibrational modes. We calculate the vibrational energy as a function of the distance between the quark and the antiquark. This is then treated as an effective potential $V_n(r)$ inserted into the bound-state equation. Besides a summary, Sec. VI also includes a discussion of the validity of the Schrödinger approximation, and some of the properties of the QCS model beyond those studied in this work.

II. THE QUARK-CONFINING STRING

A. Review and notations (Refs. 7, 9)

The string we consider is a $(1+1)$-dimensional world sheet, of infinite extent, embedded in $(3+1)$-dimensional Minkowski space. It may be described by the coordinate function $R^\mu(u^\alpha)$, where μ is a Minkowski index (0, 1, 2, 3) and $\{u^0, u^1\}$ gives an arbitrary coordinatization of the world sheet. The flat geometry of Minkowski space induces a Riemannian geometry in the internal coordinate space $\{u^\alpha,\ \alpha=0,1\}$. Let τ^μ_α be the tangent vector to the sheet in the u^α direction,

$$\tau^\mu_\alpha = \partial_\alpha R^\mu \equiv \frac{\partial R^\mu}{\partial u^\alpha} \equiv R^\mu|_\alpha \ . \tag{2.1}$$

Then we define

$$g_{\alpha\beta} = \tau_\alpha \cdot \tau_\beta \ ,$$
$$g = \det g_{\alpha\beta} \ . \tag{2.2}$$

$g_{\alpha\beta}$ is the induced metric in the internal coordinate space, and it will be used along with its inverse, $g^{\alpha\beta}$, to transform between covariant and contravariant tensors.

At each point on the sheet, we introduce two unit (spacelike) normal vectors n^μ_l ($l=1,2$), $n_1 \cdot n_2 = n_l \cdot \tau_\alpha = 0$. We have ($\eta_l{}^2 = -1$)

$$\eta^{\mu\nu} = \tau^\mu_\alpha \tau^{\alpha\nu} - \sum_l n^\mu_l n^\nu_l \ , \tag{2.3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $(1,-1,-1,-1)$. Also

$$\tau^\mu_\alpha|_\beta = \frac{\partial}{\partial u^\beta}\tau^\mu_\alpha = \{{}^{\gamma}_{\alpha\beta}\}\tau^\mu_\gamma + \sum_l h_{l,\alpha\beta} n^\mu_l \ , \tag{2.4a}$$

$$n^\mu_l|_\alpha = h_{l,\alpha\beta}\tau^{\beta\mu} + \epsilon_{lm} n_{m\mu}\nu_\alpha \ . \tag{2.4b}$$

$\{{}^{\gamma}_{\alpha\beta}\}$ is the usual Christoffel symbol,

$$\{{}^{\gamma}_{\alpha\beta}\} = \tfrac{1}{2} g^{\gamma\rho}(g_{\alpha\rho}|_\beta + g_{\beta\rho}|_\alpha - g_{\alpha\beta}|_\rho) \ ,$$

and $h_{l,\alpha\beta}$ is a symmetric tensor for each l, whose principal values are the inverse radii of curvature of the sheet in the n_l direction. The torsion $\nu_\alpha = +n_2|_\alpha \cdot n_1$ and $\epsilon_{12} = -\epsilon_{21} = 1$. We shall use a vertical bar to represent ordinary derivatives and a double verticle bar to represent covariant derivatives:

$$A^\alpha|_\beta = \frac{\partial A^\alpha}{\partial u^\beta} \ ,$$
$$A^\alpha\|_\beta = A^\alpha|_\beta + \{{}^{\alpha}_{\beta\gamma}\}A^\gamma \ .$$

We also have a covariant derivative for Dirac spinors,[10]

$$D_\alpha = \partial_\alpha - \frac{i}{2}\sigma^{\mu\nu}\tau^\beta{}_\mu \tau_{\alpha\nu}\|_\beta \ . \tag{2.5}$$

The QCS model is a model of interacting color quarks and gauge fields defined on the string. The system is defined by the action

$$S = \int_{-\infty}^{\infty} d^2u\sqrt{-g}\left\{\sum_j \bar\psi_j\left[\tau\!\!\!/^\alpha\left(\frac{i}{2}\overleftrightarrow{\partial}_\alpha - eB_{a\alpha}T^a\right) - M_j\right]\psi_j - \tfrac{1}{4}F_{a\alpha\beta}F_a{}^{\alpha\beta}\right\} . \tag{2.6}$$

$\psi_j(u)$, for each flavor j, is a four-component Dirac field which is also in the fundamental representation of the color gauge group SU(3); $\tau\!\!\!/^\alpha = \gamma_\mu \tau^{\alpha\mu}$. $d^2u\sqrt{-g}$ is the invariant volume element on the sheet. T^a, $a=1,\ldots,8$, are the matrix generators of SU(3) color,

$$[T^a, T^b] = if_{abc}T^c \ . \tag{2.7}$$

$B_{a\alpha}(u)$ ($\alpha=0,1$) are color gauge fields in the internal coordinate space,

$$F_{a\alpha\beta} = B_{a\beta}|_\alpha - B_{a\alpha}|_\beta + ef_{abc}B_{b\alpha}B_{c\beta} \ , \tag{2.8}$$

and may be written as $F_{a\alpha\beta} = E_a e_{\alpha\beta}$, where $e_{\alpha\beta}$ is the unit antisymmetric tensor in two dimensions ($e_{01} = \sqrt{-g}$) and E_a is the color electric flux (a scalar).

The action (2.6) is invariant under color gauge transformations:

$$T^a B_{a\alpha}(u) \to U(u)[T^a B_{a\alpha}(u)]U^{-1}(u) + \frac{i}{e}U|_\alpha(u)U^{-1}(u) \ , \tag{2.9a}$$

where $U(u)$ is any element of SU(3). The action (2.6) is also invariant under general coordinate transformations of the two-dimensional $\{\mu^\alpha\}$ space,

$$u^\alpha \to V^\alpha(u) \ ,$$
$$\psi(u) \to \psi(V) \ ,$$
$$R^\mu(u) \to R^\mu(V) \ , \tag{2.9b}$$
$$B_{a\alpha}(u) \to \frac{\partial V^\beta}{\partial u^\alpha}B_{a\beta}(V) \ .$$

Coordinate invariance is manifest once we note the identity

$$\tau\!\!\!/^\alpha\overleftrightarrow{\partial}_\alpha = \tau\!\!\!/^\alpha\overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha\tau\!\!\!/^\alpha = \tau\!\!\!/^\alpha\overrightarrow{D}_\alpha - \overleftarrow{D}_\alpha\tau\!\!\!/^\alpha \ .$$

The equations of motion of the theory follow directly from the variation of the action (2.6) with respect to ψ, B, and R^μ:

$$[\tau\!\!\!/^\alpha(iD_\alpha - eT^aB_{a\alpha}) - M_j]\psi_j = \left[i\tau\!\!\!/^\alpha\partial_\alpha + \frac{i}{2}\tau\!\!\!/^\alpha\|_\alpha - e\tau\!\!\!/^\alpha T^a B_{a\alpha} - M_j\right]\psi_j = 0 \ , \tag{2.10}$$

$$F_a^{\alpha\beta}\|_\alpha + ef_{abc}B_{b\alpha}F_c^{\beta\alpha} = e\sum_j \bar\psi_j\tau\!\!\!/^\beta T_a\psi_j \equiv ej_a^\beta \ , \tag{2.11}$$

$$0 = P^{\alpha\mu}\|_\alpha \ , \tag{2.12}$$

where $P^{\alpha\mu}$ is the covariant energy-momentum tensor

$$P^{\alpha\mu} = T^{\alpha\beta}\tau_\beta{}^\mu + \sum_l n^\mu_l V^\alpha_l \ , \tag{2.13a}$$

with

$$T^{\alpha\beta} = \tfrac{1}{2}E_a{}^2 g^{\alpha\beta} + \sum_j \bar\psi_j\tau\!\!\!/^\alpha\left(\frac{i}{2}\overleftrightarrow{\partial}^\beta - eB_a{}^\beta T^a\right)\psi_j \ , \tag{2.13b}$$

$$V_l^\alpha = \sum_j \bar{\psi}_j \not{n}_l \left(\frac{i}{2} \overleftrightarrow{\partial}^\alpha - eB_a{}^\alpha T^a\right) \psi_j \ . \qquad (2.13c)$$

Equations (2.10) and (2.11) are the natural generalizations of the corresponding flat-space field equations to a curved two-dimensional sheet. The differential operator

$$\not{\tau}^\alpha (D_\alpha + ieB_{a\alpha}T^a) = \not{\tau}^\alpha \partial_\alpha + \tfrac{1}{2}\not{\tau}^\alpha\|_\alpha + ieB_{a\alpha}\not{\tau}^\alpha T^a$$

is the covariant spinor derivative with respect to both the general coordinate group and SU(3) color. The Dirac matrices $\not{\tau}^0, \not{\tau}^1$ replaces the γ^0, γ^1 which would be present in flat two-dimensional Minkowski space. The term $\tau^{\alpha\mu}\|_\alpha$ is the mean curvature vector of the sheet. Its presence in Eq. (2.10) reflects the sensitivity of the spin of the quark field to the curvature of the string. A spinor forced to move along the curved string will precess.

The equation of motion of the gauge field (2.11) can be rewritten as

$$E_{a|\alpha} + ef_{abc}B_{b\alpha}E_c = -ee_{\alpha\beta}j_a^\beta \ .$$

This determines $B_{a\alpha}$ up to gauge transformations, in terms of the quark color charge density; as we expect, the (1+1)-dimensional gauge field has no independent dynamical degrees of freedom.

The string equation of motion is simply the local conservation of energy-momentum. We note that the tangential components in Minkowski space of the equation $P^{\alpha\mu}\|_\alpha = 0$ follow from the field equations (2.10) and (2.11). This is a trivial consequence of coordinate invariance, since variations of $R^\mu(u^\alpha)$ along the sheet are equivalent to coordinate transformations. The nontrivial string equations of motion are the two normal components of $P^{\alpha\mu}\|_\alpha = 0$:

$$h_{l\alpha\beta}T^{\alpha\beta} + V_l^\alpha\|_\alpha - \epsilon_{lm}\nu_\alpha V_m^\alpha = 0 \ . \qquad (2.14)$$

The left-hand side is the net force density on the sheet arising from the field stresses on it, and must vanish if the sheet is freely moving.

We remark, as discussed in Ref. 7, that the color electric field energy plays precisely the same role in the stress tensor $T^{\alpha\beta}$ as does the string constant $1/2\pi\alpha'$ in the conventional string. The term $\frac{1}{2}E_a{}^2 g^{\alpha\beta}$ provides a constant energy per unit length along the string.

