Application of the Wigner distribution function to partially coherent light

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The paper presents a review of the Wigner distribution function (WDF) and of some of its applications to optical problems, especially in the field of partial coherence. The WDF describes a signal in space and in spatial frequency simultaneously and can be considered the local spatial-frequency spectrum of the signal. Although derived in terms of Fourier optics, the description of an optical signal by means of its WDF closely resembles the ray concept in geometrical optics; the WDF thus presents a link between partial coherence and radiometry. Properties of the WDF and its propagation through linear optical systems are considered; again, the description of systems by WDF's can be interpreted directly in geometric-optical terms. Some examples are included to show how the WDF can be applied to practical problems that arise in the field of partial coherence.

1. INTRODUCTION

In 1932 Wigner introduced a distribution function in mechanics[1] that permitted a description of mechanical phenomena in a phase space. Such a Wigner distribution function (WDF) was introduced in optics by Walther in 1968,[2] to relate partial coherence to radiometry. A few years later, the WDF was introduced in optics again[3] (especially in the area of Fourier optics), and, since then, a great number of applications of the WDF have been reported. It is the aim of this paper to review the WDF and some of its applications to optical problems, especially in the field of partial coherence.

In Section 2 we describe how we represent partially coherent light. We introduce its positional power spectrum (or cross-spectral density function) and the spatial Fourier transform of that function: the directional power spectrum.

The WDF for partially coherent light is defined in Section 3, where its concept is elucidated with some simple examples and where some of its most important properties are given. Although derived in terms of Fourier optics, we see that the description of a signal by means of its WDF closely resembles the ray concept in geometrical optics and that the properties of the WDF have clear physical meanings.

In Section 4 we introduce a modal expansion for the cross-spectral density function of partially coherent light and derive a similar expansion for the WDF. This modal expansion allows us to formulate more properties of the WDF, especially in the form of inequalities.

The transformation of the WDF when a signal propagates through an optical system is described in Section 5. An optical system is treated there in two distinct forms: (1) as a black box, with an input plane and an output plane, for which an input–output relationship in terms of the WDF's is formulated and (2) as a continuous medium, for which a transport equation for the WDF is derived. We observe again that both the input–output relationship and the transport equation can be given a geometric-optical interpretation.

Finally, in Section 6, we describe some applications of the WDF to problems that arise in the field of partial coherence.

We conclude this introduction with some remarks about the signals with which we are dealing. We consider scalar optical signals, which can be described by, say, $\tilde{\phi}(x, y, z, t)$, where x, y, z denote space variables and t represents the time variable. Very often, we consider signals in a plane $z =$ constant, in which case we can omit the longitudinal space variable z from the formulas. Furthermore, we restrict ourselves to the one-dimensional case in which the signals are functions only of the transverse space variable x; in general, the extension to two dimensions is straightforward. The signals with which we are dealing are thus described by a function $\tilde{\phi}(x, t)$.

2. DESCRIPTION OF PARTIALLY COHERENT LIGHT

Let partially coherent light be described by a temporally stationary stochastic process $\tilde{\phi}(x, t)$; the ensemble average of the product $\tilde{\phi}(x_1, t_1)\tilde{\phi}^*(x_2, t_2)$ is then only a function of the time difference $t_1 - t_2$:

$$E\tilde{\phi}(x_1, t_1)\tilde{\phi}^*(x_2, t_2) = \tilde{\Gamma}(x_1, x_2, t_1 - t_2), \tag{2.1}$$

where the asterisk denotes complex conjugation. The function $\tilde{\Gamma}(x_1, x_2, \tau)$ is known as the (mutual) *coherence function*[4–7] of the process $\tilde{\phi}(x, t)$. The (mutual) *power spectrum*[6,7] or *cross-spectral density function*[8] $\Gamma(x_1, x_2, \omega)$ is defined as the temporal Fourier transform of the coherence function

$$\Gamma(x_1, x_2, \omega) = \int \tilde{\Gamma}(x_1, x_2, \tau)\exp(i\omega\tau)\mathrm{d}\tau. \tag{2.2}$$

(Unless otherwise stated, all integrations in this paper extend from $-\infty$ to $+\infty$.) The basic property[7,8] of the power spectrum is that it is a *nonnegative definite Hermitian* function of x_1 and x_2, i.e., $\Gamma(x_1, x_2, \omega) = \Gamma^*(x_2, x_1, \omega)$ and $\iint f(x_1, \omega)\Gamma(x_1, x_2, \omega)f^*(x_2, \omega)\mathrm{d}x_1\mathrm{d}x_2 \geq 0$ for any function $f(x, \omega)$.

