LE MATEMATICHE
Vol. LXII (2007) – Fasc. II, pp. 127–146

APPLICATION OF THE RESIDUE THEOREM TO BILATERAL HYPERGEOMETRIC SERIES

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The application of the residue theorem to bilateral hypergeometric series identities is systematically reviewed by exemplifying three classes of summation theorems due to Dougall (1907), Jackson (1949, 1952) and Slater-Lakin (1953).

1. Introduction and Preliminaries

The Cauchy residue theorem is fundamental in the theory of complex variables and important in the evaluation of improper real integrals (cf. [10, §72-§75] for example). The purpose of the present paper is to illustrate the application of the residue theorem to bilateral hypergeometric series identities by exemplifying three classes of summation theorems due to Dougall [4], Jackson [6, 7] and Slater-Lakin [9].

To make the paper self-contained, we first recall the concept of residue and the residue theorem, which can be found in [10, §67-§68].

Definition 1 (Residue)**.** Let $f(z)$ be an analytic function in $0 < |z-a| < R$ with $z = a$ being an isolated singular point of it. Then the integral

$$\frac{1}{2\pi i}\oint_C f(z)\,dz \quad \text{where} \quad C: |z-a| = \rho \quad \text{and} \quad 0 < \rho < R$$

Entrato in redazione 1 gennaio 2007

AMS 2000 Subject Classification: Primary 33C20, Secondary 33D15

Keywords: The residue theorem; Contour integral; Bilateral hypergeoemtric series

is called the residue of $f(z)$ at $z=a$ and denoted by $\operatorname*{Res}_{z=a} f(z)$. It is not difficult to check that if $f(z)$ has a pole of order n at $z=a$, then its residue can be evaluated through

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \frac{(z-a)^n f(z)}{(n-1)!}. \tag{1}$$

Theorem 2 (Residue theorem)**.** *Let C be a simple closed contour within and on which a function $f(z)$ is analytic except for a finite number of singular points $a_1, a_2, \cdots, a_n$ interior to C. Then there holds the formula*

$$\frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z).$$

Corollary 3 (Partial fraction decomposition [11, §3.2])**.** Let $f(z)$ be a meromorphic function with all the singular points $\{a_k\}_{k=1}^{\infty}$ being simple poles subject to $0<|a_1|<|a_2|<\cdots$. Denote by b_k the residue of $f(z)$ at a_k. For each n, if there exists a contour C_n containing exactly $\{a_k\}_{k=1}^{n}$ such that $f(z)=o(R_n)$ on C_n as $n\to\infty$ and the minimum distance R_n from the origin to C_n in the complex plane is proportional to the circumference $\ell(C_n)$ of C_n. Then there holds the partial fraction expansion formula:

$$\frac{f(z)}{z} = \frac{f(0)}{z} + \sum_{k=1}^{+\infty} \frac{b_k}{a_k(z-a_k)}.$$

Proof. Consider the auxiliary function $F(\xi) := \frac{f(\xi)}{\xi(\xi-z)}$. Then all the singular points of $F(\xi)$ are simple poles z, 0 and $\{a_k\}_{k\geq 1}$ with the respective residues of $F(\xi)$ being given by $\frac{f(z)}{z}$, $-\frac{f(0)}{z}$ and $\{\frac{b_k}{a_k(a_k-z)}\}_{k=1}^{\infty}$. Let C_n be the circle of radius R_n centered at the origin which contains $z, 0, a_1, a_2, \cdots, a_n$ as interior points. Then the residue theorem tells us that

$$\frac{1}{2\pi i} \oint_{C_n} \frac{f(\xi)}{\xi(\xi-z)}\,\mathrm{d}\xi = \frac{f(z)-f(0)}{z} + \sum_{k=1}^{n} \frac{b_k}{a_k(a_k-z)}.$$

In order to confirm the corollary, it suffices to show that $\lim_{n\to\infty} \oint_{C_n} \frac{f(\xi)}{\xi(\xi-z)}\,\mathrm{d}\xi = 0$. This is justified by the following estimation:

$$\left| \oint_{C_n} \frac{f(\xi)}{\xi(\xi-z)}\,\mathrm{d}\xi \right| \leq \ell(C_n) \times \max_{\xi\in C_n} \left| \frac{f(\xi)}{\xi(\xi-z)} \right| \leq \frac{o(R_n)}{R_n - |z|}.$$

Before proceeding to deal with bilateral hypergeometric series, we display the following simple examples:

$$\sum_{n=1}^{+\infty}(-1)^n\frac{2z}{z^2-n^2\pi^2}=\frac{z-\sin z}{z\sin z}, \tag{2a}$$

$$\sum_{n=1}^{+\infty}\frac{1}{\lambda^2+n^2}=\frac{-1}{2\lambda^2}+\frac{\pi}{2\lambda}\coth(\pi\lambda), \tag{2b}$$

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{\lambda^2+n^2}=\frac{1}{2\lambda^2}+\frac{\pi}{2\lambda}\frac{1}{\sinh(\pi\lambda)}, \tag{2c}$$

$$\sum_{n=1}^{+\infty}\frac{\coth(n\pi)}{n^3}=\frac{7\pi^3}{180}, \tag{2d}$$

$$\sum_{n=0}^{+\infty}(-1)^n\frac{\operatorname{sech}\pi(n+\frac{1}{2})}{(2n+1)^5}=\frac{\pi^5}{768}, \tag{2e}$$

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(2n-1)\{\cosh\frac{\pi}{m}(n-\frac{1}{2})+\cos\frac{\pi}{m}(n-\frac{1}{2})\}}=\frac{\pi}{8}. \tag{2f}$$

All these six formulae can be shown by the residue theorem with the integral contours being circles or squares centered at the origin and the respective integrands being specified as:

$$\mathscr{F}_a(z):=\frac{z-\sin z}{z\sin z}, \qquad \mathscr{F}_b(z):=\frac{\cot(\pi z)}{\lambda^2+z^2},$$

$$\mathscr{F}_c(z):=\frac{\csc(\pi z)}{\lambda^2+z^2}, \qquad \mathscr{F}_d(z):=\frac{\cot(\pi z)\coth(\pi z)}{z^3},$$

$$\mathscr{F}_e(z):=\frac{\pi\sec(\pi z)}{z^5\cosh(\pi z)}, \qquad \mathscr{F}_f(z):=\frac{\pi\sec(\pi z)}{z\{\cosh\frac{\pi z}{m}+\cos\frac{\pi z}{m}\}}.$$

Now we take (2e) as example to illustrate the method. As exercises, the reader can show the others. Define the function by

$$f(z):=\frac{\pi\sec(\pi z)}{z^5\cosh(\pi z)}=\frac{\pi}{z^5\cos(\pi z)\cos(\pi z i)}.$$

It is not hard to see that $f(z)$ has the pole of order 5 at $z=0$ and the simple poles $z=n+\frac{1}{2}$ and $z=(n+\frac{1}{2})i$ with $n\in\mathbb{Z}$. Denote by $[z^n]f(z)$ the coefficient of z^n in the Laurent series of $f(z)$. Then we can compute, by means of (1), the following residues:

$$\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= \frac{\mathfrak{D}^4}{4!}\{z^5 f(z)\}_{z=0} = \frac{\mathfrak{D}^4}{4!}\left\{\frac{\pi}{\cos(\pi z)\cos(\pi z i)}\right\}_{z=0}\\
&= [z^4]\frac{\pi}{\{1-(\pi z)^2/2!+(\pi z)^4/4!+\cdots\}\{1+(\pi z)^2/2!+(\pi z)^4/4!+\cdots\}} = \frac{\pi^5}{6};
\end{aligned}$$

$$\operatorname*{Res}_{z=n+1/2} f(z) = \lim_{z\to n+1/2}\frac{\pi(z-n-1/2)}{z^5\cos(\pi z)\cosh(\pi z)} = \frac{2^5(-1)^{n+1}}{(2n+1)^5\cosh\pi(n+1/2)};$$

$$\operatorname*{Res}_{z=(n+1/2)i} f(z) = \lim_{z\to(n+1/2)i}\frac{\pi\{z-(n+1/2)i\}}{z^5\cos(\pi z)\cosh(\pi z)} = \frac{2^5(-1)^{n+1}}{(2n+1)^5\cosh\pi(n+1/2)}.$$