We expect that this dynamical generation of the string constant will have profound effects on the spectrum of states in the theory. In a completely classical picture, with quarks in localized wave packets, we expect that noncolor-singlet states will have infinite energy, since $\frac{1}{2}E_a{}^2$ will be nonzero along the entire infinite string, while color-singlet states will have a nonzero string constant only *between* quark and antiquark (Fig. 1). The string can carry energy-momentum only in the region between $q\bar{q}$, and thus the quarks appear to be at the ends of the (physical) string.

For a baryon, each of the two ends of the physical string must also end with a quark.

B. Coordinate and gauge choices

The gauge invariances (2.9a) and (2.9b) of the theory reflect the presence of nonphysical, "gauge" degrees of freedom in the action (2.6). Indeed, all components of $B_{a\alpha}$, and the tangential components of the string motion are nonphysical. One may approach the quantization of the theory either by quantizing "covariantly" and showing that nonphysical degrees of freedom decouple, or by eliminating the redundant degrees of freedom from the outset at the expense of manifest gauge, coordinate, and Lorentz invariance. We shall take the latter approach.

We will not discuss a fully quantized theory here. Our strategy will be to remove all nonphysical degrees of freedom through gauge choices, with the focus of maximally simplifying the Dirac structure in the rest frame of a heavy $q\bar{q}$ system.

Our notation is such that $a, b, c, d, \ldots$ refer to the group index, $\mu, \nu, \lambda, \rho = 0, 1, 2, 3$ refer to the Minkowski space index, and $\alpha, \beta, \gamma, \delta = 0, 1$ refer to the string parameters. The $\mu = 0$ and $\alpha = 0$ components are timelike. The flavor index j shall be suppressed for convenience.

We shall choose coordinates ($u^0 = t$, $u^1 = \sigma$) by requiring

$$R_\mu(u^\alpha) = (t, \vec{R}(\sigma, t)) \ , \qquad (2.15)$$

$$g_{01} = \tau_0 \cdot \tau_1 = -\dot{\vec{R}} \cdot \vec{R}' = 0 \ . \qquad (2.16)$$

Hence the induced metric $g_{\alpha\beta}$ becomes

$$g_{\alpha\beta} = \begin{pmatrix} 1 - \dot{\vec{R}}^2 & 0 \\ 0 & -\vec{R}'^2 \end{pmatrix} = \begin{pmatrix} \Gamma^{-2} & 0 \\ 0 & -h^2 \end{pmatrix} \qquad (2.17)$$

and

$$g^{\alpha\beta} = \begin{pmatrix} \Gamma^2 & 0 \\ 0 & -h^{-2} \end{pmatrix} \ . \qquad (2.18)$$

In this coordinate system, the coordinate "time" u^0 has been chosen to agree with the time, R^0, in Minkowski space (in a particular Lorentz frame). σ has been chosen so that the instantaneous spacial velocity of the string at a point of fixed σ is normal to the string. The coordinate system is still not uniquely fixed as we still have invariance under $\sigma \to f(\sigma)$; σ will be specified more precisely in Sec. III. The choice of coordinates we have made manifestly breaks Lorentz invariance.

We now turn to the specification of an SU(3) gauge in the coordinate system $\{u^\alpha\}$ we have chosen. Let us choose the axial gauge

$$\vec{B}_1 = 0 \ . \tag{2.19}$$

The gluon equation (2.11) becomes ($\vec{j}^{\,\alpha} = \overline{\psi}\not{t}^\alpha \vec{T}\psi$)

$$\sqrt{-g}\ \vec{F}^{\alpha 0}\|_\alpha = (\sqrt{-g}\,\partial^1 \vec{B}_0)' = e\sqrt{-g}\ \vec{j}^{\,0} \ , \tag{2.20}$$

$$(\sqrt{-g}\,\vec{F}^{01})^{\cdot} = e\sqrt{-g}\ \vec{j}^{\,1} - e\vec{B}_0 \times \vec{F}^{10}\sqrt{-g} \ , \tag{2.21}$$

where we introduce the notation $\dot{f} = \partial_0 f = \partial_t f$, $f' = \partial_1 f = \partial_\sigma f$, and $(\vec{B}\times\vec{h})_a = f_{abc} B_b h_c$. Recall that $\partial_\alpha(\sqrt{-g} f^\alpha) = \sqrt{-g} f^\alpha\|_\alpha$. Equation (2.20) can be immediately integrated to give

$$\vec{B}_0 = e\int d\sigma' [-g(\sigma')]^{1/2} \vec{j}^{\,0} G(\sigma,\sigma') \ , \tag{2.22}$$

where the Green's function $G(\sigma,\sigma')$ obeys

$$-\partial_\sigma\{[-g(\sigma)]^{-1/2}\partial_\sigma G(\sigma,\sigma')\} = \delta(\sigma-\sigma') \ . \tag{2.23}$$

Using Eq. (2.20) and (2.21), we obtain the vector current that is conserved, $\partial_\alpha \vec{J}^{\,\alpha} = 0$,

$$\vec{J}^{\,0} = \sqrt{-g}\ \vec{j}^{\,0} \ , \tag{2.24}$$

$$\vec{J}^{\,1} = \sqrt{-g}\ \vec{j}^{\,1} + \tfrac{1}{2}\sqrt{-g}\{\vec{F}^{01}, \vec{B}_0\}_c \ , \tag{2.25}$$

where the symmetrization $\{\vec{F}^{01},\vec{B}\}_c = \vec{F}^{01}\times\vec{B} - \vec{B}\times\vec{F}^{01}$ is introduced in anticipation of the requirement that J^1 is Hermitian in the quantized theory.

C. Transformation of the Dirac field

In principle, the quantization of the QCS model can be realized using the Dirac constraint method.[11] This problem has not been solved. In our discussion we will focus on the quantum mechanics of the Dirac fields. Canonically, the momentum conjugate to the Dirac field is

$$\frac{\delta\mathcal{L}}{\delta\dot{\psi}} = \frac{i}{2}\sqrt{-g}\ \overline{\psi}\not{t}^0 \ , \tag{2.26}$$

which would (formally) lead to the anticommutation relation

$$\{\psi(\sigma,t), \sqrt{-g}\ \overline{\psi}\not{t}^0(\sigma',t)\}_t = \delta(\sigma-\sigma') \ . \tag{2.27}$$

The factor $\sqrt{-g}\not{t}^0$ involving the string degrees of freedom immensely complicates the analysis of the model in terms of ψ. Its origin is easy to understand: $\psi^\dagger\psi$ transforms as the zeroth component of a four-vector in Minkowski space, but as a scalar in the internal (σ,t) space. $\sqrt{-g}\overline{\psi}\not{t}^0\psi$, like $\delta(\sigma-\sigma')$, transforms as the zeroth component of a vector density in the internal space. This suggests the introduction of a new Dirac field χ, related to ψ by a local transformation $S(\sigma,t)$

$$\psi(\sigma,t) = S(\sigma,t)\chi(\sigma,t) \tag{2.28}$$

such that

$$\{\chi(\sigma,t), \chi^\dagger(\sigma',t)\} = \delta(\sigma-\sigma') \ . \tag{2.29}$$

In our coordinate system we can write

$$\tau_{0\mu} = \dot{R}_\mu = (1, v\hat{m}) \ , \tag{2.30a}$$

$$\tau_{1\mu} = R'_\mu = (0, h\hat{r}) \ , \tag{2.30b}$$

$$n_{1\mu} = \Gamma(v, \hat{m}) \ , \tag{2.30c}$$

$$n_{2\mu} = (0, \hat{r}\times\hat{m}) \ , \tag{2.30d}$$

where $\hat{r}$ and $\hat{m}$ are unit tangent and spatial normal vectors, respectively, $\hat{r}^2 = \hat{m}^2 = 1$, $\hat{m}\cdot\hat{r} = 0$, $\Gamma = (1-v^2)^{-1/2}$. We introduce the transformation $\psi(\sigma,t) = U(\sigma,t)\chi(\sigma,t)$, $U = SW$, where S is a local boost transformation and W is a local rotation transformation such that

$$U^{-1}\not{t}^0 U = \gamma^0\Gamma \ , \tag{2.31a}$$

$$U^{-1}\not{t}^1 U = \frac{1}{h}\gamma^1 \ , \tag{2.31b}$$

$$U^{-1}\not{n}_1 U = -\gamma^2 \ , \tag{2.31c}$$

$$U^{-1}\not{n}_2 U = -\gamma^3 \ . \tag{2.31d}$$

We find it convenient to rotate to the string frame (via W). It is straightforward (although rather lengthy) to show that the Lagrangian of the QCS model can be written in terms of χ as ($\gamma^0 \equiv \beta$)

$$\mathcal{L} = \chi^\dagger\left[i\partial_t + \frac{i\alpha_1}{(\Gamma h)^{1/2}}\partial_1\left(\frac{1}{(\Gamma h)^{1/2}}\right) - \frac{M\beta}{\Gamma} - eB_0^a T^a + \frac{\Gamma}{2}\sigma_1(\hat{r}\cdot\hat{m}\times\dot{\hat{m}}) + \frac{\gamma_5}{2h}(\hat{r}\cdot\hat{m}\times\hat{m}')\right]\chi + \frac{\Gamma}{2h}\vec{B}_0'^{\,2} \ . \tag{2.32}$$

The effective Lagrangian in this form is convenient for the nonrelativistic approximation. We note that the derivation of the form (2.32) does not require the explicit construction of U. We need only

$$2U^{-1}\dot{U} = -\frac{\alpha_1\Gamma v v'}{h} + \alpha_2\Gamma^2\dot{v} + \alpha_3\Gamma v(\hat{r}\cdot\hat{m}\times\dot{\hat{m}}) - \frac{\dot{h}}{h} - \frac{i\sigma_3\Gamma v'}{h} - i\sigma_1\Gamma(\hat{r}\cdot\hat{m}\times\dot{\hat{m}}) + \frac{i\sigma_2 v}{h}(\hat{r}\cdot\hat{m}\times\hat{m}') \tag{2.33}$$

and

$$2U^{-1}U' = \alpha_1\Gamma\dot{h} + \alpha_2\Gamma^2 v' + \alpha_3\Gamma v(\hat{r}\cdot\hat{m}\times\hat{m}') - \frac{h'}{h} - i\sigma_1\Gamma(\hat{r}\cdot\hat{m}\times\hat{m}') + i\sigma_2(\hat{r}\cdot\hat{m}\times\hat{r}') + i\sigma_3\Gamma\hat{r}\cdot\hat{m}' \ , \tag{2.34}$$

which follow from Eq. (2.31). The string equation (2.14) in this new frame is summarized in Appendix A. In this notation the Dirac and gluon equations (2.10) and (2.11) become

$$\left[i\partial_t + \frac{i\alpha_\perp}{\Gamma h}\partial_\sigma + \frac{i\alpha_\perp}{2}\left(\frac{1}{\Gamma h}\right)' - \frac{M\beta}{\Gamma} - eB_0 + \frac{\Gamma}{2}\sigma_1(\hat{r}\cdot\hat{m}\times\dot{\hat{m}}) + \frac{\gamma_5}{2h}(\hat{r}\cdot\hat{m}\times\hat{m}')\right]\chi = 0\ , \quad (2.35)$$

$$\left(\frac{\Gamma}{h}B_0^{a\prime}\right)' + e\chi^\dagger T^a\chi = 0\ . \quad (2.36)$$

The Dirac field χ obeys the anticommutation relation (2.29).