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Instead of describing a stochastic process in a space domain by means of its power spectrum $\Gamma(x_1, x_2, \omega)$, we can represent it equally well in a spatial-frequency domain by means of the spatial Fourier transform $\bar{\Gamma}(u_1, u_2, \omega)$ of the power spectrum

$$\bar{\Gamma}(u_1, u_2, \omega) = \iint \Gamma(x_1, x_2, \omega)\exp[-i(u_1x_1 - u_2x_2)]\mathrm{d}x_1\mathrm{d}x_2. \tag{2.3}$$

(Throughout we represent the spatial Fourier transform of a function by the same symbol as the function itself but marked with a bar on top of the symbol.) Unlike the power spectrum $\Gamma(x_1, x_2, \omega)$, which expresses the coherence of the light at two different positions, its spatial Fourier transform $\bar{\Gamma}(u_1, u_2, \omega)$ expresses the coherence of the light in two different directions. Therefore we call $\Gamma(x_1, x_2, \omega)$ the *positional* power spectrum[9] and $\bar{\Gamma}(u_1, u_2, \omega)$ the *directional* power spectrum[9] of the light. It is evident that the directional power spectrum $\bar{\Gamma}(u_1, u_2, \omega)$ is a nonnegative definite Hermitian function of u_1 and u_2.

Apart from the pure space representation of a stochastic process by means of its positional power spectrum or the pure spatial-frequency representation by means of its directional power spectrum, we can describe a stochastic process in space and spatial frequency *simultaneously*. In this paper we therefore use the WDF, which is introduced in Section 3. Since, in the present discussion, the explicit temporal-frequency dependence is of no importance, we shall, for the sake of convenience, omit the temporal-frequency variable ω from the formulas in the remainder of the paper.

3. WIGNER DISTRIBUTION FUNCTION

It is sometimes convenient to describe an optical signal not in a space domain by means of its positional power spectrum but in a spatial-frequency domain by means of its directional power spectrum. The directional power spectrum globally shows how the energy of the signal is distributed as a function of direction (i.e., spatial frequency). However, instead of in this *global* distribution of the energy, one is often more interested in the *local* distribution of the energy as a function of spatial frequency. A similar local distribution occurs in music, for instance, in which a signal is usually described not by a time function nor by the Fourier transform of that function but by its musical score.

The score is indeed a picture of the local distribution of the energy of the musical signal as a function of frequency. The horizontal axis of the score clearly represents a time axis, and the vertical one a frequency axis. When a composer writes a score, he prescribes the frequencies of the notes that should be present at a certain time. We see that the musical score is something that might be called the *local frequency spectrum* of the musical signal.

The need for a description of the signal by means of a local frequency spectrum arises in other disciplines too. Geometrical optics, for instance, is usually treated in terms of rays, and the signal is described by giving the directions of the rays that should be present at a certain position. It is not difficult to translate the concept of the musical score to geometrical optics: We simply have to consider the horizontal (time) axis as a position axis and the vertical (frequency) axis as a direction axis. A musical note then represents an optical light ray passing through a point at a certain position and having a certain direction.

Another discipline in which we can apply the idea of a local frequency spectrum is in mechanics: The position and the momentum of a particle are given in a phase space. It was in mechanics that Wigner introduced in 1932 a distribution function[1] that provided a description of mechanical phenomena in the phase space.

In this section we define the WDF in optics, we elucidate its concept by some simple examples, and we give some of its properties.

A. Definition of the Wigner Distribution Function

The WDF of a stochastic process can be defined in terms of the positional power spectrum by

$$F(x, u) = \int \Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x')\exp(-iux')\mathrm{d}x' \tag{3.1a}$$

or, equivalently, in terms of the directional power spectrum, by

$$F(x, u) = \frac{1}{2\pi}\int \bar{\Gamma}(u + \tfrac{1}{2}u', u - \tfrac{1}{2}u')\exp(iu'x)\mathrm{d}u'. \tag{3.1b}$$

A distribution function according to definitions (3.1) was first introduced in optics by Walther,[2] who called it the *generalized radiance*.