Let C_n be the square encircled by four lines $z=\pm n$ and $z=\pm ni$. Then the contour integral $\frac{1}{2\pi i}\oint_{C_n} f(z)\,\mathrm{d}z$ tends to zero when $n\to\infty$ because of the asymptotic relation

$$\left|\frac{\pi}{z^5\cos\pi z\cos\pi zi}\right| \sim \frac{1}{|z|^5} \quad \text{for} \quad z\in C_n.$$

According to Theorem 2, the sum of the residues of $f(z)$ in the whole complex plane results in zero:

$$2\sum_{n=-\infty}^{+\infty}\frac{(-1)^n\,2^5}{(2n+1)^5\cosh\pi(n+\frac{1}{2})} = \frac{\pi^5}{6}$$

which is obviously equivalent to (2e). □

2. Dougall's Bilateral Series Formulae

This section will prove the two formulae on bilateral ${}_2H_2$ and ${}_5H_5$-series due to Dougall [4]. Following the notation of [1, Chapter 1], the unilateral and bilateral series read respectively as

$$\begin{aligned}
{}_pF_q\left[\begin{matrix}a_1, a_2, \cdots, a_p\\ b_1, b_2, \cdots, b_q\end{matrix}\,\middle|\, z\right] &= \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!},\\
{}_pH_q\left[\begin{matrix}a_1, a_2, \cdots, a_p\\ b_1, b_2, \cdots, b_q\end{matrix}\,\middle|\, z\right] &= \sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}z^n;
\end{aligned}$$

where the shifted factorial is defined through the Γ-function

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{with} \quad \Gamma(x) = \int_0^\infty \tau^{x-1}e^{-\tau}\,\mathrm{d}\tau \quad \text{for} \quad \Re(x)>0.$$

In particular, we have explicitly

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

There are several important properties of the Γ-function (cf. [1, §1.2 and §1.5] for example). We shall frequently use, without explanation, the shifted and duplicative relations

$$\Gamma(x-n) = \frac{(-1)^n}{(1-x)_n}\Gamma(x) \quad \text{and} \quad \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+1/2)$$

as well as the reflection formulae:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \quad \text{and} \quad \Gamma(\tfrac{1}{2}+x)\Gamma(\tfrac{1}{2}-x) = \frac{\pi}{\cos(\pi x)}.$$

In addition, all the singular points of Γ-function are simple poles $z = -n$ for $n \in \mathbb{N}_0$ with the residues being given by

$$\operatorname*{Res}_{z=-\lambda-n} \Gamma(z+\lambda) = \lim_{z\to-\lambda-n} (z+\lambda+n)\frac{\Gamma(1+n+z+\lambda)}{(z+\lambda)_{n+1}} = \frac{(-1)^n}{n!}.$$

Theorem 4 (Dougall [4]). *For four complex numbers a,b,c,d subject to the condition $\Re(c+d-a-b) > 1$, there holds the nonterminating bilateral series identity:*

$${}_2H_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}\,\middle|\, 1\right] = \Gamma\left[\begin{matrix} 1-a, 1-b, c, d, c+d-a-b-1 \\ c-a, d-a, c-b, d-b \end{matrix}\right].$$

Proof. Define the complex function by

$$f(z) := \pi\cot(\pi z) \times \Gamma\left[\begin{matrix} a+z, & b+z \\ c+z, & d+z \end{matrix}\right].$$

All of its singular points $z = n,\ -a-m,\ -b-m$ with $n \in \mathbb{Z}$ and $m \in \mathbb{N}_0$ are simple poles. In view of (1), we compute the sum of the residues of $f(z)$ at the poles $z = n$:

$$\begin{aligned} \mathrm{R}_\star &:= \sum_{n=-\infty}^{+\infty} \operatorname*{Res}_{z=n} f(z) = \sum_{n=-\infty}^{+\infty} \lim_{z\to n}(z-n)f(z) \\ &= \sum_{n=-\infty}^{+\infty} \Gamma\left[\begin{matrix} a+n, & b+n \\ c+n, & d+n \end{matrix}\right] = \Gamma\left[\begin{matrix} a, & b \\ c, & d \end{matrix}\right] {}_2H_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}\,\middle|\, 1\right]. \end{aligned}$$

Similarly, the sum of the residues of $f(z)$ at the poles $z=-a-m$ may be reformulated as:

$$\begin{aligned}
\mathrm{R}_a &:= \sum_{m=0}^{+\infty} \operatorname*{Res}_{z=-a-m} f(z) = \sum_{m=0}^{+\infty} \lim_{z\to -a-m} (z+a+m) f(z) \\
&= -\pi\cot(\pi a) \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \Gamma\begin{bmatrix} b-a-m \\ c-a-m, d-a-m \end{bmatrix} \\
&= \frac{-\pi}{\tan(\pi a)} \Gamma\begin{bmatrix} b-a \\ c-a, d-a \end{bmatrix} {}_2F_1\begin{bmatrix} 1+a-c,\ 1+a-d \\ 1+a-b \end{bmatrix} 1 \Bigg] .
\end{aligned}$$

Evaluating the last ${}_2F_1$-series by means of the Gauss summation theorem [2, §1.3]

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, 1\right] = \Gamma\begin{bmatrix} c,\ c-a-b \\ c-a,\ c-b \end{bmatrix} \quad \text{with} \quad \Re(c-a-b) > 0,$$

we get the following closed expression for R_a:

$$\mathrm{R}_a = \frac{-\pi}{\tan(\pi a)} \Gamma\begin{bmatrix} b-a, 1+a-b, c+d-a-b-1 \\ c-a, d-a, c-b, d-b \end{bmatrix}.$$

Observing that $f(z)$ is symmetric with respect to a and b, we write down the sum of the residues of $f(z)$ at the poles $z=-b-m$:

$$\mathrm{R}_b := \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-b-m} f(z) = \frac{-\pi}{\tan(\pi b)} \Gamma\begin{bmatrix} a-b, 1+b-a, c+d-a-b-1 \\ c-b, d-b, c-a, d-a \end{bmatrix}.$$

Let $C_n(\varepsilon)$ be the circle of radius $n+\varepsilon$ centered at the origin with the $\varepsilon > 0$ being chosen such that $C_n(\varepsilon)$ does not pass any pole of $f(z)$.