Physically, the transformation $U(\sigma, t)$ brings us from the space frame to the string frame. To derive the string equation from the effective Lagrangian density (2.32), a Lagrange multiplier term must be introduced for the constraint $\vec{R}\cdot\vec{R}'=0$. We shall use the string equation (2.9) given in Appendix A. This Lagrange multiplier term has no effect on the derivation of the Hamiltonian, or the Dirac and gluon equations, as is obvious.

III. THE NONRELATIVISTIC LIMIT

A. The nonrelativistic Hamiltonian

We consider the case where the quark mass M is large compared to the coupling constant e, which has the dimension of mass. In the bound-state problem, it is the condition [for color SU(N), $N=3$]

$$\frac{e^2}{2}\,\frac{N^2-1}{2N}\,\frac{1}{M^2} = \frac{2e^2}{3M^2} \ll 1 \quad (3.1)$$

that allows us to treat the system in the Schrödinger approximation. In this limit the relative quark motion is nonrelativistic. Throughout this work we consider only quarks with nonrelativistic velocities in the Lorentz frame we have chosen. This restricts our study to the low-energy states.

The motion of the string away from regions where the quark wave function is large may be relativistic or nonrelativistic. We can see this qualitatively from the structure of the energy-momentum tensor on the string. T^{00}, the "mass" density of the string, is of order $M\chi^\dagger\beta\chi$ where χ is large and goes to $E^2/2g^{00}\sim O(e^2)$ where χ is small. The "tension" of the string $(T^{11})\sim E^2/2g^{11} + (i/2)\chi^\dagger\alpha_1\overleftrightarrow{\partial}_1\chi$ is always of order e^2. When χ is small, the velocity of sound $(T^{11}/T^{00})^{1/2}$ is $O(1)$. Therefore, the string can rapidly adjust to changes in the position of the quarks and, we expect, will be *straight* in the lowest-energy states. String excitations ("vibrations") will, however, involve relativistic motion of the string in regions (between quarks) where $\chi^\dagger\chi$ is small. In this section, we consider completely nonrelativistic motion:

$$v = |\dot{\vec{R}}| \ll 1\ . \quad (3.2)$$

In the nonrelativistic approximation we may choose σ so that

$$h^2 = \vec{R}'^2 \simeq 1 \quad (3.3)$$

and

$$\frac{1}{\Gamma} \simeq 1 - \frac{v^2}{2} \quad (3.4)$$

so that both h and Γ can be replaced by unity except when multiplied to the mass term. The effective Lagrangian density (2.32) becomes

$$\mathcal{L} \simeq \chi^\dagger\left[i\partial_t + i\alpha_1\partial_\sigma - M\beta\left(1-\frac{v^2}{2}\right) - eB_0\right]\chi + \frac{\vec{B}_0'^2}{2}\ . \quad (3.5)$$

Equation (2.23) becomes, to leading order

$$\partial_\sigma{}^2 G(\sigma,\sigma') = -\delta(\sigma-\sigma')$$

or (3.6)

$$G(\sigma,\sigma') = -\tfrac{1}{2}|\sigma-\sigma'|$$

so that Eq. (2.22) also simplifies to $(J^a = J^{a0})$

$$B_0^a = e\int d\sigma' G(\sigma,\sigma')\chi^\dagger(\sigma')T^a\chi(\sigma') = e\int d\sigma' G(\sigma,\sigma')J^a(\sigma')\ . \quad (3.7)$$

Now we can derive the nonrelativistic Hamiltonian for the QCS model:

$$H = \int d\sigma\left(\frac{\delta}{\delta\dot{\chi}}\dot{\chi} + \frac{\delta}{\delta\dot{\vec{R}}}\cdot\dot{\vec{R}} - \mathcal{L}\right) = \int d\sigma\left[\chi^\dagger\left(\beta M - i\alpha_1\partial_\sigma + eB_0 + \frac{\beta M v^2}{2}\right)\chi - \frac{(B_0^{a\prime})^2}{2}\right]\ . \quad (3.8)$$

Since B_0^a is given by Eq. (3.7),

$$\int B_0^{a\prime\,2}d\sigma = e^2\int d\sigma' d\sigma'' J^a(\sigma')G(\sigma',\sigma'')J^a(\sigma'') + e^2\int d\sigma' d\sigma'' J^a(\sigma')J^a(\sigma'') \times \partial_\sigma G(\sigma,\sigma')G(\sigma,\sigma'')\big|_\sigma\ , \quad (3.9)$$

the Hamiltonian becomes

$$H = \int d\sigma\,\chi^\dagger\left(\beta M + \frac{\beta M v^2}{2} - i\alpha_1\partial_\sigma\right)\chi + \frac{e^2}{2}\int d\sigma d\sigma' J^a(\sigma)G(\sigma,\sigma')J^a(\sigma')\ , \quad (3.10)$$

where we have dropped the surface term in Eq. (3.10). Dropping this term is valid *only* if the net color charge vanishes. The surface term is

$$\lim_{\Lambda\to\infty} \frac{e^2}{4}\int d\sigma' d\sigma'' J^a(\sigma')J^a(\sigma'')[\epsilon(\sigma-\sigma')|\sigma-\sigma''|]\Big|_{\sigma=-\Lambda}^{\sigma=\Lambda}$$

$$\simeq \lim_{\Lambda\to\infty}\frac{e^2}{2}\Lambda\int d\sigma' J^a(\sigma')\int d\sigma'' J^a(\sigma'')$$

$$\simeq \lim_{\Lambda\to\infty}\frac{e^2\Lambda}{2} Q^a Q^a \; .$$

If $Q^a \neq 0$ for any a, this term contributes an infinite energy piece to the Hamiltonian (3.8). Hence only color-singlet states can have finite energies:

$$Q^a|\text{color singlet}\rangle = Q^a|S\rangle = 0 \; . \tag{3.11}$$

Using the commutator of χ, Eq. (2.29), we obtain

$$[Q^a, J^b(\sigma)] = if^{abc}J^c(\sigma) \; . \tag{3.12}$$

Equations (3.11) and (3.12) together give another amusing statement,

$$\langle S_1|J^c(\sigma,t)|S_2\rangle = 0 \; .$$

To argue that there is quark confinement, we must show that the quarks and/or the antiquarks in a color-singlet state can be pulled apart only at a cost of infinite energy. This is provided by the Coulomb (last) term of the Hamiltonian (3.10). As we shall see, the Coulomb potential between a quark and another quark or antiquark is proportional to the distance between them. This linear potential completes the quark confinement. Of course, as we pull a quark away from a color-singlet state, new quark-antiquark pairs can be created as the potential energy grows.

The Hamiltonian (3.10) is composed of three (the quark, the string, and the Coulomb) terms

$$\begin{aligned} H &= H_q + H_s + H_C \\ &= \int d\sigma\, \chi^\dagger(M\beta - i\alpha_1\partial_1)\chi + \int d\sigma\,\chi^\dagger\beta\chi\frac{Mv^2}{2} \\ &\quad + \frac{e^2}{2}\int d\sigma d\sigma' \chi^\dagger(\sigma)T^a\chi(\sigma)G(\sigma,\sigma')\chi^\dagger(\sigma')T^a\chi(\sigma') \; . \end{aligned} \tag{3.13}$$

The quark equation of motion follows from the Hamiltonian. This can be checked by taking the nonrelativistic limit of the quark equation of motion (2.10) and the gluon equation of motion (2.11). The nonrelativistic Hamiltonian has a clear physical interpretation. The first term describes the motion of a quark field along the string. The second term is the kinetic energy of the nonrelativistic string, which involves only the mass densities of the quarks. This immediately implies that the string is straight because, where $\chi^\dagger\beta\chi$ is zero, the string energy has no kinetic term. The last term is the remnant of quark-gluon interaction. The segment of the string between $\vec{R}(\sigma)$ and $\vec{R}(\sigma')$ (i.e., between quarks) has a nonzero color electric flux line. This contributes to the Hamiltonian an energy piece proportional to e^2.

In the Schrödinger approximation, the spin structures of the quarks play no role. Since it is important for us to retain the quark-antiquark structure, we must extract this from the first term of the Hamiltonian.

Let k and s be the momentum and spin labels where

$$\chi(\sigma,t) = \sum_{k,s}[U(k,s)b(k,s)e^{-iEt} + V(k,s)d^\dagger(k,s)e^{+iEt}] \; . \tag{3.14}$$

Choose U and V to be the nonrelativistic solutions of the wave equation

$$(-i\alpha_1\partial_\sigma + \beta M)W = EW \, , \tag{3.15}$$

where

$$W = \begin{cases} U & \text{for } E>0 \, , \\ V & \text{for } E<0 \, . \end{cases}$$

Solving the equation, we obtain

$$U = \begin{pmatrix} 1 \\ \dfrac{-i\sigma_1\partial_\sigma}{2M}\end{pmatrix} U(k,s), \quad E = M + \frac{k^2}{2M}$$

$$V = \begin{pmatrix} \dfrac{i\sigma_1\partial_\sigma}{2M} \\ 1\end{pmatrix} V(k,s), \quad E = -\left(M + \frac{k^2}{2M}\right)$$

where σ_1 is the first Pauli matrix. For a free quark field moving along a string, we have

$$\chi(\sigma,0) = \sum_{k,s}\begin{pmatrix} U(k,s)b(k,s) \\ V(k,s)d^\dagger(k,s)\end{pmatrix} + \text{corrections}$$

$$= \begin{pmatrix} b(\sigma) \\ d^\dagger(\sigma)\end{pmatrix} \; . \tag{3.16}$$

Using this solution for $\chi(\sigma,t)$ we obtain (tr = transpose)

$$\chi^\dagger\beta\chi = b^\dagger(\sigma)b(\sigma) + d^\dagger(\sigma)d(\sigma) \; , \tag{3.17}$$

$$\begin{aligned} J^a &= \chi^\dagger T^a\chi \\ &= b^\dagger(\sigma)T^a b(\sigma) - d^\dagger(\sigma)T^a_{\text{tr}}\, d(\sigma) \; , \end{aligned} \tag{3.18}$$

where normal ordering has been introduced; following from the anticommutator of χ, we have

$$\{b_i(\sigma), b_j^\dagger(\sigma')\}_t = \delta_{ij}\delta(\sigma-\sigma') \; , \tag{3.19}$$

$$\{d_i(\sigma), d_j^\dagger(\sigma')\}_t = \delta_{ij}\delta(\sigma-\sigma') \; . \tag{3.20}$$

All other anticommutators among the quarks and antiquarks vanish. The index i,j refers to the color ($i,j = 1,2,3$). The quark-gluon interaction term is approximated as an instantaneous Coulomb potential term.