The WDF $F(x, u)$ represents a stochastic signal in space and (spatial) frequency simultaneously and is thus a member of a wide class of phase-space distribution functions.[10,11] It forms an intermediate signal description between the pure space representation $\Gamma(x_1, x_2)$ and the pure frequency representation $\bar{\Gamma}(u_1, u_2)$. Furthermore, this simultaneous space–frequency description closely resembles the ray concept in geometrical optics, in which the position and direction of a ray are also given simultaneously. In a way, $F(x, u)$ is the amplitude of a ray passing through the point x and having a frequency (i.e., direction) u.

B. Examples of Wigner Distribution Functions

Before we mention some properties of the WDF, we first illustrate its concept with some simple examples.

(1) Spatially *incoherent* light can be described by its positional power spectrum, which reads as $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x') = p(x)\delta(x')$, where the intensity $p(x)$ is a nonnegative function. The corresponding WDF takes the form $F(x, u) = p(x)$; note that it is a function only of the space variable x and that it does not depend on u.

(2) As a second example, we consider light that is *dual* to incoherent light, i.e., light whose frequency behavior is similar to the space behavior of incoherent light and vice versa. Such light is *spatially stationary* light. The positional power spectrum of spatially stationary light reads as $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x') = s(x')$; its directional power spectrum thus reads as $\bar{\Gamma}(u + \frac{1}{2}u', u - \frac{1}{2}u') = \bar{s}(u)\delta(u')$, where the nonnegative function $\bar{s}(u)$ is the Fourier transform of $s(x')$:

$$\bar{s}(u) = \int s(x')\exp(-iux')\mathrm{d}x'. \tag{3.2}$$

Note that, indeed, the directional power spectrum of spatially stationary light has a form that is similar to the positional power spectrum of incoherent light. The duality between incoherent light and spatially stationary light is, in fact, the van Cittert–Zernike theorem.

The WDF of spatially stationary light reads as $F(x, u) = \bar{s}(u)$; note that it is a function only of the frequency variable u and that it does not depend on x. It thus has the same form as the WDF of incoherent light, except that it is rotated through 90° in the space–frequency plane.

(3) Incoherent light and spatially stationary light are special cases of so-called *quasi-homogeneous* light.[9,12,13] Such quasi-homogeneous light can be locally considered as spatially stationary, having, however, a slowly varying intensity. It can be represented by a positional power spectrum such as $\Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x') \simeq p(x)s(x')$, where p is a slowly varying function compared with s. The WDF of quasi-homogeneous light takes the form of a product: $F(x, u) \simeq p(x)\bar{s}(u)$; both $p(x)$ and $\bar{s}(u)$ are nonnegative, which implies that the WDF is nonnegative. The special case of incoherent light arises for $\bar{s}(u) = 1$, whereas for spatially stationary light we have $p(x) = 1$.

(4) Let us consider *Gaussian* light, also known as Gaussian Schell-model light,[14,15] whose positional power spectrum reads as follows:

$$\Gamma(x_1, x_2) = \frac{\sqrt{2\sigma}}{\rho} \exp\left\{-\frac{\pi}{2\rho^2}\left[\sigma(x_1 + x_2)^2 + \frac{1}{\sigma}(x_1 - x_2)^2\right]\right\} \qquad (\rho > 0, \quad 0 < \sigma \le 1); \quad (3.3)$$

the positive factor ρ is a mere scaling factor, whereas σ is a measure of the coherence of the Gaussian light. The nonnegative definiteness of the power spectrum requires that σ be bounded by 0 and 1; $\sigma = 1$ leads to Gaussian light that is completely coherent, whereas $\sigma \rightarrow 0$ leads to the incoherent limit. The WDF of such Gaussian light takes the form

$$F(x, u) = 2\sigma \exp\left[-\sigma\left(\frac{2\pi}{\rho^2}x^2 + \frac{\rho^2}{2\pi}u^2\right)\right] \qquad (\rho > 0, 0 < \sigma \le 1), \quad (3.4)$$

which is again Gaussian both in x and in u.