For sufficient large $|z|$, there holds the asymptotic relation (cf. [1, §1.4]):

$$\Gamma\begin{bmatrix} a+z, & b+z \\ c+z, & d+z \end{bmatrix} \sim |z|^{\Re(a+b-c-d)}. \tag{3}$$

When $n \to \infty$, we have also for any $z = x+yi \in C_n(\varepsilon)$:

$$|\cot(\pi z)|^2 = \cot\pi z \cot\pi\bar{z} = \frac{\cos 2\pi x + \cos 2\pi yi}{\cos 2\pi x - \cos 2\pi yi} = \mathscr{O}(1).$$

Therefore for sufficient large n, the following inequality holds

$$\begin{aligned}
\left| \frac{1}{2\pi i} \oint_{C_n(\varepsilon)} f(z)\, \mathrm{d}z \right| &\leq \mathscr{O}\Big\{ (n+\varepsilon) \max_{z\in C_n(\varepsilon)} |f(z)| \Big\} \\
&\leq \mathscr{O}\Big\{ (n+\varepsilon)^{\Re(1+a+b-c-d)} \Big\}.
\end{aligned}$$

When $\Re(c+d-a-b)>1$, we have consequently the limiting relation:

$$\frac{1}{2\pi i}\oint_{C_n(\varepsilon)} f(z)\,\mathrm{d}z \to 0 \quad \text{as} \quad n\to\infty. \tag{4}$$

According to Theorem 2, the sum of the residues of $f(z)$ at all the poles is equal to zero: $\mathrm{R}_\star+\mathrm{R}_a+\mathrm{R}_b=0$, which reads explicitly as:

$$\Gamma\begin{bmatrix}a, & b\\ c, & d\end{bmatrix} {}_2H_2\left[\begin{matrix}a, & b\\ c, & d\end{matrix}\,\middle|\,1\right] = \frac{\pi}{\tan(\pi a)}\Gamma\begin{bmatrix}b-a,1+a-b,c+d-a-b-1\\ c-a,d-a,c-b,d-b\end{bmatrix}$$
$$+\frac{\pi}{\tan(\pi b)}\Gamma\begin{bmatrix}a-b,1+b-a,c+d-a-b-1\\ c-b,d-b,c-a,d-a\end{bmatrix}.$$

The last expression can be simplified, by means of the reflective property of the Γ-function, as

$$\Gamma\begin{bmatrix}a,b\\ c,d\end{bmatrix} {}_2H_2\left[\begin{matrix}a,b\\ c,d\end{matrix}\,\middle|\,1\right] = \Gamma\begin{bmatrix}c+d-a-b-1\\ c-b,d-b,c-a,d-a\end{bmatrix}\left\{\frac{\pi^2\cot\pi a}{\sin\pi(b-a)}+\frac{\pi^2\cot\pi b}{\sin\pi(a-b)}\right\}$$
$$=\Gamma\begin{bmatrix}c+d-a-b-1\\ c-b,d-b,c-a,d-a\end{bmatrix}\frac{\pi^2}{\sin\pi a\sin\pi b} = \Gamma\begin{bmatrix}a,1-a,b,1-b,c+d-a-b-1\\ c-b,d-b,c-a,d-a\end{bmatrix}.$$

This is a restatement of the bilateral ${}_2H_2$-series identity displayed in Theorem 4. □

We prove next the ${}_5H_5$-series identity discovered by Dougall in 1907. The difference between the proof we are going to present and the original one due to Dougall lies in the simplification of the trigonometric expression, where our approach is more accessible.

Theorem 5 (Dougall [4]). *For five complex numbers a,b,c,d,e satisfying the condition $\Re(1+2a-b-c-d-e)>0$, there holds the following nonterminating well-poised bilateral hypergeometric series identity:*

$$ {}_5H_5\left[\begin{matrix}1+a/2, & b, & c, & d, & e\\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\,\middle|\,1\right]$$
$$=\Gamma\begin{bmatrix}1-b,1-c,1-d,1-e,1+a-b,1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e\\ 1+a,1-a,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e\end{bmatrix}.$$

Proof. Similar to the proof of Theorem 4, define the function

$$g(z):=\frac{\pi(a+2z)}{a\tan(\pi z)}\Gamma\begin{bmatrix}b+z,\ c+z,\ d+z,\ e+z\\ 1+a-b+z,1+a-c+z,1+a-d+z,1+a-e+z\end{bmatrix}.$$

It is obvious that all the singular points of $g(z)$ are simple poles $z=n$, $-b-m$, $-c-m$, $-d-m$, $-e-m$ where $n\in\mathbb{Z}$ and $m\in\mathbb{N}_0$. The sum of residues of $g(z)$ at $z=n$ can be expressed in terms of bilateral series:

$$\begin{aligned}
\mathscr{R}_a &:= \sum_{n=-\infty}^{+\infty} \operatorname*{Res}_{z=n} g(z) = \sum_{n=-\infty}^{+\infty} \lim_{z\to n}(z-n)g(z)\\
&= \sum_{n=-\infty}^{+\infty} \frac{a+2n}{a}\Gamma\left[\begin{matrix} b+n, c+n, d+n, e+n \\ 1+a-b+n, 1+a-c+n, 1+a-d+n, 1+a-e+n\end{matrix}\right]\\
&= \Gamma\left[\begin{matrix} b,\ c,\ d,\ e, \\ 1+a-b, 1+a-c, 1+a-d, 1+a-e\end{matrix}\right] {}_5H_5\left[\begin{matrix} 1+a/2, & b, & c, & d, & e \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\,\Big|\,1\right].
\end{aligned}$$

The sum of residues of $g(z)$ at $z=-b-m$ can be reformulated as follows:

$$\begin{aligned}
\mathscr{R}_b &:= \sum_{m=0}^{+\infty} \operatorname*{Res}_{z=-b-m} g(z) = \sum_{m=0}^{+\infty} \lim_{z\to -b-m}(z+b+m)g(z)\\
&= \frac{\pi}{a}\sum_{m=0}^{+\infty} \frac{(-1)^m(2b-a+2m)}{m!\tan(\pi b)}\Gamma\left[\begin{matrix} c-b-m, d-b-m, e-b-m \\ 1+a-2b-m, 1+a-b-c-m, 1+a-b-d-m, 1+a-b-e-m\end{matrix}\right]\\
&= \frac{\pi(2b-a)}{a\tan(\pi b)}\Gamma\left[\begin{matrix} c-b,\ d-b,\ e-b \\ 1+a-2b,\ 1+a-b-c,\ 1+a-b-d,\ 1+a-b-e\end{matrix}\right]\\
&\times {}_5F_4\left[\begin{matrix} 2b-a, & 1+(2b-a)/2, & b+c-a, & b+d-a, & b+e-a \\ & (2b-a)/2, & 1+b-c, & 1+b-d, & 1+b-e\end{matrix}\,\Big|\,1\right].
\end{aligned}$$

Recalling the Dougall-Dixion formula [2, P27]

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix} a, & 1+a/2, & b, & c, & d \\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d\end{matrix}\,\Big|\,1\right]\\
&= \Gamma\left[\begin{matrix} 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-b-c-d \\ 1+a,\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d\end{matrix}\right]
\end{aligned}$$

provided that $\Re(1+a-b-c-d)>0$ for convergence, we can evaluate the following ${}_5F_4(1)$-series:

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix} 2b-a, & 1+(2b-a)/2, & b+c-a, & b+d-a, & b+e-a \\ & (2b-a)/2, & 1+b-c, & 1+b-d, & 1+b-e\end{matrix}\,\Big|\,1\right]\\
&= \Gamma\left[\begin{matrix} 1+b-c, & 1+b-d, & 1+b-e, & 1+2a-b-c-d-e \\ 1+2b-a, & 1+a-c-d, & 1+a-e-d, & 1+a-c-e\end{matrix}\right].
\end{aligned}$$

This leads us to the following closed expression:

$$\begin{aligned}
\mathscr{R}_b &= \frac{\pi^3\cot(\pi b)\sin\pi(a-2b)}{a\sin\pi(b-c)\sin\pi(b-d)\sin\pi(b-e)}\\
&\times\ \Gamma\left[\begin{matrix} 1+2a-b-c-d-e \\ 1+a-b-c, 1+a-b-d, 1+a-b-e, 1+a-c-d, 1+a-c-e, 1+a-d-e\end{matrix}\right].
\end{aligned}$$