Performing the Foldy-Wouthuysen transformation on Eq. (3.10) to remove the α_1 matrix, we obtain the final nonrelativistic Hamiltonian,

$$H=\int d\sigma\Big[b_i^\dagger(\sigma)\Big(M-\frac{1}{2M}\partial_\sigma{}^2\Big)b_i(\sigma)+d_i^\dagger(\sigma)\Big(M-\frac{1}{2M}\partial_\sigma{}^2\Big)d_i(\sigma)\Big]+\int d\sigma[b_i^\dagger(\sigma)b_i(\sigma)+d_i^\dagger(\sigma)d_i(\sigma)]\frac{Mv^2}{2}+\frac{e^2}{2}\int d\sigma d\sigma' J^a(\sigma)G(\sigma,\sigma')J^a(\sigma') , \qquad (3.21)$$

where $J^a(\sigma)$ is given by Eq. (3.18). The color index is summed in the Hamiltonian. $d(\sigma)$ and $b(\sigma)$ annihilate antiquarks and quarks, and $d^\dagger(\sigma)$ and $b^\dagger(\sigma)$ create them, respectively. Explicitly

$$b(\sigma)=\begin{pmatrix} b_1(\sigma)\\ b_2(\sigma)\\ b_3(\sigma)\end{pmatrix} . \qquad (3.22)$$

It is straightforward to extend to more than one flavor, each with mass M_j. The Hamiltonian (3.21) is valid provided $2e^2/3M_j{}^2 \ll 1$ for each flavor j. In the presence of flavor

$$J^a \to J^a=\sum_j J_j^a . \qquad (3.23)$$

This implies that a string with mixed flavors can also be an eigenstate of the Hamiltonian.

The Hamiltonian (3.21) describes both mesons and baryons. For a meson state with energy E_M

$$|M\rangle=\int d\sigma d\sigma'\phi(\sigma,\sigma')b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle ,$$
$$H|M\rangle=E_M|M\rangle . \qquad (3.24)$$

For a baryon state with energy E_B

$$|B\rangle=\int d\sigma d\sigma' d\sigma''\phi(\sigma,\sigma',\sigma'')b_1^\dagger(\sigma)b_2^\dagger(\sigma')b_3^\dagger(\sigma'')|0\rangle ,$$
$$H|B\rangle=E_B|B\rangle . \qquad (3.25)$$

Proper color symmetrization must be incorporated.

Let us turn our attention to the string motion. Physically the string can translate, rotate, and vibrate. Since segments of the string may move at relativistic velocities during vibrations, we can discuss only rotations and (Galilean) translations in the nonrelativistic limit. The string equation given in Appendix A is simplified considerably in the nonrelativistic limit where all spin effects drop out. In the leading-order approximation,

$$T^{00}h_{l,00}\simeq (M\chi^\dagger\chi)h_{l,00}\simeq 0 \qquad (3.26)$$

or

$$h_{1,00}=\Gamma\dot v\sim\dot v=0 , \qquad (3.27a)$$
$$h_{2,00}\simeq\ddot{\vec{R}}\cdot(\hat r\times\hat m)=0 . \qquad (3.27b)$$

A constant translational motion is clearly a solution. For rotation we consider

$$\vec R(\sigma,t)=\sigma\hat r(t)+\vec R_0 , \qquad (3.28)$$

then the string equation (3.27) becomes ($\hat r^2=1$)

$$\ddot{\hat r}\times\hat m=0 ,$$
$$\ddot{\hat r}\cdot(\hat r\times\hat m)=0 ,$$

where a constant rotation is a solution.

IV. THE CHARMONIUM PICTURE

In this paper we limit ourselves to investigate the meson state (3.24). Physically it is clear that the string can translate, rotate, and vibrate with the quark and the antiquark at the two ends. In this section we shall study the string in the absence of vibrational modes; in this case, the string is essentially straight. To show that the charmonium equation (1.1) emerges from the string picture, we proceed via three steps: First, we quantize the rotational modes, this gives simply the standard discrete angular momentum; then we obtain the Schrödinger equation for the probability amplitude along the string. Finally we derive the relation between the bound-state wave function in Minkowski space and this amplitude along the string.

A. The string modes

To separate the translation and the rotation, let

$$\vec R(\sigma,t)=\sigma\hat r(t)+\vec x(t) , \qquad (4.1)$$

where $\hat r(t)$ is a unit vector $\hat r^2=1$. This is the form of the solution from the string equation. The derivatives are

$$\dot{\vec R}(\sigma,t)=\sigma\dot{\hat r}(t)+\dot{\vec x}(t) , \qquad (4.2a)$$
$$\vec R'(\sigma,t)=\hat r(t) . \qquad (4.2b)$$

The constraint (2.20) becomes

$$\dot{\vec R}\cdot\vec R'=\frac{\sigma}{2}(\hat r^2)^\cdot+\dot{\vec x}\cdot\hat r=\dot{\vec x}\cdot\hat r=0 . \qquad (4.3)$$

Now, the string term in the Hamiltonian can be written as

$$H_S=\int d\sigma[n_b(\sigma)+n_d(\sigma)]\frac{M}{2}(\sigma^2\dot{\hat r}^2+2\sigma\dot{\hat r}\cdot\dot{\vec x}+\dot{\vec x}^2) , \qquad (4.4)$$

where

$$n_b(\sigma)=b_i^\dagger b_i(\sigma)\ ,$$

$$n_d(\sigma)=d_i^\dagger d_i(\sigma)\ .$$

Since $\dot{\hat{r}}(t)$ and $\dot{\vec{x}}(t)$ both are independent of σ, Eq. (4.4) can be rewritten as

$$H_S=\tfrac{1}{2}I\,\dot{\hat{r}}^2+Z\dot{\hat{r}}\cdot\dot{\vec{x}}+\tfrac{1}{2}\mu\dot{\vec{x}}^2\ , \tag{4.5}$$

where

$$I=\int d\sigma M\sigma^2 n(\sigma)\ , \tag{4.6}$$

$$Z=\int d\sigma M\sigma n(\sigma)\ , \tag{4.7}$$

$$\mu=\int d\sigma M n(\sigma)\ , \tag{4.8}$$

where

$$n(\sigma)=n_b(\sigma)+n_d(\sigma)\ . \tag{4.9}$$

I, Z, and μ are the moment of inertia, the first moment, and the mass operators, respectively. The Hamiltonian (3.25) now becomes, in the absence of vibrational modes,

$$H=\int d\sigma\left[b_i^\dagger\left(M-\frac{1}{2M}\partial_\sigma{}^2\right)b_i+d_i^\dagger\left(M-\frac{1}{2M}\partial_\sigma{}^2\right)d_i\right]+\frac{e^2}{2}\int d\sigma d\sigma' n_b^a(\sigma)|\sigma-\sigma'|n_d^a(\sigma')+\tfrac{1}{2}I\,\dot{\hat{r}}^2+Z\dot{\hat{r}}\cdot\dot{\vec{x}}+\tfrac{1}{2}\mu\dot{\vec{x}}^2\ , \tag{4.10}$$

where only the relevant (for meson) Coulomb term is kept, and

$$n_b^a(\sigma)=b^\dagger(\sigma)T^a b(\sigma)\ , \tag{4.11}$$

$$n_d^a(\sigma)=d^\dagger(\sigma)\,T_{tr}^a\,d(\sigma)\ . \tag{4.12}$$

In the effective Lagrangian (3.5), we write the v^2 term in terms of Eqs. (4.6)–(4.8). We derive the canonical momenta:

$$\vec{\Pi}=\frac{\delta\mathcal{L}}{\delta\dot{\vec{x}}}=\mu\dot{\vec{x}}+Z\dot{\hat{r}}\ , \tag{4.13}$$

$$\vec{p}=\frac{\delta\mathcal{L}}{\delta\dot{\hat{r}}}=Z\dot{\vec{x}}+I\dot{\hat{r}}\ . \tag{4.14}$$

Equation (4.5) can now be written in terms of the momenta,

$$H_S=\tfrac{1}{2}(\dot{\vec{x}},\dot{\hat{r}})W\begin{pmatrix}\dot{\vec{x}}\\ \dot{\hat{r}}\end{pmatrix}=\tfrac{1}{2}(\dot{\vec{x}},\dot{\hat{r}})\begin{pmatrix}\mu & Z\\ Z & I\end{pmatrix}\begin{pmatrix}\dot{\vec{x}}\\ \dot{\hat{r}}\end{pmatrix}=\tfrac{1}{2}(\vec{\Pi},\vec{p})W^{-1}\begin{pmatrix}\vec{\Pi}\\ \vec{p}\end{pmatrix}\ , \tag{4.15}$$

where W^{-1} is the inverse of the matrix operator W.

B. Quantization of the angular momentum

Following Eqs. (4.3), (4.13), and (4.14), we have the following constraints:

$$\hat{r}\cdot\vec{p}=0\ , \tag{4.16a}$$

$$\hat{r}\cdot\vec{\Pi}=0\ . \tag{4.16b}$$

As a consequence, the canonical commutation relations of $\hat{r}$ and $\vec{x}$ with their respective momenta $\vec{p}$ and $\vec{\Pi}$ are nontrivial. The commutators must be introduced in such a way that the constraints (4.16) remain valid.