(5) Completely *coherent* light is our final example. Its positional power spectrum $\Gamma(x_1, x_2) = q(x_1)q^*(x_2)$ has the form of a product of a function with its complex-conjugate version.[7] The WDF of coherent light thus takes the form

$$f(x, u) = \int q(x + \tfrac{1}{2}x')q^*(x - \tfrac{1}{2}x')\exp(-iux')\mathrm{d}x'. \quad (3.5)$$

We denote the WDF of coherent light throughout by the lower-case letter f.

C. Properties of the Wigner Distribution Function

Let us now consider some properties of the WDF. We consider only the most important ones; they can all be derived directly from the definitions (3.1). Additional properties of the WDF, especially of the WDF in the completely coherent case, as defined by Eq. (3.5), can be found elsewhere.[3,16–23]

(1) The definition (3.1a) of the WDF $F(x, u)$ has the form of a *Fourier transformation* of the positional power spectrum $\Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x')$, with x' and u as conjugated variables and with x as a parameter. The positional power spectrum can thus be reconstructed from the WDF simply by applying an inverse Fourier transformation; a similar property holds for the directional power spectrum. The latter property follows from the general remark that space and frequency, or position and direction, play equivalent roles in the WDF: If we interchange the roles of x and u in any expression containing a WDF, we get an expression that is the dual of the original one. Thus, when the original expression describes a property in the space domain, the dual expression describes a similar property in the frequency domain and vice versa.

(2) The WDF is *real*. Unfortunately, it is not necessarily nonnegative; this prohibits a direct interpretation of the WDF as an energy density function (or radiance function). Friberg has shown[24] that it is not possible to define a radiance function that satisfies all the physical requirements from radiometry; in particular, as we see, the WDF has the physically unattractive property that it may take negative values.

(3) If the signal is *limited* to a certain space or frequency interval and vanishes outside that interval, then its WDF is limited to the same interval. Indeed, if the positional power spectrum $\Gamma(x_1, x_2)$ vanishes for x_1 or $x_2 > x_0$, say, then its WDF $F(x, u)$ vanishes for $x > x_0$ as well; a similar property holds for the directional power spectrum. Hence, for instance, for a light source with a finite extent, the WDF vanishes outside the source, which is surely a physically attractive property.

(4) A space or frequency *shift* of the signal yields the same shift for its WDF. Indeed, if the WDF $F(x, u)$ corresponds to the positional power spectrum $\Gamma(x_1, x_2)$, then $F(x - x_0, u)$ corresponds to $\Gamma(x_1 - x_0, x_2 - x_0)$; a similar property holds for the directional power spectrum.

(5) Several integrals of the WDF have clear physical meanings and can be interpreted as radiometric quantities. The integral over the frequency variable, for instance,

$$\frac{1}{2\pi}\int F(x, u)\mathrm{d}u = \Gamma(x, x) \quad (3.6a)$$

represents the *positional intensity* of the signal, whereas the integral over the space variable

$$\int F(x, u)\mathrm{d}x = \bar{\Gamma}(u, u) \quad (3.6b)$$

yields the *directional intensity* of the signal, which is, apart from the usual factor $\cos^2\theta$ (where θ is the angle of observation), proportional to the *radiant intensity*.[12,13] The *total energy* of the signal follows from the integral over the entire space–frequency plane

$$\frac{1}{2\pi}\iint F(x, u)\mathrm{d}x\mathrm{d}u = \int \Gamma(x, x)\mathrm{d}x = \frac{1}{2\pi}\int \bar{\Gamma}(u, u)\mathrm{d}u. \quad (3.6c)$$

The normalized second-order moment of the WDF with respect to the space variable x yields the *effective width* d_x of the positional intensity $\Gamma(x, x)$:

$$\frac{\dfrac{1}{2\pi}\iint x^2 F(x, u)\mathrm{d}x\mathrm{d}u}{\dfrac{1}{2\pi}\iint F(x, u)\mathrm{d}x\mathrm{d}u} = \frac{\int x^2\Gamma(x, x)\mathrm{d}x}{\int \Gamma(x, x)\mathrm{d}x} = d_x^{\,2}; \quad (3.7)$$

a similar relation holds for the effective width d_u of the directional intensity $\bar{\Gamma}(u, u)$. The *radiant emittance*[12,13] is equal to the integral

$$J_z(x) = \frac{1}{2\pi} \int \frac{(k^2 - u^2)^{1/2}}{k} F(x, u)\mathrm{d}u, \tag{3.8a}$$

in which k represents the usual wave number $2\pi/\lambda$. When we combine the radiant emittance J_z with the integral

$$J_x(x) = \frac{1}{2\pi} \int \frac{u}{k} F(x, u)\mathrm{d}u, \tag{3.8b}$$

we can construct the two-dimensional vector

$$\mathbf{J}(x) = [J_x(x), J_z(x)], \tag{3.8c}$$

which is known as the *geometrical vector flux*.[25] The total *radiant flux*[12] follows from integrating the radiant emittance over the space variable x.