Observe that $g(z)$ is symmetric in b, c, d and e. We get the sum of residues of $g(z)$ at the poles $z=-c-m$ by interchanging b and c in "R_b" as follows:

$$\mathscr{R}_c := \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-c-m} g(z) = \frac{\pi^3 \cot(\pi c) \sin\pi(a-2c)}{a \sin\pi(c-b)\sin\pi(c-d)\sin\pi(c-e)} \times \Gamma\left[\begin{matrix} 1+2a-b-c-d-e \\ 1+a-b-c, 1+a-c-d, 1+a-c-e, 1+a-b-d, 1+a-b-e, 1+a-d-e \end{matrix}\right].$$

Interchanging b and d in "R_b", we find the sum of residues of $g(z)$ at $z=-d-m$:

$$\mathscr{R}_d := \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-d-m} g(z) = \frac{\pi^3 \cot(\pi d) \sin\pi(a-2d)}{a \sin\pi(d-b)\sin\pi(d-c)\sin\pi(d-e)} \times \Gamma\left[\begin{matrix} 1+2a-b-c-d-e \\ 1+a-b-d, 1+a-c-d, 1+a-d-e, 1+a-b-c, 1+a-b-e, 1+a-c-e \end{matrix}\right].$$

Interchanging b and e in "R_b", we get the sum of residues of $g(z)$ at $z=-e-m$:

$$\mathscr{R}_e := \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-e-m} g(z) = \frac{\pi^3 \cot(\pi e) \sin\pi(a-2e)}{a \sin\pi(e-b)\sin\pi(e-c)\sin\pi(e-d)} \times \Gamma\left[\begin{matrix} 1+2a-b-c-d-e \\ 1+a-b-e, 1+a-c-e, 1+a-d-e, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right].$$

Denote by $C_n(\varepsilon)$ the circle $|z| = n+\varepsilon$ where $\varepsilon > 0$ is chosen such that $C_n(\varepsilon)$ does not pass through any pole of $g(z)$. By means of the same argument as for (4), we have

$$\begin{aligned} \left| \frac{1}{2\pi i} \oint_{C_n(\varepsilon)} g(z)\,\mathrm{d}z \right| &\leq \mathscr{O}\Big\{ (n+\varepsilon) \max_{z\in C_n} |g(z)| \Big\} \\ &\leq \mathscr{O}\Big\{ (n+\varepsilon)^{-2\Re(1+2a-b-c-d-e)} \Big\} \end{aligned}$$

which leads us to the limiting relation: $\lim_{n\to\infty} \frac{1}{2\pi i} \oint_{C_n(\varepsilon)} g(z)\,\mathrm{d}z = 0$. In view of Theorem 2, the sum of residues of $g(z)$ over all the poles vanishes:

$$\mathscr{R}_a + \mathscr{R}_b + \mathscr{R}_c + \mathscr{R}_d + \mathscr{R}_e = 0.$$

Writing the last relation explicitly as

$$\Gamma\left[\begin{matrix} b, c, d, e \\ 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix}\right] {}_5H_5\left[\begin{matrix} 1+a/2, b, c, d, e \\ a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix} \,\middle|\, 1\right] \tag{5a}$$

$$= \Delta \times \frac{\pi^3}{a} \Gamma\left[\begin{matrix} 1+2a-b-c-d-e \\ 1+a-b-e, 1+a-c-e, 1+a-d-e, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right]. \tag{5b}$$

where Δ is given by the trigonometric sum

$$\begin{aligned}\Delta \;:=\; & \frac{\cot(\pi b)\sin\pi(a-2b)}{\sin\pi(c-b)\sin\pi(d-b)\sin\pi(e-b)}\\ + & \frac{\cot(\pi c)\sin\pi(a-2c)}{\sin\pi(b-c)\sin\pi(d-c)\sin\pi(e-c)}\\ + & \frac{\cot(\pi d)\sin\pi(a-2d)}{\sin\pi(b-d)\sin\pi(c-d)\sin\pi(e-d)}\\ + & \frac{\cot(\pi e)\sin\pi(a-2e)}{\sin\pi(b-e)\sin\pi(c-e)\sin\pi(d-e)}.\end{aligned}$$

If we can show that Δ has the following closed form:

$$\Delta = \frac{\sin(\pi a)}{\sin(\pi b)\sin(\pi c)\sin(\pi d)\sin(\pi e)} \tag{6a}$$

$$= \pi^{-3}\Gamma\begin{bmatrix} b,c,d,e,1-b,1-c,1-d,1-e \\ 1-a, \quad a \end{bmatrix} \tag{6b}$$

then substituting this into (5b) and simplifying the result, we establish Dougall's bilateral ${}_5H_5$-series identity displayed in Theorem 5.

It remains to confirm (6a-6b). For this reason, consider the *rational function* defined by

$$U(z) := \frac{e^{iz}\cot z\sin(a+2z)}{\sin(b+z)\sin(c+z)\sin(d+z)\sin(e+z)}.$$

Then $U(z)$ can be decomposed in partial fractions

$$U(z) = \frac{A}{\sin z} + \frac{B}{\sin(b+z)} + \frac{C}{\sin(c+z)} + \frac{D}{\sin(d+z)} + \frac{E}{\sin(e+z)} \tag{7}$$

where the coefficients A, B, C, D and E are determined as follows:

$$A = \frac{\sin a}{\sin b\sin c\sin d\sin e},$$

$$B = \frac{e^{-ib}\cot b\sin(a-2b)}{\sin(b-c)\sin(b-d)\sin(b-e)},$$

$$C = \frac{e^{-ic}\cot c\sin(a-2c)}{\sin(c-b)\sin(c-d)\sin(c-e)},$$

$$D = \frac{e^{-id}\cot d\sin(a-2d)}{\sin(d-b)\sin(d-c)\sin(d-e)},$$

$$E = \frac{e^{-ie}\cot e\sin(a-2e)}{\sin(e-b)\sin(e-c)\sin(e-d)}.$$

Keep in mind of the limiting relation

$$\lim_{M\to\infty}\frac{\sin Mi}{\sin(\lambda+Mi)}=\lim_{M\to\infty}\frac{i\sinh M}{\sin\lambda\cosh M+i\cos\lambda\sinh M}=e^{\lambda i}.$$

Multiplying across (7) by $\sin z$ and then letting $z=Mi$, we find that

$$\lim_{M\to\infty}U(Mi)\sin(Mi)=0$$

which can be restated explicitly as

$$A+Be^{bi}+Ce^{ci}+De^{di}+Ee^{ei}=0.$$

From this equation, we derive (6a-6b) after having replaced each parameter by its π-times. □

3. Jackson's Bilateral Series Identities

By specializing hypergeometric transformations, M. Jackson derived several bilateral series identities. We prove two typical ones through the residue theorem.