Physically the momentum $\vec{\Pi}$ generates translation normal to r and the momentum $\vec{p}$ generates rotation. To satisfy Eq. (4.16a) it is convenient to introduce a new operator $\vec{L}$ such that

$$\vec{p}=-\hat{r}\times\vec{L}\ . \tag{4.17}$$

The commutator of $\vec{x}$ and $\vec{\Pi}$ is modified to satisfy Eq. (4.16b):

$$[x_i,\Pi_j]=i(\delta_{ij}-\hat{r}_i\hat{r}_j)\ . \tag{4.18}$$

The operator $\vec{L}$ obeys the commutators of angular momentum,

$$[L_i,\hat{r}_j]=i\epsilon_{ijk}\hat{r}_k\ , \tag{4.19}$$

$$[L_i,L_j]=i\epsilon_{ijk}L_k\ . \tag{4.20}$$

To shorten the discussion, let us write down a differential operator representation for $\vec{L}$ and $\vec{\Pi}$:

$$L_i=-i\epsilon_{ijk}\hat{r}_j\frac{\partial}{\partial\hat{r}_k}\ , \tag{4.21}$$

$$\Pi_i=-i(\delta_{ij}-\hat{r}_i\hat{r}_j)\frac{\partial}{\partial x_j}\ . \tag{4.22}$$

Commutators (4.18), (4.19), and (4.20) are satisfied, where $\vec{L}$ is the standard orbital angular momentum operator. All other commutators among $\vec{x}$, $\hat{r}$, $\vec{\Pi}$ and $\vec{L}$ vanish except

$$[L_i,\vec{\Pi}_j]\neq 0\ .$$

The constraint (4.16b) remains valid, since

$$[L_i,\hat{r}\cdot\vec{\Pi}]=0$$

and

$$[H_S,\hat{r}\cdot\vec{\Pi}]=0\ ,$$

where H_S can now be written in terms of $\vec{L}$ ($\vec{p}^2=\vec{L}^2$),

$$H_S=\frac{1}{2(I\mu-Z^2)}[\mu\vec{L}^2+I\vec{\Pi}^2+2Z\vec{\Pi}\cdot(\hat{r}\times\vec{L})]. \tag{4.23}$$

Since we are only interested in the bound-state problem, we can choose the frame where the translational momentum $\vec{\Pi}$ vanishes (see Appendix B for an alternative approach). The Hamiltonian for the meson bound state then becomes

$$H_M = \int d\sigma \left[b_i^\dagger\left(M - \frac{1}{2M}\,\partial_\sigma{}^2\right)b_i + d_i^\dagger\left(M - \frac{1}{2M}\,\partial_\sigma{}^2\right)d_i\right] + \frac{e^2}{2}\int d\sigma d\sigma' n_b^a(\sigma)|\sigma-\sigma'|n_d^a(\sigma) + \frac{\mu\, l(l+1)}{2(I\mu - Z^2)}\,, \tag{4.24}$$

where $l = 0, 1, 2, \ldots$ is the angular momentum eigenvalue. We have restricted ourselves to consider the case where the quark and the antiquark both are of the charm flavor. We are now ready to derive the charmonium equation (1.1).

C. The Schrödinger equation

The equation of motion for the bound-state wave function $\phi(\sigma, \sigma')$ where

$$H_M|M\rangle = H_M \int d\sigma d\sigma' \phi(\sigma,\sigma')b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle = E|M\rangle \tag{3.24'}$$

can now be evaluated using the anticommutators for b_i and d_i,

$$\begin{aligned}\int d\rho\, b_i^\dagger(\rho)\left(M - \frac{1}{2M}\,\partial_\rho{}^2\right)b_i(\rho)|M\rangle &= \int d\rho d\sigma d\sigma' \phi(\sigma,\sigma')b_i^\dagger(\rho)\\ &\quad\times\left(M - \frac{1}{2M}\,\bar\partial_\rho{}^2\right)b_i(\rho)b_j(\sigma)d_j^\dagger(\sigma')|0\rangle\\ &= \int d\sigma d\sigma' b_j^\dagger(\sigma)\left(M - \frac{1}{2M}\,\bar\partial_\sigma{}^2\right)\phi(\sigma,\sigma')d_j^\dagger(\sigma')|0\rangle\\ &= \int d\sigma d\sigma'\left[\left(M - \frac{1}{2M}\,\partial_\sigma{}^2\right)\phi(\sigma,\sigma')\right]b_j^\dagger(\sigma)d_j(\sigma')|0\rangle\,,\end{aligned} \tag{4.25}$$

and similarly for the antiquark term,

$$\frac{e^2}{2}\int d\rho d\rho'|\rho-\rho'|n_b^a(\rho)n_d^a(\rho')|M\rangle = \int d\sigma d\sigma'\left(\frac{e^2}{2}\,\frac{N^2-1}{2N}\,|\sigma-\sigma'|\phi(\sigma,\sigma')\right) \times b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle\,, \tag{4.26}$$

where $N=3$ for color SU(3).

To calculate the angular momentum term, we use Eqs. (4.6), (4.7), and (4.8)

$$[I, b^\dagger(\sigma)] = M\sigma^2 b^\dagger(\sigma)\,,$$

$$[Z, b^\dagger(\sigma)] = M\sigma b^\dagger(\sigma)\,,$$

$$[\mu, b^\dagger(\sigma)] = M b^\dagger(\sigma)\,,$$

and similarly for $d(\sigma)$. Therefore

$$\frac{\mu}{I\mu - Z^2}|M\rangle = \left(\mu\int_0^\infty dy\, \exp[-y(I\mu - Z^2)]\right)|M\rangle = \int d\sigma d\sigma'\,\frac{2M}{M^2(\sigma-\sigma')^2} \times\phi(\sigma,\sigma')b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle\,, \tag{4.27}$$

where

$$\begin{aligned}(I\mu - Z^2)|M\rangle &= \int d\sigma d\sigma'\phi(\sigma,\sigma')[2(\sigma^2+\sigma'^2)-(\sigma+\sigma')^2] \times M^2 b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle\\ &= \int d\sigma d\sigma'\phi(\sigma,\sigma')M^2(\sigma-\sigma')^2 \times b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle\end{aligned}$$

is used. Combining Eqs. (3.24), (4.25), (4.26), and (4.27) then gives

$$E|M\rangle = \int d\sigma d\sigma'\left[\left(2M - \frac{1}{2M}\,\frac{\partial^2}{\partial\sigma^2} - \frac{1}{2M}\,\frac{\partial^2}{\partial\sigma'^2} + \frac{2e^2}{3}|\sigma-\sigma'| + \frac{l(l+1)}{M(\sigma-\sigma')^2}\right)\phi(\sigma,\sigma')\right]b_i^\dagger(\sigma)d_i^\dagger(\sigma')|0\rangle\,. \tag{4.28}$$

Introducing the center-of-mass coordinate $z = \frac{1}{2}(\sigma+\sigma')$ and $r = \sigma - \sigma'$ we finally obtain the Schrödinger equation for the meson bound-state wave function $\phi(\sigma,\sigma')$

$$E\,\phi(r,z) = \left(2M - \frac{1}{M}\,\frac{\partial^2}{\partial r^2} - \frac{1}{4M}\,\frac{\partial^2}{\partial z^2} + \frac{2e^2}{3}\,|r| + \frac{l(l+1)}{Mr^2}\right)\phi(r,z)\,. \tag{4.29}$$

Earlier we set $\vec\Pi = 0$. From Eq. (4.16), it is clear that the longitudinal component of the quark translational momentum is still not fixed. We obtain zero total momentum by choosing $\phi(r,z) = \phi(r)$, or

$$-\frac{1}{4M}\,\frac{\partial^2}{\partial z^2}\,\phi(r,z) \to 0 \tag{4.30}$$

so that ϕ becomes a function of r only. Equation (4.30) completes our choice of the center-of-momentum frame.

D. The charmonium equation

To conclude that Eq. (4.29) is the charmonium Eq. (1.1), we need the identification

$$\phi(r) = f(r) \equiv r\Phi(r)\,, \tag{4.31}$$

where, in both cases, r measures the distance between the quark and the antiquark. In Eq. (1.1),

$r \geq 0$. In Eq. (4.29), $r = \sigma - \sigma'$ and $-\infty < r < \infty$. Hence, it appears that Eq. (4.29) allows solutions where

$$\phi(r)|_{r=0} \neq 0 \tag{4.32}$$

in addition to the solutions of Eq. (1.1), where

$$f(r)|_{r=0} = 0 \ . \tag{4.33}$$

[Equation (4.33) is required since the probability amplitude $\Phi(r) = f(r)/r$ must remain finite at $r = 0$.] Actually the identification is correct, and therefore the additional solutions (4.32) are not allowed.

First, we interpret $\phi(r)$ to be the probability amplitude along the string. To see this, we calculate the following correlation function of $|M\rangle$ along the string:

$$\langle M | n_b(\sigma) n_d(\sigma') | M\rangle = \phi^2(\sigma - \sigma') \ ,$$

where $|M\rangle$ is normalized,

$$\langle M | M\rangle = \int dr\, \phi^2(r) = 1 \ .$$

Next we want to calculate the probability distribution in the real physical space. Since the string is embedded in Minkowski space, it also can be calculated from $|M\rangle$. For $l \neq 0$ solutions in Eq. (4.29), $\phi(r=0)$ must vanish. Hence we can consider only the $S(l=0)$ wave case. The correlation function in Minkowski space is given by

$$\xi = \langle M | n_b(\vec{y}) n_d(\vec{z}) | M\rangle \ ,$$

where

$$n_{b,d}(\vec{y}) = \int \delta^3(\vec{y} - \vec{R}(\sigma))\, n_{b,d}(\sigma)\, d\sigma \ .$$

It is straightforward to obtain

$$\xi(\vec{y}, \vec{z}) \propto \frac{\phi^2(|\vec{y} - \vec{z}|)}{|\vec{y} - \vec{z}|^2} = \frac{\phi^2(r)}{r^2}$$

and $\phi(r)$ must vanish at $r = 0$ so that the correlation function $\xi(r)$ remains finite at $r = 0$. This completes our derivation of Eq. (1.1) from the quark-confining string, where $k = 2e^2/3$. The eigenfunctions for the angular momentum are the standard spherical harmonics $Y_{l,m}(\theta, \phi)$.