(6) An important relationship between the WDF's of two signals and the power spectra of these signals reads as

$$\frac{1}{2\pi} \iint F_1(x, u)F_2(x, u)\mathrm{d}x\mathrm{d}u = \iint \Gamma_1(x_1, x_2)\Gamma^*{}_2(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

$$= \frac{1}{4\pi^2} \iint \bar{\Gamma}_1(u_1, u_2)\bar{\Gamma}^*{}_2(u_1, u_2)\mathrm{d}u_1\mathrm{d}u_2. \tag{3.9}$$

This relationship has an application in averaging one WDF with another one, which averaging always yields a nonnegative result. We show this in Section 4, after having introduced the modal expansion for the power spectrum.

4. MODAL EXPANSIONS

To derive more properties of the WDF, we introduce *modal expansions* for the power spectrum and the WDF.

A. Modal Expansion of the Positional Power Spectrum

We represent the positional power spectrum $\Gamma(x_1, x_2)$ by its modal expansion[26] [see also, for instance, Refs. 15 and 27, in which a modal expansion of the (nonnegative definite Hermitian) mutual intensity $\bar{\Gamma}(x_1, x_2, 0)$ is given]:

$$\Gamma(x_1, x_2) = \frac{1}{\rho} \sum_{m=0}^{\infty} \lambda_m q_m(x_1/\rho)q^*{}_m(x_2/\rho); \tag{4.1}$$

a similar expansion holds for the directional power spectrum. For the mathematical subtleties of this modal expansion, we refer to the standard mathematical literature.[28,29] In the modal expansion (4.1), the functions q_m are the *eigenfunctions*, and the numbers λ_m are the *eigenvalues* of the integral equation

$$\int \Gamma(x_1, x_2)q_m(x_2/\rho)\mathrm{d}x_2 = \lambda_m q_m(x_1/\rho) \qquad (m = 0, 1, \ldots); \tag{4.2}$$

the positive factor ρ is, again, a mere scaling factor. Since the kernel $\Gamma(x_1, x_2)$ is *Hermitian* and under the assumption of *discrete* eigenvalues, the eigenfunctions can be made orthonormal:

$$\int q_m(\xi)q^*{}_n(\xi)\mathrm{d}\xi = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \qquad (m, n = 0, 1, \ldots). \tag{4.3}$$

Moreover, since the kernel $\Gamma(x_1, x_2)$ is *nonnegative definite* Hermitian, the eigenvalues are nonnegative. Note that the light is completely coherent if there is only *one* nonvanishing eigenvalue. As a matter of fact, the modal expansion (4.1) expresses the partially coherent light as a superposition of coherent modes.

As an example, we remark that the eigenvalues of the Gaussian light [Eq. (3.3)] take the form[15,30,31]

$$\lambda_m = \frac{2\sigma}{1 + \sigma}\left(\frac{1 - \sigma}{1 + \sigma}\right)^m \qquad (0 < \sigma \leq 1, m = 0, 1, \ldots), \tag{4.4}$$

whereas the eigenfunctions q_m are just the *Hermite functions* ψ_m defined, for instance, by means of the generating function

$$\exp[\pi\xi^2 - 2\pi(\xi - w)^2] = 2^{-1/4} \sum_{m=0}^{\infty} (m!)^{-1/2}(4\pi)^{m/2}w^m\psi_m(\xi). \tag{4.5}$$

Note that, for $\sigma = 1$, the eigenvalue λ_0 is the only nonvanishing eigenvalue and that the Gaussian light is completely coherent.