Theorem 6 (Jackson [6, Eq 2.3])**.** *For six complex numbers a,b,c,e,f,g with $e+f=1+2a$ and $2g=1+b+c$ satisfying $\Re(1+2a-b-c)>0$, there holds the bilateral series identity:*

$$ {}_3H_3\left[\begin{matrix}a,b,c\\ e,f,g\end{matrix}\,\middle|\,1\right]=\frac{2^{b+c-2a}}{\pi}\Gamma\left[\begin{matrix}e,f,g,1-a,1-b,1-c,1+a-g\\ \frac{1+e-b}{2},\frac{1+e-c}{2},\frac{1+f-b}{2},\frac{1+f-c}{2}\end{matrix}\right]$$
$$\times\left\{\sin(\pi a)\cos\tfrac{\pi(b-c)}{2}+\sin\pi(e-a)\cos\tfrac{\pi(b+c)}{2}\right\}.$$

Proof. This identity is the common generalization of the two identities due to Watson [2, §3.3] and Whipple [2, §3.4]. Define the function

$$F(z):=\frac{\pi}{\sin(\pi z)}\Gamma\left[\begin{matrix}b+z,\quad c+z\\ 1-a-z,e+z,f+z,g+z\end{matrix}\right].$$

Then all the singular points of $F(z)$ are simple poles $z=n$, $-b-m$ and $-c-m$ with $n\in\mathbb{Z}$ and $m\in\mathbb{N}_0$. From (1), we get the sum of residues of $F(z)$ at $z=n$ as follows:

$$\mathrm{R}_\star:=\sum_{n=-\infty}^{+\infty}\operatorname*{Res}_{z=n}F(z)=\sum_{n=-\infty}^{+\infty}(-1)^n\Gamma\left[\begin{matrix}b+n,\ c+n\\ 1-a-n,e+n,f+n,g+n\end{matrix}\right]$$
$$=\Gamma\left[\begin{matrix}b,\ c\\ 1-a,\ e,\ f,\ g\end{matrix}\right]{}_3H_3\left[\begin{matrix}a,&b,&c\\ e,&f,&g\end{matrix}\,\middle|\,1\right].$$

Instead, the sum of residues of $F(z)$ at $z=-b-m$ results in the closed expression:

$$
\begin{aligned}
\mathrm{R}_b := \sum_{m=0}^{+\infty} \operatorname*{Res}_{z=-b-m} F(z) &= \frac{-\pi}{\sin(\pi b)} \sum_{m=0}^{+\infty} \frac{1}{m!} \Gamma\left[\begin{matrix} c-b-m \\ 1-a+b+m, e-b-m, f-b-m, g-b-m \end{matrix}\right] \\
&= \frac{-\pi}{\sin(\pi b)} \Gamma\left[\begin{matrix} c-b \\ 1-a+b, e-b, f-b, g-b \end{matrix}\right] {}_3F_2\left[\begin{matrix} 1+b-e,\ 1+b-f,\ 1+b-g \\ 1+b-a,\ 1+b-c \end{matrix} \,\middle|\, 1\right] \\
&= \frac{-\pi}{\sin(\pi b)} \Gamma\left[\begin{matrix} \frac{1}{2},\ 1+\frac{b-c}{2},\ c-b,\ 1+a-g \\ e-b, f-b, g-b, \frac{1+e-c}{2}, \frac{1+f-c}{2}, \frac{2+b-e}{2}, \frac{2+b-f}{2} \end{matrix}\right] \\
&= \frac{2^{b+c-2a} \pi\, \Gamma(1+a-g)}{\Gamma[\frac{1+e-b}{2}, \frac{1+e-c}{2}, \frac{1+f-b}{2}, \frac{1+f-c}{2}]} \frac{\sin\pi(\frac{b-e}{2}) \sin\pi(\frac{b-f}{2})}{\sin(\pi b) \sin\frac{\pi(b-c)}{2}}
\end{aligned}
$$

where we have utilized the ${}_3F_2$-series identity due to Watson [2, §3.3]:

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1+a+b}{2},\ 2c \end{matrix} \,\middle|\, 1\right] = \Gamma\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2}+c,\ \frac{1+a+b}{2},\ c-\frac{a+b-1}{2} \\ \frac{1+a}{2},\ \frac{1+b}{2},\ \frac{1-a}{2}+c,\ \frac{1-b}{2}+c \end{matrix}\right]
$$

provided that $\Re(1+2c-a-b)>0$ for convergence. In view of the symmetry, we obtain the sum of residues of $F(z)$ at the poles $z=-c-m$ by interchanging b and c in "R_b" as follows:

$$
\mathrm{R}_c := \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-c-m} F(z) = \frac{2^{b+c-2a} \pi\, \Gamma(1+a-g)}{\Gamma[\frac{1+e-b}{2}, \frac{1+e-c}{2}, \frac{1+f-b}{2}, \frac{1+f-c}{2}]} \frac{\sin\pi(\frac{c-e}{2}) \sin\pi(\frac{c-f}{2})}{\sin(\pi c) \sin\frac{\pi(c-b)}{2}}.
$$

Similar to the proofs of Dougall's formulae, when $\Re(1+2a-b-c)>0$, we can properly choose the contour C_n such that

$$
\frac{1}{2\pi i} \oint_{C_n} F(z)\, \mathrm{d}z \to 0 \quad \text{as} \quad n \to \infty.
$$

According to Theorem 2, the sum of residues of $F(z)$ over all the poles vanishes: $\mathrm{R}_\star + \mathrm{R}_b + \mathrm{R}_c = 0$, which can explicitly be restated as

$$
\begin{aligned}
\Gamma\left[\begin{matrix} b,\ c \\ 1-a, e, f, g \end{matrix}\right] {}_3H_3\left[\begin{matrix} a, b, c \\ e, f, g \end{matrix} \,\middle|\, 1\right] &= \frac{2^{b+c-2a} \pi\, \Gamma(1+a-g)}{\Gamma[\frac{1+e-b}{2}, \frac{1+e-c}{2}, \frac{1+f-b}{2}, \frac{1+f-c}{2}]} \\
&\times \left\{ \frac{\sin\pi(\frac{b-e}{2}) \sin\pi(\frac{b-f}{2})}{\sin(\pi b) \sin\frac{\pi(c-b)}{2}} + \frac{\sin\pi(\frac{c-e}{2}) \sin\pi(\frac{c-f}{2})}{\sin(\pi c) \sin\frac{\pi(b-c)}{2}} \right\}.
\end{aligned}
$$

Simplifying the trigonometric fractions inside the braces $\{\cdots\}$

$$
\left\{ \sin(\pi a) \cos\frac{\pi(b-c)}{2} + \sin\pi(e-a) \cos\frac{\pi(b+c)}{2} \right\} \Big/ \sin(\pi b) \sin(\pi c),
$$

we find the bilateral series identity displayed in (8). □

Theorem 7 (Jackson [7, Eq 1.2]). *For four complex numbers a,b,c,d satisfying the condition $\Re(1+a-2b)>0$, there holds the well-poised bilateral series identity:*

$$
\begin{aligned}
&{}_6H_6\left[\begin{matrix}1+a/2, & \frac12+a-b, & b+c, & b+d, & b-c, & b-d\\ a/2, \frac12+b, 1+a-b-c, 1+a-b-d, 1+a-b+c, 1+a-b+d\end{matrix}\,\middle|\,-1\right]\\
&=\Gamma\left[\begin{matrix}\frac12+b,1-b\pm c,1-b\pm d,1+a-b\pm c,1+a-b\pm d\\ \frac12+a-b,1+a-2b,1\pm a,1-b+\frac{a\pm c\pm d}{2}\end{matrix}\right]\\
&\times\frac{\cos\pi(a-b)\cos(\pi b)+\cos(\pi c)\cos(\pi d)}{2^{1+2a-4b}\cos\pi(a-b)}
\end{aligned}
$$

where $\Gamma(u\pm v)$ denotes the Γ-function product with parameters $u+v$ and $u-v$.

Proof. Similar to the proof of Theorem 6, consider the function

$$G(z):=\frac{\pi(a+2z)}{a\sin(\pi z)}\Gamma\left[\begin{matrix}\frac12+a-b+z, \ b+c+z, \ b+d+z, \ b-c+z, \ b-d+z\\ \frac12+b+z, 1+a-b-c+z, 1+a-b-d+z, 1+a-b+c+z, 1+a-b+d+z\end{matrix}\right].$$

It is not difficult to see that all the singular points of $G(z)$ are simple poles $z=n$, $-m-a+b-1/2$, $-m-b-c$, $-m-b+c$, $-m-b-d$ and $-m-b+d$ with $n\in\mathbb{Z}$ and $m\in\mathbb{N}_0$.