V. VIBRATIONAL MODES

In this section, we calculate the vibrational states of the string. The coupled string and the quark equations are very nonlinear so that there is no hope of solving them completely. We content ourselves with the following approximation scheme. First, we solve the string equation (via the Bohr-Sommerfeld method) to obtain the vibrational mode energies as functions of r, the distance between the quark and the antiquark. These are then inserted into the meson equation as an effective potential $V_{nl}(r)$ between the quark and the antiquark:

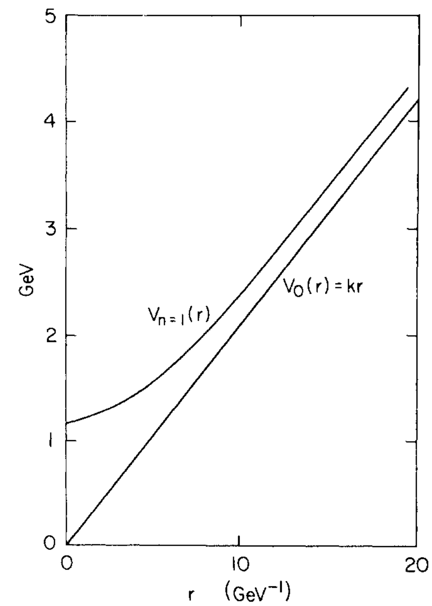

FIG. 2. The effective potential $V_n(r)$ for $n = 0, 1$.

$$Ef(r) = \left[2M - \frac{1}{M}\frac{\partial^2}{\partial r^2} + V_{nl}(r)\right] f(r) \ , \tag{5.1}$$

where n is the vibrational-mode quantum number. For $n = 0$, $V_{0l}(r)$ is simply the orbital-angular-momentum term plus the linear-potential term in Eq. (4.29). This approximation is valid if the vibrational energies are bigger than the quark longitudinal-mode energies. In the presence of both vibrational and rotational modes, $V_{nl}(r)$ is, in general, rather complicated. We shall assume

$$V_{nl}(r) = V_n(r) + \frac{l(l+1)}{Mr^2} \tag{5.2}$$

and keep in mind that due to the negligence of the vibration-rotation coupling, states with $n \neq 0$ and $l \neq 0$ in general have errors larger than that of the other states. In Sec. V A, we provide a naive guess of $V_n(r)$. In Sec. V B we estimate via Bohr-Sommerfeld quantization the form of $V_n(r)$ for the case of infinite quark mass. Since for $r > 2M/k$, the string is heavier than the quark masses, the above approximation is obviously inadequate. The finite-mass correction is introduced in Sec. V C. $V_n(r)$ for $n = 0, 1$ are plotted in Fig. 2. The two lowest $(l = 0)$ vibrational energies from various estimates are given in Table I for comparison. Their wave functions at the origin are also given in the form $|\Phi(0)|^2/E^2$ normalized to that of the ground state.

It is clear that the energy levels differ a lot less than the wave functions at the origin for different estimates, as is expected.

TABLE I. Various estimates of the first two vibrational levels ($n=1, l=0$) and their wave functions at the origin. The latter are expressed as $|\Phi(0)|^2/E_v^2$, where E_v is the mass of the level. For comparison, we include $\psi(3.1)$ and $\psi'(3.7)$ as well and the above expression is normalized to that of $\psi(3.1)$.

$V_n(r)$ The effective potential	E Mass in GeV	$\|\Phi(0)\|^2/E^2$ Normalized to that of $\psi(3.1)$
kr ($n=0$)	3.10	1
	3.68	0.71
$kr+\frac{\pi}{r}$ ($n=1$)	4.15	<0.01
	4.55	<0.01
$kr\left(1+\frac{2\pi}{kr^2}\right)^{1/2}$ ($n=1$)	3.82	0.23
	4.26	0.23
Eqs. (5.41b) and (5.41c) ($n=1$)	3.96	0.35
	4.41	0.32

A. The string motion

For massive quarks, the quark and the antiquark essentially sit at the two ends of the string. In the nonrelativistic limit (i.e., $2e^2/3M^2 \ll 1$), we can simplify the string equation (2.14) (Appendix A) by the following approximation:

$$T^{00} \simeq \frac{\Gamma^4 B_0'^2}{2h^2} + M\chi^\dagger \beta\chi , \tag{5.3}$$

$$T^{11} \simeq \frac{\Gamma^2 B_0'^2}{2h^4} , \quad T^{01} \simeq T^{10} \simeq V_i^\alpha \simeq 0 . \tag{5.4}$$

Physically, the ends of the QCS model are dominated by the quark-mass terms while the string itself is dominated by the color electric flux line where Γ and h are not necessarily close to unity. The quark effects along the string are neglected. Using Eqs. (3.6) and (3.7), we obtain

$$T^{00} \simeq kg^{00} + M\chi^\dagger\beta\chi , \quad T^{11} \simeq kg^{11} \tag{5.5}$$

so that the string equation is given by

$$(\sqrt{-g}\,T^{00}\dot{\vec{R}})^{\cdot} + (\sqrt{-g}\,T^{11}\vec{R}')' \simeq 0 . \tag{5.6}$$

Away from the ends of the string, we have

$$\partial_\alpha(\sqrt{-g}\;\tau_i^\alpha) = \partial_\alpha(\sqrt{-g}\,g^{\alpha\beta}\partial_\beta R_i) = 0 . \tag{5.7}$$

First let us make a very crude estimate of the vibrational-mode energies as a function of the distance r between the quark and the antiquark. This function is then inserted into the bound-state equation as an effective potential. This illustrates our approach and also indicates the areas where better estimates are required. Since the quark mass is heavy enough so that the ends of the string are moving at nonrelativistic speed, we assume that they are essentially fixed. Next, we assume (wrongly) that the string speed is also very slow and the amplitude of the vibration is small so that Eq. (5.7) reduces to

$$\ddot{\vec{R}} - \vec{R}'' = 0 . \tag{5.8}$$

Then the vibrational energy is given by

$$V_n^c(r) - kr = \frac{n\pi}{r} , \tag{5.9}$$

where $n=0,1,2,\ldots$. The superscript c is to remind us that this is a very crude estimate. Adding this term to the meson equation (4.29) gives ($k=2e^2/3$)

$$(E-2M)f(r) = \left[-\frac{1}{M}\frac{\partial^2}{\partial r^2} + kr + \frac{n\pi}{r} + \frac{l(l+1)}{Mr^2}\right]f(r) , \tag{5.10}$$

where the vibrational energy acts like a repulsive (three-space dimensional) Coulomb potential. The two lowest eigenvalues $E(N,n,l=0)$ are given in Table I, as are their wave functions at the origin. This is clearly an overestimate since $V_n^c(r)$ increases without bound as $r\to 0$. Physically $V_n(r)$ is expected to remain finite and smooth as $r\to 0$. The approximation (5.8) breaks down as $r\to 0$ since $\dot{\vec{R}}$ and $\vec{R}'$ are no longer small.

In the remainder of this section, we treat Eq. (5.6) semiclassically for the vibrational modes. First we assume quark masses to be infinitely heavy so that the ends of the string are fixed. Next we improve this approximation by introducing the finite-quark-mass effect. A numerical treatment of Eq. (5.6) gives essentially the same results.

B. Bohr-Sommerfeld method

Let us assume the quark mass to be very heavy so that the ends of the string are fixed:

$$R_1^\mu(t)=\left(t,0,0,\frac{r}{2}\right)\ ,\tag{5.11a}$$

$$R_2^\mu(t)=\left(t,0,0,-\frac{r}{2}\right)\ .\tag{5.11b}$$

The equation of motion (5.7)

$$\partial_\alpha(\sqrt{-g}\,\tau_\mu^\alpha)=0$$

has the general solution (parameters u^0, u^1)

$$R^\mu(u^0,u^1)=S^\mu(u^0)+Q^\mu(u^1)\tag{5.12}$$

in the light-cone coordinates

$$g_{00}=s^2=(\partial_u S)^2=0\ ,\tag{5.13}$$

$$g_{11}=q^2=(\partial_u Q)^2=0\ .\tag{5.14}$$

Let us choose $S^0=u^0$ and $Q^0=u^1$ so that

$$R_\mu=(u^0+u^1,\vec{S}+\vec{Q})\ .\tag{5.15}$$

Equations (5.13) and (5.14) become

$$\vec{s}^{\,2}=\vec{q}^{\,2}=1\tag{5.16}$$

and $t=u^0+u^1$. Let $\sigma=(u^0-u^1)/2$ and $\sigma_\pm(t)$ be the ends of the string. Equation (5.11) becomes ($\perp\to 1,2$)

$$S_\perp\left(\frac{t}{2}+\sigma_\pm(t)\right)+Q_\perp\left(\frac{t}{2}-\sigma_\pm(t)\right)=0\ ,\tag{5.17}$$

$$S_3\left(\frac{t}{2}+\sigma_\pm(t)\right)+Q_3\left(\frac{t}{2}-\sigma_\pm(t)\right)=\pm\frac{r}{2}\ .\tag{5.18}$$

From Eqs. (5.17) and (5.18) and their time derivatives, it is clear that $\sigma_\pm$ are constants. From reflection symmetry, we can write

$$\sigma_+=-\sigma_-=\frac{a}{2}\ ,\tag{5.19}$$

where the constant a is to be determined. $\vec{S}$ and $\vec{Q}$ have the following periodicity properties:

$$S_\perp(u+2a)=S_\perp(u)=-Q_\perp(u-a)\ ,\tag{5.20}$$

$$S_3(u+2a)-S_3(u)=r\ ,\tag{5.21}$$

$$Q_3(u)=\frac{r}{2}-S_3(u+a)\ .\tag{5.22}$$

Introducing a new function $F_\perp(\theta)$ with period 2π, $F_\perp(\pi u/a)=(\pi/a)S_\perp(u)$, we have, using Eq. (5.16)

$$S_3(u)=\frac{a}{\pi}\int_0^{\pi u/a}d\theta[1-F_\perp'(\theta)^2]^{1/2}\ ,\tag{5.23}$$

where the prime denotes derivative. Combining Eq. (5.21) and (5.23) gives

$$\frac{r}{2a}=\frac{1}{2\pi}\int_0^{2\pi}d\theta[1-F_\perp'(\theta)^2]^{1/2}\ .\tag{5.24}$$

The energy can be computed in terms of a easily,

$$\begin{aligned}U&=\int_{-a/2}^{a/2}d\rho\sqrt{-g}\,k\tau_0^0\\&=k\int_{-a/2}^{a/2}d\rho\sqrt{-g}\,(g^{00}\tau_{0,0}+g^{01}\tau_{1,0})\\&=2ka\ ,\end{aligned}\tag{5.25}$$

where

$$R_\mu=\left(t,\vec{S}\left(\frac{t}{2}+\sigma\right)+\vec{Q}\left(\frac{t}{2}-\sigma\right)\right)$$

and

$$\sqrt{-g}=1-\vec{S}'\cdot\vec{Q}'\ .$$

From Eqs. (5.16), (5.20), (5.22), and (5.23), we can write Q_3 as

$$Q_3(u)=\frac{r}{2}-\frac{a}{\pi}\int_0^{\pi+\pi u/a}d\theta[1-F_\perp'(\theta)^2]^{1/2}\ .\tag{5.26}$$

We also note that the largest period T of $R_\perp$ (hence lowest frequency) is $4a$.