B. Modal Expansion of the Wigner Distribution Function

When we substitute the modal expansion (4.1) into the definition [Eq. (3.1a)], the WDF can be expressed as

$$F(x, u) = \sum_{m=0}^{\infty} \lambda_m f_m\left(\frac{x}{\rho}, \rho u\right), \tag{4.6}$$

where

$$f_m(\xi, \eta) = \int q_m(\xi + \tfrac{1}{2}\xi')q^*{}_m(\xi - \tfrac{1}{2}\xi')\exp(-i\eta\xi')\mathrm{d}\xi' \qquad (m = 0, 1, \ldots) \tag{4.7}$$

are the WDF's of the eigenfunctions q_m, as in the completely coherent case [see definition (3.5)]. By applying relation (3.9) and using the orthonormality property [Eq. (4.3)], it can easily be seen that the WDF's f_m satisfy the orthonormality relation

$$\frac{1}{2\pi} \iint f_m(\xi, \eta)f_n(\xi, \eta)\mathrm{d}\xi\mathrm{d}\eta = \left|\int q_m(\xi)q^*{}_n(\xi)\mathrm{d}\xi\right|^2$$

$$= \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \qquad (m, n = 0, 1, \ldots). \tag{4.8}$$

As an example, again, it can be shown that the WDF's of the Hermite functions ψ_m, and thus the WDF's f_m that appear in the modal expansion of Gaussian light, take the form[32]

$$f_m(\xi, \eta) = 2(-1)^m \exp\left[-\left(2\pi\xi^2 + \frac{\eta^2}{2\pi}\right)\right]L_m\left[2\left(2\pi\xi^2 + \frac{\eta^2}{2\pi}\right)\right] \qquad (m = 0, 1, \ldots), \tag{4.9}$$

where L_m are the Laguerre polynomials.[33]

The modal expansion (4.6) allows us to formulate some interesting inequalities for the WDF.

C. Inequalities for the Wigner Distribution Function

(1) Using the expansion (4.6), it is easy to see that de Bruijn's inequality[34]

$$\frac{1}{2\pi}\iint\left(\frac{2\pi}{\rho^2}x^2+\frac{\rho^2}{2\pi}u^2\right)^n F(x,u)\mathrm{d}x\mathrm{d}u \geq n!\frac{1}{2\pi}\iint F(x,u)\mathrm{d}x\mathrm{d}u \tag{4.10}$$

holds not only in the completely coherent case but also for the WDF of partially coherent light. In the special case $n = 1$, relation (4.10) reduces to $(2\pi/\rho^2)d_x^2 + (\rho^2/2\pi)d_u^2 \geq 1$, which leads to the ordinary uncertainty relation[6] $2d_xd_u \geq 1$, by choosing $\rho^2 = 2\pi d_x/d_u$. The equality sign in this ordinary uncertainty relation occurs for completely coherent Gaussian light [$\sigma = 1$ in Eq. (3.4)]; for all other signals, the product of the effective widths in the space and the frequency direction is larger. We thus conclude that the coherent Gaussian WDF occupies the smallest possible area in the space–frequency plane. A more sophisticated uncertainty principle for partially coherent light, which will take into account the overall degree of coherence of the light, is derived in Subsection 6.A.

(2) Using the relationship (3.9) and expanding the power spectra $\Gamma_1(x_1, x_2)$ and $\Gamma_2(x_1, x_2)$ in the form (4.1), it can readily be shown that

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)\mathrm{d}x\mathrm{d}u \geq 0. \tag{4.11}$$

Thus, as we remarked before, averaging one WDF with another always yields a nonnegative result. In particular, the averaging with the WDF of completely coherent Gaussian light is of some practical importance[34–36] since the coherent Gaussian WDF occupies the smallest possible area in the space–frequency plane, as we concluded before.

(3) An upper bound for the expression that arises in relation (4.11) can be found by applying Schwarz's inequality[6]:

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)\mathrm{d}x\mathrm{d}u \leq \left[\frac{1}{2\pi}\iint F_1^2(x,u)\mathrm{d}x\mathrm{d}u\right]^{1/2}\left[\frac{1}{2\pi}\iint F_2^2(x,u)\mathrm{d}x\mathrm{d}u\right]^{1/2}. \tag{4.12a}$$

The right-hand side of relation (4.12a) again has an upper bound, which leads to the inequality