The sum of residues of $G(z)$ at the poles $z=n$ reads as follows:

$$
\begin{aligned}
\mathscr{R}_a:=\sum_{n=-\infty}^{\infty}\operatorname*{Res}_{z=n}G(z)&=\Gamma\left[\begin{matrix}1/2+a-b,b+c,b+d,b-c,b-d\\ 1/2+b,1+a-b-c,1+a-b-d,1+a-b+c,1+a-b+d\end{matrix}\right]\\
&\times\ {}_6H_6\left[\begin{matrix}1+a/2,1/2+a-b,b+c,b+d,b-c,b-d\\ a/2,1/2+b,1+a-b-c,1+a-b-d,1+a-b+c,1+a-b+d\end{matrix}\,\middle|\,-1\right].
\end{aligned}
$$

The sum of residues of $G(z)$ at the poles $z=-m-a+b-\frac12$ can be expressed in terms of well-poised ${}_6F_5(-1)$-series as follows:

$$
\begin{aligned}
\mathscr{R}_b&:=\sum_{m=0}^{\infty}\operatorname*{Res}_{z=-m-a+b-\frac12}G(z)\\
&=\sum_{m=0}^{\infty}\frac{\pi(1+a-2b+2m)}{m!\,a\cos\pi(a-b)}\Gamma\left[\begin{matrix}2b-a\pm c-m-\frac12,2b-a\pm d-m-\frac12\\ 2b-a-m,\frac12\pm c-m,\frac12\pm d-m\end{matrix}\right]\\
&=\frac{\pi(1+a-2b)}{a\cos\pi(a-b)}\Gamma\left[\begin{matrix}2b-a\pm c-\frac12,2b-a\pm d-\frac12\\ 2b-a,\frac12\pm c,\frac12\pm d\end{matrix}\right]\\
&\times\ {}_6F_5\left[\begin{matrix}1+a-2b, & \frac{3+a-2b}{2}, & \frac12\pm c, & \frac12\pm d\\ & \frac{1+a-2b}{2}, & \frac32+a-2b\pm c, & \frac32+a-2b\pm d\end{matrix}\,\middle|\,-1\right].
\end{aligned}
$$

Reformulating the last ${}_6F_5(-1)$-series through the limiting version of Whipple's transformation [2, §4.4]

$$
{}_6F_5\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d, & e \\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} \,\Big|\, -1\right]
= \Gamma\left[\begin{matrix} 1+a-d, 1+a-e \\ 1+a, 1+a-d-e \end{matrix}\right] {}_3F_2\left[\begin{matrix} 1+a-b-c, \ d, \ e \\ 1+a-b, \ 1+a-c \end{matrix} \,\Big|\, 1\right]
$$

and then evaluating ${}_3F_2$-series by means of Dixon's formula [2, §3.1]

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} \,\Big|\, 1\right] = \Gamma\left[\begin{matrix} 1+\frac{a}{2}, 1+a-b, 1+a-c, 1+\frac{a}{2}-b-c \\ 1+a, 1+\frac{a}{2}-b, 1+\frac{a}{2}-c, 1+a-b-c \end{matrix}\right]
$$

where $\Re(1+\frac{a}{2}-b-c) > 0$, we get the following closed expression:

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} 1+a-2b, & \frac{3+a-2b}{2}, & \frac{1}{2}\pm c, & \frac{1}{2}\pm d \\ & \frac{1+a-2b}{2}, & \frac{3}{2}+a-2b\pm c, & \frac{3}{2}+a-2b\pm d \end{matrix} \,\Big|\, -1\right] \\
&= \Gamma\left[\begin{matrix} \frac{3}{2}+a-2b+c, \frac{3}{2}+a-2b+d \\ 2+a-2b, 1+a-2b+c+d \end{matrix}\right] {}_3F_2\left[\begin{matrix} 1+a-2b-c-d, \frac{1}{2}-c, \frac{1}{2}-d \\ \frac{3}{2}+a-2b-c, \frac{3}{2}+a-2b-d \end{matrix} \,\Big|\, 1\right] \\
&= \frac{\pi}{2^{1+2a-4b}} \Gamma\left[\begin{matrix} \frac{3}{2}+a-2b\pm c, \frac{3}{2}+a-2b\pm d \\ 1+a-2b, 2+a-2b, \frac{2+a-2b\pm c\pm d}{2} \end{matrix}\right].
\end{aligned}
$$

Substituting this into the expression for $\mathscr{R}_b$ and simplifying the result, we finally obtain

$$
\begin{aligned}
\mathscr{R}_b \;=\; & \frac{2^{4b-2a-1}\pi^3}{a\Gamma[1+a-2b, (2+a-2b\pm c\pm d)/2]} \\
\times\; & \frac{\sin\pi(2b-a)\cos(\pi c)\cos(\pi d)}{\cos\pi(a-b)\cos\pi(a-2b\pm c)\cos\pi(a-2b\pm d)}.
\end{aligned}
$$

Analogously, the sum of residues of $G(z)$ at the poles $z = -m-b-c$ can also be expressed in terms of well-poised ${}_6F_5(-1)$-series:

$$
\begin{aligned}
\mathscr{R}_c &:= \sum_{m=0}^{\infty} \operatorname*{Res}_{z=-m-b-c} G(z) = \sum_{m=0}^{\infty} \frac{\pi(2b+2c-a+2m)}{m!\, a\sin\pi(b+c)} \\
&\times \Gamma\left[\begin{matrix} \frac{1}{2}+a-2b-c-m, -2c-m, -c\pm d-m \\ \frac{1}{2}-c-m, 1+a-2b-m, 1+a-2b-2c-m, 1+a-2b-c\pm d-m \end{matrix}\right] \\
&= \frac{\pi(2b+2c-a)}{a\sin\pi(b+c)} \Gamma\left[\begin{matrix} \frac{1}{2}+a-2b-c, \ -2c, -c\pm d \\ \frac{1}{2}-c, \ 1+a-2b, \ 1+a-2b-2c, \ 1+a-2b-c\pm d \end{matrix}\right] \\
&\times {}_6F_5\left[\begin{matrix} 2b+2c-a, & 1+b+c-\frac{a}{2}, & \frac{1}{2}+c, & 2b-a, & 2b+c-a\pm d \\ b+c-\frac{a}{2}, & \frac{1}{2}+2b+c-a, & 1+2c, & 1+c\pm d & \end{matrix} \,\Big|\, -1\right].
\end{aligned}
$$

The last ${}_6F_5(-1)$-series can also be reformulated by the limiting version of Whipple's transformation and then be evaluated by means of Dixon's formula as follows:

$$\begin{aligned}
&{}_6F_5\left[\begin{matrix}2b+2c-a, 1+b+c-\frac{a}{2}, & \frac{1}{2}+c, & 2b-a, 2b+c-a\pm d\\ b+c-\frac{a}{2}, & \frac{1}{2}+2b+c-a, & 1+2c, \quad 1+c\pm d\end{matrix}\middle| -1\right]\\
&=\Gamma\left[\begin{matrix}1+2c, 1+c-d\\ 1+2b+2c-a, 1+a+c-2b-d\end{matrix}\right]{}_3F_2\left[\begin{matrix}2b+c+d-a, 2b-a, \frac{1}{2}+d\\ 1+c+d, \frac{1}{2}+2b+c-a\end{matrix}\middle| 1\right]\\
&=\sqrt{\pi}\times\Gamma\left[\begin{matrix}1+c, 1+c\pm d, \frac{1}{2}+2b+c-a\\ 1+2b+2c-a, \frac{1-a+2b+c\pm d}{2}, \frac{2+a-2b+c\pm d}{2}\end{matrix}\right].
\end{aligned}$$