We now have all solutions to the classical equations of motion (5.7) subject to the boundary conditions (5.11) parametrized in terms of the arbitrary function $F_\perp(\theta)$. Unfortunately, this does not extend naturally to a parametrization of all of phase space so that we cannot quantize the system simply in terms of these modes. We content ourselves with a Bohr-Sommerfeld estimate of the energy levels of one of the simplest possible modes—that is, we choose a form for $F_\perp(\theta)$ that depends on *one* parameter and then quantize the values of that parameter via the Bohr-Sommerfeld method.

Now we are ready to introduce Bohr-Sommerfeld quantization

$$\begin{aligned}2n\pi&=\oint_{\text{orbit}}(H+L)dt\\&=UT+S\\&=2ak(4a)-k\int_{-2a}^{2a}dt\int_{-a/2}^{a/2}d\rho\sqrt{-g}\ ,\end{aligned}\tag{5.27}$$

where the last term is the action. Defining

$$\theta_1=\pi t/2a\ ,$$

$$\theta_2=\pi\rho/a\ ,$$

$$\vec{F}'=(F_\perp',(1-F_\perp'^{\,2})^{1/2})\ .$$

Equation (5.27) becomes

$$2n\pi=4a^2k\left[1-\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\times\int_{-\pi/2}^{\pi/2}\frac{d\theta_2}{2\pi}\vec{F}'(\theta_1+\theta_2)\cdot\vec{F}'(\pi+\theta_1-\theta_2)\right]\ .\tag{5.28}$$

Consider the simplest example for $F_\perp$ [see Fig. 3(a)]

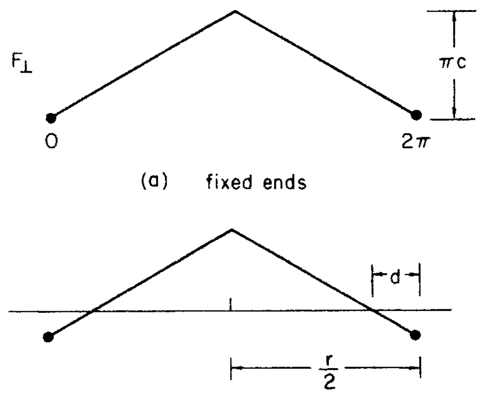

FIG. 3. The trial shape of the vibrating string for $n=1$. In (a) the ends of the string are fixed. In (b) the ends of the string (where the quark masses are) move in such a way that the string's center of mass is fixed.

$$\frac{F_\perp'(\theta)}{c}=\begin{cases}1, & 0<\theta<\pi\\ -1, & \pi<\theta<2\pi\,.\end{cases} \qquad (5.29)$$

For this oversimplified case, Eq. (5.24) can be solved immediately,

$$2a(1-c^2)^{1/2}=r\,, \qquad (5.30)$$

and Eq. (5.28) gives

$$2n\pi=k(2a)^2c^2\,. \qquad (5.31)$$

From Eqs. (5.30) and (5.31) we finally obtain the vibrational plus potential energy V_n for very large quark mass,

$$V_n(r)=2ka_n = kr\left(1+\frac{2n\pi}{kr^2}\right)^{1/2}. \qquad (5.32)$$

For large r, we have

$$V_n(r)\underset{r\to\infty}{\sim} kr+\frac{n\pi}{r}\,,$$

which is precisely the effective potential given in Eq. (5.9). For small r, we have (for $n\neq 0$)

$$V_n(r)\underset{r\sim 0}{\sim}(2kn\pi)^{1/2}\left(1+\frac{kr^2}{4n\pi}\right)\,,$$

which is finite, as one might expect. Using the potential (5.32) for Eqs. (5.1) and (5.2) we calculate the lowest two vibrational levels and their wave functions at the origin. These are shown in Table I.

C. Finite-mass correction

Since the quark mass is finite, the above approximation breaks down when kr becomes comparable to M. As we shall see, the charm string has $M/k\sim 5$ GeV^{-1}. Hence the finite-mass effect cannot be neglected. To include this, we simply extend the above approximation with a linear trial function used in Sec. V B. Let us consider the QCS model in the center-of-mass frame. With the same trial function, we have [see Fig. 3(b)]

$$k\int_0^r d\sigma|R_\perp(\sigma)|(1+c^2)^{1/2}+2M(-dc)=0\,. \qquad (5.33)$$

This can be solved for d,

$$d(r,c)=\frac{1}{4}\frac{kr^2(1+c^2)^{1/2}}{2M+kr(1+c^2)^{1/2}}\,. \qquad (5.34)$$

In the Bohr-Sommerfeld quantization formula (5.27), $2n\pi=UT+S$, T remains $4a$ while

$$U=\int_{-a/2-b}^{a/2+b}d\sigma(2k)=2k(a+2b) \qquad (5.35)$$

and

$$S=-k\int_{-2a}^{2a}dt\int_{-a/2-b}^{a/2+b}d\sigma(1-\vec{S}'\cdot\vec{Q}')\,, \qquad (5.36)$$

where b is the correction to a due to the finite-mass effect. To the leading order, we simply extend the relation (5.30) to (where r is replaced by ω)

$$2(a+2b)(1-c^2)^{1/2}=\omega+2d=r \qquad (5.37a)$$

or

$$2b(1-c^2)^{1/2}=d\,. \qquad (5.37b)$$

It is then a simple matter to obtain the following quantization formula

$$2n\pi=4kc^2[a(a+2b)-2b(a-2b)]\,. \qquad (5.38)$$

Eliminating a,b using Eq. (5.37), we obtain, using Eqs. (5.34) and (5.38),

$$V_n(r)=U=kr(1-c^2)^{-1/2} = kr\left\{1+\frac{2n\pi}{k[(r-2d)^2+4d^2]}\right\}^{1/2}, \qquad (5.39)$$

where d is given by Eq. (5.34) and c^2 by

$$c^2=\frac{2n\pi}{2n\pi+k[(r-2d)^2+4d^2]}\,. \qquad (5.40)$$

The coupled set of equations (5.34) and (5.40) determine c and d in terms of r so that $d=d(r)$.

Since $c^2\to 0$ as $r\to\infty$ and $c^2\to 1$ as $r\to 0$, d is relatively insensitive to c^2 and hence it is easy to estimate $d(r)$. We note that

$$V_n(r)\underset{r\to\infty}{\sim}kr+\frac{2n\pi}{r} \underset{r\to 0}{\sim}(2n\pi k)^{1/2}\left(1+\frac{kr}{2\sqrt{2}\,M}\right)\,.$$

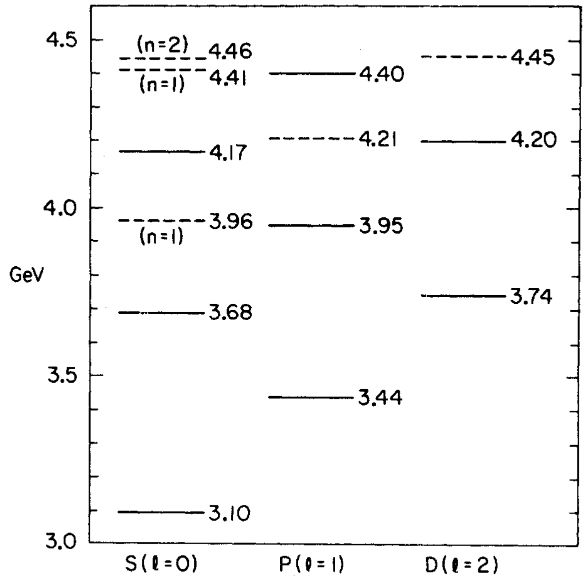

FIG. 4. The nonrelativistic spectroscopy of the charm string. $\psi(3.10)$ and $\psi(3.68)$ are fitted to obtain M =1.154 GeV and k =0.21 GeV^2. The dashed lines are the vibrational levels absent in the charmonium model. Levels with $E > 4.5$ GeV or $l > 2$ are not shown.

Putting together Eqs. (5.1), (5.2), (5.34), (5.39), and (5.40), we obtain the nonrelativistic charm-string equation:

$$\left[2M - \frac{1}{M}\frac{\partial^2}{\partial r^2} + V_n(r) + \frac{l(l+1)}{Mr^2}\right] f(r) = Ef(r) , \tag{5.41a}$$

where

$$\begin{aligned} V_n(r) &= kr\left\{1 + \frac{2n\pi}{k[r^2 - 4rd + 8d^2]}\right\}^{1/2} \\ &= kr(2 - \alpha_n{}^2)^{-1/2} \end{aligned} \tag{5.41b}$$

and

$$d(M, r, k, n) = \frac{kr^2\alpha_n}{4(2M + kr\alpha_n)} , \tag{5.41c}$$

where n is the vibrational quantum number. For $n=0$, $\alpha_n{}^2 = 1$ and V_n reduces to the linear potential in the charmonium model (1.1). d is introduced as a correction due to the finite quark mass. $d \to 0$ as $M \to \infty$. The effective potential $V_n(r)$ is plotted in Fig. 2 for $n = 0, 1$. The two lowest vibrational levels and their wave functions at the origin are given in Table I. The spectroscopy of Eq. (5.41) is shown in Fig. 4.

After the series of approximations we have adopted to derive Eq. (5.41) from the QCS model, the resulting picture resembles closely that of a string with a quark and an antiquark at its two ends.[12]

The two parameters of the charm string are the charmed-quark mass M and the color coupling in the form k. They are taken to be

$$M = 1.154 \text{ GeV},$$

$$k = 0.21 \text{ GeV}^2 .$$

We observe that the wave functions at the origin for the vibrational levels are in general smaller than those of the radial excitation levels. Unfortunately, our method is too crude for the determination of the wave functions (the energies of the levels are more reliable). Since physically the vibrational mode of the string between the quark and the antiquark tends to push them apart, we expect the quark-antiquark annihilation probability to be smaller for the vibrational levels.

VI. DISCUSSIONS AND REMARKS

In order to compare quantitatively the spectroscopy of the charm string with experiments, we must include the leading-order relativistic effects, possible S-D mixing, threshold effects (of charmed mesons and baryons in the e^+e^- channel), and vibration-rotation couplings. These investigations are beyond the scope of this paper. Instead we restrict ourselves to a simpler task: to check the validity of the nonrelativistic approximation employed in this work.

To be specific, let us consider some typical contributions to the spin-orbit splitting. Using the same string variable (4.1), we pick up from the Lagrangian terms that contribute to the Hamiltonian in the form $\vec{L}\cdot\vec{S}$,

$$\chi^\dagger \frac{\vec{\sigma}}{2}\chi\left[\dot{\vec{R}}\times\ddot{\vec{R}} + (\dot{\hat{r}}\times\hat{r})v^2 + (\dot{\vec{x}}\cdot\dot{\hat{r}})\dot{\vec{R}}\times\hat{r} - \frac{i}{4M}\dot{\hat{r}}\times\dot{\hat{r}}\right] - \frac{i}{2M}\chi^\dagger\frac{\vec{\sigma}}{2}\dot{\vec{R}}\times\dot{\hat{r}}\partial_1\chi , \tag{6.1}$$

where we have rotated back to the space axis. The first term involves the string acceleration $\ddot{\vec{R}}$ and is identified as a quark precession term. It is straightforward to evaluate their contributions (in leading-order approximation) to the bound-state Hamiltonian in Eq. (1.1),

$$H \to H + \frac{\vec{L}\cdot\vec{S}}{M^3r^3}\left[\frac{2l(l+1)-1}{r} - \partial_r\right] . \tag{6.2}$$

For the lowest P state, which has a mass $E = 3.45$ GeV, the total splitting due to Eq. (6.2) can be evaluated straightforwardly:

$$E_{J=2} - E_{J=0} \sim 0.14 \text{ GeV}. \tag{6.3}$$

This is small in comparison to the binding energy of the state, $E - 2M \sim 1.1$ GeV. Hence our Schrödinger approximation for the low-lying levels are justified *a posteriori*. Of course, the complete fine structure involves other terms as well (e.g., tensor splitting). The splitting (6.3) should be taken as an indication of the size of the relativistic corrections we expect from the charm string.

Spin-spin splittings do not arise directly as a result of transverse vector exchange in this model. Quark spin interactions arise only from quark-antiquark annihilation terms in the Dirac Hamiltonian and from virtual exchange of transverse string vibrations. Though such terms have not yet been evaluated, it is expected that they will be small corrections to the nonrelativistic limit. We remark that our estimate of the $n=2$ level may not be as good as that of the $n=1$ levels. Hence it is possible that the 4.46-GeV level in the e^+e^- channel is degenerate with the 4.41-GeV level.

Since the vibrational levels are above the charm (i.e., $D\bar{D}$) threshold, they are expected to decay predominantly into charmed mesons.

We note that the presence of vibrational levels in the ψ spectroscopy is actually more general than the quark-confining string model. To obtain a linear potential from field theory, the color electric flux must be confined, dynamically or by hand, to a tube (or a vortex). The resulting tube can, in general, move and vibrate in space. Such motion gives the extra vibrational states in quantum mechanics. The rigidity of such a tube is determined by the dynamics of any particular picture. Hence it is the energies (not the presence) of the vibrational levels that is characteristic of the QCS model.

To summarize, we have shown that the charmonium model with a linearly rising potential can be derived from a relativistic, gauge and reparametrization invariant field-theoretic (albeit unconventional) model. Furthermore, in the QCS model, relativistic invariance requires the introduction of string dynamics, which provides additional vibrational modes that are absent in the charmonium model. This already occurs in the Schrödinger limit. With the inclusion of all the relativistic corrections, we expect many additional terms contributing to Eqs. (1.1) and (5.41). Their effects on the mass shifts of the low-lying energy levels are roughly an order of magnitude smaller than the binding energies. However, their effects on the wave function may be more drastic. It is also very important to evaluate the effects due to S-D mixing, opening of thresholds, and various decay channels. Keeping this in mind, the charm-string spectrum seems to agree with experimental data[1] quite well.

Though we have discussed only the ψ spectrum as a test of the QCS model, it is clear that the model can be considered successful only if it can be applied to the rest of the hadronic spectrum and interactions. As a model of hadrons, the QCS model appears to have the unique property of color-quark confinement without the presence of massless color gluons or of pure gluonic states. Further, we expect that physical states with light quarks will lie on essentially straight Regge trajectories with slope $\alpha' \sim 1/2\pi k \sim 0.8\ \mathrm{GeV}^{-2}$.[13]

ACKNOWLEDGMENT

We thank our colleagues at SLAC for many useful discussions.

APPENDIX A

The string equation (2.14)

$$T^{\alpha\beta} h_{l,\alpha\beta} + V^{\alpha}_{l,\|\alpha} - \epsilon_{lm} V_{m,}{}^{\alpha} \nu_\alpha = 0$$

in the coordinates defined by Eq. (2.31) looks rather complicated. Using Eqs. (2.30), (2.31), and (2.33), we obtain

$$T^{00} = \frac{\Gamma^3}{2h} \chi^\dagger \Big[i\overleftrightarrow{\partial}_t - 2eB_0 + \Gamma(\hat{r}\cdot\hat{m}\times\dot{\hat{m}})\sigma_1 - \sigma_2(\hat{r}\cdot\hat{m}\times\dot{\hat{r}}) + \frac{\Gamma v'}{h}\sigma_3 \Big]\chi + \frac{\Gamma^4}{2h^2}\vec{B}_0'^{\,2} ,$$

$$T^{01} = -\frac{\Gamma}{2h^3}\chi^\dagger \Big[i\overleftrightarrow{\partial}_\sigma + \Gamma\sigma_1(\hat{r}\cdot\hat{m}\times\hat{m}') - \sigma_2(\hat{r}\cdot\hat{m}\times\hat{r}') - \frac{\Gamma\dot{h}}{v}\sigma_3 \Big]\chi ,$$

$$T^{10} = \frac{\Gamma^2}{2h^2}\chi^\dagger \Big[i\alpha_1\overleftrightarrow{\partial}_t - 2eB_0\alpha_1 - \Gamma^2\dot{v}\sigma_3 + \Gamma v\sigma_2(\hat{r}\cdot\hat{m}\times\dot{\hat{m}}) + \Gamma(\hat{r}\cdot\hat{m}\times\dot{\hat{m}})\gamma_5 \Big]\chi ,$$

$$T^{11} = \frac{-1}{2h^4}\chi^\dagger \Big[i\alpha_1\overleftrightarrow{\partial}_\sigma - \Gamma^2 v'\sigma_3 + \Gamma v\sigma_2(\hat{r}\cdot\hat{m}\times\hat{m}') + \Gamma(\hat{r}\cdot\hat{m}\times\hat{m}')\gamma_5 \Big]\chi ,$$

$$V_1{}^{,0} = -\frac{\Gamma^2}{2h}\chi^\dagger \Big[i\alpha_2\overleftrightarrow{\partial}_t - 2eB_0\alpha_2 - \frac{\Gamma v v'}{h}\sigma_3 - \Gamma v(\hat{r}\cdot\hat{m}\times\dot{\hat{m}})\sigma_1 - \gamma_5(\hat{r}\cdot\hat{m}\times\dot{\hat{r}}) \Big]\chi ,$$

$$V_1{}^{,1} = \frac{1}{2h^3}\chi^\dagger \Big[i\alpha_2\overleftrightarrow{\partial}_\sigma + \Gamma\dot{h}\sigma_3 - \Gamma v\sigma_1(\hat{r}\cdot\hat{m}\times\hat{m}') - \gamma_5(\hat{r}\cdot\hat{m}\times\hat{r}') \Big]\chi ,$$

$$V_2{}^{,0} = -\frac{\Gamma^2}{2h}\chi^\dagger \Big(i\alpha_3\overleftrightarrow{\partial}_t - 2eB_0\alpha_3 + \frac{\Gamma v v'}{h}\sigma_2 + \Gamma^2\dot{v}\sigma_1 + \frac{\Gamma v'}{h}\gamma_5 \Big)\chi ,$$

$$V_2{}^{,1} = \frac{1}{2h^3}\chi^\dagger \Big(i\alpha_3\overleftrightarrow{\partial}_\sigma - \Gamma\dot{h}\sigma_2 + \Gamma^2 v'\sigma_1 - \frac{\Gamma\dot{h}}{v}\gamma_5 \Big)\chi ,$$

$$h_{1,00}=\Gamma\dot{v}\ ,$$

$$h_{1,01}=\Gamma v'\ ,$$

$$h_{1,11}=\Gamma h\hat{m}\cdot\hat{r}'=-\Gamma\dot{h}/v\ ,$$

$$h_{2,00}=v(\hat{r}\cdot\hat{m}\times\dot{\hat{m}})\ ,$$

$$h_{2,01}=v(\hat{r}\cdot\hat{m}\times\hat{m}')\ ,$$

$$h_{2,11}=h(\hat{r}\cdot\hat{m}\times\hat{r}')\ ,$$

$$\nu_0=+\Gamma(\hat{r}\cdot\hat{m}\times\dot{\hat{m}})$$

$$\nu_1=+\Gamma(\hat{r}\cdot\hat{m}\times\hat{m}')\ .$$

Notice that T^{00} has a term (the time derivative) of order M; all other $T^{\alpha\beta}$, V_i^α, $h_{i,\alpha\beta}$, and ν_α are of lower order in M. Note that this string equation must be derived from the original Lagrangian (2.6). To derive this equation from the Lagrangian (2.32), a Lagrangian multiplier term for $\dot{\vec{R}}\cdot\vec{R}'=0$ must be taken into account in Eq. (2.32).

APPENDIX B

We note that with $\vec{\Pi}=0$, the quantization of angular momentum can be easily obtained via Dirac's method.[11] Starting from the canonical Poisson bracket

$$\{\hat{r}_i,p_j\}_P=\delta_{ij}$$

under second-class constraints

$$\phi_1=\hat{r}\cdot\vec{p}\simeq 0\ ,$$

$$\phi_2=\hat{r}^2-1\simeq 0$$

we obtain the Dirac brackets

$$\{p_i,p_j\}_D=p_i\hat{r}_j-p_j\hat{r}_i\ ,$$

$$\{\hat{r}_i,p_j\}_D=\delta_{ij}-\hat{r}_i\hat{r}_j\ ,$$

$$\{\hat{r}_i,\hat{r}_j\}_D=0\ .$$

Introducing $\vec{L}=\hat{r}\times\vec{p}$, we obtain

$$i\{L_i,L_j\}_D=i\epsilon_{ijk}L_k\ .$$

Quantization follows from replacing the left-hand side by a commutator.

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