Substituting this into the expression for $\mathscr{R}_c$ and simplifying the result, we have

$$\begin{aligned}
\mathscr{R}_c &= \frac{2^{4b-2a-1}\pi^3}{a\Gamma[1+a-2b, (2+a-2b\pm c\pm d)/2]}\\
&\times \frac{\sin\pi(a-2b-2c)\cos\frac{a-2b-c\pm d}{2}\pi}{\sin(\pi c)\sin\pi(b+c)\sin\pi(c\pm d)\cos\pi(a-2b-c)}.
\end{aligned}$$

Interchanging c and d in "$\mathscr{R}_c$", we get the sum of residues of $G(z)$ at the poles $z=-m-b-d$:

$$\begin{aligned}
\mathscr{R}_d := \sum_{m=0}^{\infty}\operatorname*{Res}_{z=-b-d-m} G(z) &= \frac{2^{4b-2a-1}\pi^3/a}{\Gamma[1+a-2b, \frac{2+a-2b\pm c\pm d}{2}]}\\
&\times \frac{\sin\pi(a-2b-2d)\cos\frac{a-2b\pm c-d}{2}\pi}{\sin(\pi d)\sin\pi(b+d)\sin\pi(d\pm c)\cos\pi(a-2b-d)}.
\end{aligned}$$

Replacing c by $-c$ in "$\mathscr{R}_c$", we obtain the sum of residues of $G(z)$ at the poles $z=-m-b+c$:

$$\begin{aligned}
\mathscr{R}'_c := \sum_{m=0}^{\infty}\operatorname*{Res}_{z=-m-b+c} G(z) &= \frac{2^{4b-2a-1}\pi^3/a}{\Gamma[1+a-2b, \frac{2+a-2b\pm c\pm d}{2}]}\\
&\times \frac{\sin\pi(a-2b+2c)\cos\frac{a-2b+c\pm d}{2}\pi}{\sin(\pi c)\sin\pi(b-c)\sin\pi(d\pm c)\cos\pi(a-2b+c)}.
\end{aligned}$$

Replacing d by $-d$ in "$\mathscr{R}_d$", we find the sum of residues of $G(z)$ at the poles $z=-m-b+d$:

$$\begin{aligned}
\mathscr{R}'_d := \sum_{m=0}^{\infty}\operatorname*{Res}_{z=-m-b+d} G(z) &= \frac{2^{4b-2a-1}\pi^3/a}{\Gamma[1+a-2b, \frac{2+a-2b\pm c\pm d}{2}]}\\
&\times \frac{\sin\pi(a-2b+2d)\cos\frac{a-2b\pm c+d}{2}\pi}{\sin(\pi d)\sin\pi(b-d)\sin\pi(c\pm d)\cos\pi(a-2b+d)}.
\end{aligned}$$

When $\Re(3+3a-6b)>1$, we can devise properly the contours C_n such that the contour integral $\frac{1}{2\pi i}\oint_{C_n} G(z)\,\mathrm{d}z$ tends to zero as $n\to\infty$. According to the residue theorem, the sum of residues of $G(z)$ over all the poles vanishes. We can show that

$$\mathscr{R}_a = -\left\{\mathscr{R}_b+\mathscr{R}_c+\mathscr{R}_c'+\mathscr{R}_d+\mathscr{R}_d'\right\} = \frac{2^{4b-2a-1}\pi^3/a}{\Gamma[1+a-2b,\frac{2+a-2b\pm c\pm d}{2}]} \tag{8a}$$

$$\times\ \frac{\sin(\pi a)\{\cos\pi(a-b)\cos(\pi b)+\cos(\pi c)\cos(\pi d)\}}{\sin\pi(b\pm c)\sin\pi(b\pm d)\ \cos\pi(a-b)}. \tag{8b}$$

This is a restatement of the identity displayed in Theorem 7, where the restricton $\Re(3+3a-6b)>1$ has been replaced by $\Re(3+3a-6b)>0$, the convergent condition justified by analytic continuation.

In order to confirm (8a-8b), we have to prove the equivalent trigonometric formula:

$$\frac{\sin\pi(a-2b)\cos(\pi c)\cos(\pi d)}{\cos\pi(a-b)\cos\pi(a-2b\pm c)\cos\pi(a-2b\pm d)} \tag{9a}$$

$$-\frac{\sin\pi(a-2b-2c)\cos\frac{a-2b-c+d}{2}\pi\cos\frac{a-2b-c-d}{2}\pi}{\sin(\pi c)\sin\pi(b+c)\sin\pi(c+d)\sin\pi(c-d)\cos\pi(a-2b-c)} \tag{9b}$$

$$+\frac{\sin\pi(a-2b-2d)\cos\frac{a-2b+c-d}{2}\pi\cos\frac{a-2b-c-d}{2}\pi}{\sin(\pi d)\sin\pi(b+d)\sin\pi(c+d)\sin\pi(c-d)\cos\pi(a-2b-d)} \tag{9c}$$

$$+\frac{\sin\pi(a-2b+2c)\cos\frac{a-2b+c+d}{2}\pi\cos\frac{a-2b+c-d}{2}\pi}{\sin(\pi c)\sin\pi(b-c)\sin\pi(c+d)\sin\pi(c-d)\cos\pi(a-2b+c)} \tag{9d}$$

$$-\frac{\sin\pi(a-2b+2d)\cos\frac{a-2b+c+d}{2}\pi\cos\frac{a-2b-c+d}{2}\pi}{\sin(\pi d)\pi(b-d)\sin\pi(c+d)\sin\pi(c-d)\cos\pi(a-2b+d)} \tag{9e}$$

$$=\frac{\sin(\pi a)\{\cos\pi(a-b)\cos(\pi b)+\cos(\pi c)\cos(\pi d)\}}{\sin\pi(b+c)\sin\pi(b-c)\sin\pi(b+d)\sin\pi(b-d)\cos\pi(a-b)}. \tag{9f}$$

Consider the rational function defined by

$$V(z)=\frac{\sin(a+2z)\{\cos(a-b+z)\cos(b+z)+\cos c\cos d\}}{\sin(b-c+z)\sin(b+c+z)\sin(b-d+z)\sin(b+d+z)\cos(a-b+z)}$$

and the partial fraction decomposition:

$$\begin{aligned} V(z) \quad &= \quad \frac{A}{\cos(a-b+z)}+\frac{B}{\sin(b-c+z)}+\frac{C}{\sin(b-d+z)}\\ &+ \quad \frac{D}{\sin(b+c+z)}+\frac{E}{\sin(b+d+z)}. \end{aligned}$$

Determining the coefficients A,B,C,D,E and then letting $z=0$, we recover the identity displayed in (9). The details will not be reproduced due to space limitation. □

4. Bilateral Series due to Slater and Lakin

For the alternating bilateral hypergeometric series of higher order, Slater-Lakin [9, 1953] found the following general result.

Theorem 8. *For complex numbers* α *and* $\{\beta\}_{k=1}^{\ell}$ *subject to the condition* $\Re\{\ell(1+\alpha)-2\sum_{k=1}^{\ell}\beta_k\}>1$, *there holds the transformation formula:*

$$\begin{aligned}
&{}_{1+\ell}H_{\ell+1}\left[\begin{matrix}1+\alpha/2, & \beta_1, & \beta_2, & \cdots, & \beta_\ell\\ \alpha/2, & 1+\alpha-\beta_1, & 1+\alpha-\beta_2, & \cdots, & 1+\alpha-\beta_\ell\end{matrix}\;\middle|\;-1\right]\\
&= \sum_{k=1}^{\ell}\frac{\Gamma(1+\alpha-\beta_k)\Gamma(1-\beta_k)}{\alpha\,\Gamma(\alpha-2\beta_k)}\prod_{\substack{j=1\\ j\neq k}}^{\ell}\Gamma\left[\begin{matrix}\beta_j-\beta_k, 1+\alpha-\beta_j\\ \beta_j, 1+\alpha-\beta_j-\beta_k\end{matrix}\right]\\
&\times\; {}_{1+\ell}F_{\ell}\left[\begin{matrix}2\beta_k-\alpha,\; 1+\beta_k-\alpha/2,\; \{\beta_k+\beta_j-\alpha\}_{j\neq k}\\ \beta_k-\alpha/2,\quad \{1+\beta_k-\beta_j\}_{j\neq k}\end{matrix}\;\middle|\;-1\right].
\end{aligned}$$

Proof. For the function defined by

$$\mathscr{F}(z)=\frac{\pi(\alpha+2z)}{\alpha\sin(\pi z)}\prod_{k=1}^{\ell}\frac{\Gamma(\beta_k+z)}{\Gamma(1+\alpha-\beta_k+z)},$$

all the singular points are simple poles $z=n,\ -\beta_k-m$ with $n\in\mathbb{Z}$, $m\in\mathbb{N}_0$ and $k=1,2,\cdots,\ell$. Then the sum of residues of $\mathscr{F}(z)$ at $z=n$ reads as the following bilateral well-poised series:

$$\begin{aligned}
\mathscr{R}_\alpha &:= \sum_{n=-\infty}^{+\infty}\operatorname*{Res}_{z=n}\mathscr{F}(z)=\sum_{n=-\infty}^{+\infty}\lim_{z\to n}(z-n)\mathscr{F}(z)\\
&= \sum_{n=-\infty}^{+\infty}(-1)^n\frac{\alpha+2n}{\alpha}\prod_{k=1}^{\ell}\frac{\Gamma(\beta_k+n)}{\Gamma(1+\alpha-\beta_k+n)}\\
&= \prod_{k=1}^{\ell}\frac{\Gamma(\beta_k)}{\Gamma(1+\alpha-\beta_k)}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{\alpha+2n}{\alpha}\prod_{k=1}^{\ell}\frac{(\beta_k)_n}{(1+\alpha-\beta_k)_n}\\
&= {}_{1+\ell}H_{\ell+1}\left[\begin{matrix}1+\alpha/2,\beta_1,\beta_2,\cdots,\beta_\ell\\ \alpha/2,1+\alpha-\beta_1,1+\alpha-\beta_2,\cdots,1+\alpha-\beta_\ell\end{matrix}\;\middle|\;-1\right]\prod_{k=1}^{\ell}\frac{\Gamma(\beta_k)}{\Gamma(1+\alpha-\beta_k)}.
\end{aligned}$$

For other poles $z=-\beta_k-m$ with $k=1,2,\cdots,\ell$, the corresponding sum of residues of $\mathscr{F}(z)$ is given by the sum of well-poised hypergeometric series:

$$\begin{aligned}
\mathscr{R}_\beta &:= \sum_{k=1}^{\ell}\sum_{m=0}^{\infty} \operatorname*{Res}_{z=-\beta_k-m} \mathscr{F}(z) = \sum_{k=1}^{\ell}\sum_{m=0}^{\infty} \lim_{z\to-\beta_k-m} (z+\beta_k+m)\mathscr{F}(z) \\
&= -\sum_{k=1}^{\ell} \frac{\pi}{\alpha\sin(\pi\beta_k)} \sum_{m=0}^{\infty} \frac{\alpha-2\beta_k-2m}{m!\,\Gamma(1+\alpha-2\beta_k-m)} \prod_{j\neq k} \frac{\Gamma(\beta_j-\beta_k-m)}{\Gamma(1+\alpha-\beta_j-\beta_k-m)} \\
&= -\sum_{k=1}^{\ell} \frac{\pi}{\alpha\sin(\pi\beta_k)\Gamma(\alpha-2\beta_k)} \prod_{j\neq k} \frac{\Gamma(\beta_j-\beta_k)}{\Gamma(1+\alpha-\beta_j-\beta_k)} \\
&\times {}_{1+\ell}F_{\ell}\left[\begin{matrix}2\beta_k-\alpha, & 1+\beta_k-\alpha/2, & \{\beta_k+\beta_j-\alpha\}_{j\neq k} \\ & \beta_k-\alpha/2, & \{1+\beta_k-\beta_j\}_{j\neq k}\end{matrix}\,\middle|\, -1\right].
\end{aligned}$$

Denote by $C_n(\varepsilon)$ the circles $|z| = n+\varepsilon$ with $\varepsilon > 0$ being properly chosen such that $C_n(\varepsilon)$ does not pass through any pole of $\mathscr{F}(z)$. By using the same argument as (4), we have the following estimation:

$$\begin{aligned}
\left|\frac{1}{2\pi i}\oint_{C_n(\varepsilon)} \mathscr{F}(z)\,\mathrm{d}z\right| &\leq \mathscr{O}\Big\{(n+\varepsilon)\max_{z\in C_n}|\mathscr{F}(z)|\Big\} \\
&\leq \mathscr{O}\Big\{(n+\varepsilon)^{2-\sum_{k=1}^{\ell}\Re(1+\alpha-2\beta_k)}\Big\}.
\end{aligned}$$

When $\sum_{k=1}^{\ell}\Re(1+\alpha-2\beta_k) > 2$, the contour integral $\frac{1}{2\pi i}\oint_{C_n(\varepsilon)}\mathscr{F}(z)\,\mathrm{d}z$ tends to zero as $n\to\infty$. Hence we have $\mathscr{R}_\alpha+\mathscr{R}_\beta = 0$, which is equivalent to the transformation formula stated in Theorem 8, after having replaced the condition $\sum_{k=1}^{\ell}\Re(1+\alpha-2\beta_k) > 2$ by $\Re\{\ell(1+\alpha)-2\sum_{k=1}^{\ell}\beta_k\} > 1$, the convergence condition in view of analytic continuation. □

In addition, the residue theorem can also be employed to discover and prove summation formulae of basic hypergeometric series. For example, by computing the contour integral for the function defined by

$$\mathscr{G}(z) = z^{1+\lambda}\left[\begin{matrix}q, & q, & c/z, & qz/a \\ c, & q/a, & z, & q/z\end{matrix}\,\middle|\, q\right]_\infty \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}},$$

Chu [3] found in 1994 the following interesting bilateral q-series identity:

$$\begin{aligned}
&{}_{2+\ell}\psi_{\ell+2}\left[\begin{matrix}a, & d, & q^{n_1}x_1, & \cdots, & q^{n_\ell}x_\ell \\ c, & qd, & x_1, & \cdots, & x_\ell\end{matrix}\,\middle|\, q;q^{1-\lambda}/a\right] \\
&= \left[\begin{matrix}q, & q, & c/d, & qd/a \\ q/a, & c, & qd, & q/d\end{matrix}\,\middle|\, q\right]_\infty d^\lambda \prod_{k=1}^{\ell} \frac{(x_k/d;q)_{n_k}}{(x_k;q)_{n_k}}.
\end{aligned}$$

where $a, c, d, \{x_k\}_{k=1}^{\ell}$ are complex numbers and $\lambda, \{n_k\}_{k=1}^{\ell}$ nonnegative integers satisfying the conditions $n = \sum_{k=0}^{\ell} n_k$ and $|q/a| < |q^\lambda| < |q^n/c|$.

The examples presented in this paper have demonstrated that the residue theorem is indeed efficient to deal with hypergeometric series identities. There exist other important summation formulae (for example. Jackson [6, 7]), which can be obtained via the residue theorem. The interested reader is encouraged to try further. For other applications of the residue theorem to combinatorial computation, refer to two monographs due to Egorychev [5] and Reidel [8].

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