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)\mathrm{d}x\mathrm{d}u \leq \left[\frac{1}{2\pi}\iint F_1(x,u)\mathrm{d}x\mathrm{d}u\right]\left[\frac{1}{2\pi}\iint F_2(x,u)\mathrm{d}x\mathrm{d}u\right], \tag{4.12b}$$

where the right-hand side is simply the product of the total energies of the signals; indeed, we have the important inequality

$$\frac{1}{2\pi}\iint F^2(x,u)\mathrm{d}x\mathrm{d}u \leq \left[\frac{1}{2\pi}\iint F(x,u)\mathrm{d}x\mathrm{d}u\right]^2. \tag{4.13}$$

To prove this inequality, we first remark that, by using the expansion (4.6), the identity

$$\frac{1}{2\pi}\iint F(x,u)\mathrm{d}x\mathrm{d}u = \sum_{m=0}^{\infty}\lambda_m \tag{4.14a}$$

holds. Second, we observe the identity

$$\frac{1}{2\pi}\iint F^2(x,u)\mathrm{d}x\mathrm{d}u = \sum_{m=0}^{\infty}\lambda_m^2, \tag{4.14b}$$

which can be easily proved by applying expansion (4.6) and by using the orthonormality property (4.8). Finally, we remark that, since all eigenvalues λ_m are nonnegative, the inequality

$$\sum_{m=0}^{\infty}\lambda_m^2 \leq \left(\sum_{m=0}^{\infty}\lambda_m\right)^2 \tag{4.15}$$

holds, which completes the proof of relation (4.13). Note that the equality sign in relation (4.15), and hence in relation (4.13), holds if there is only one nonvanishing eigenvalue, i.e., in the case of complete coherence. The quotient of the two expressions that arise in relation (4.13) or relation (4.15) can therefore serve as a measure of the overall degree of coherence of the light.

5. PROPAGATION OF THE WIGNER DISTRIBUTION FUNCTION

In this section we study how the WDF propagates through linear optical systems. In Subsection 5.A we therefore consider an optical system as a *black box*, with an input plane and an output plane, whereas in Subsection 5.B we consider the system as a *continuous medium*, in which the signal must satisfy a certain differential equation.

A. Ray-Spread Function of an Optical System

We consider the propagation of the WDF through linear systems. A linear system can be represented in four different ways, depending on whether we describe the input and the output signals in the space or in the frequency domain. We thus have four equivalent input–output relationships, which for completely coherent light read as

$$q_0(x_0) = \int h_{xx}(x_0, x_i)q_i(x_i)\mathrm{d}x_i, \tag{5.1a}$$

$$\bar{q}_0(u_0) = \int h_{ux}(u_0, x_i)q_i(x_i)\mathrm{d}x_i, \tag{5.1b}$$

$$q_0(x_0) = \frac{1}{2\pi}\int h_{xu}(x_0, u_i)\bar{q}_i(u_i)\mathrm{d}u_i, \tag{5.1c}$$

$$\bar{q}_0(u_0) = \frac{1}{2\pi}\int h_{uu}(u_0, u_i)\bar{q}_i(u_i)\mathrm{d}u_i. \tag{5.1d}$$

The first relation (5.1a) is the usual system representation in the space domain by means of the *point-spread function* h_{xx}; we remark that the function h_{xx} is the response of the system in the space domain when the input signal is a point source. The last relation (5.1d) is a similar system representation in the frequency domain, where the function h_{uu} is the reponse of the system in the frequency domain when the input signal is a plane wave; therefore we can call h_{uu} the *wave-spread function* of the system. The remaining two relations (5.1b) and (5.1c) are hybrid system representations since the input and the output signals are described in different domains; therefore we can call the functions h_{ux} and h_{xu} *hybrid spread functions*.

For partially coherent light, the four input-output relationships that are equivalent to relations (5.1a)–(5.1d) read as

$$\Gamma_0(x_1, x_2) = \iint h_{xx}(x_1, \xi_1)\Gamma_i(\xi_1, \xi_2)h^*{}_{xx}(x_2, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{5.2a}$$

$$\bar{\Gamma}_0(u_1, u_2) = \iint h_{ux}(u_1, \xi_1)\Gamma_i(\xi_1, \xi_2)h^*{}_{ux}(u_2, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{5.2b}$$

$$\Gamma_0(x_1, x_2) = \frac{1}{4\pi^2}\iint h_{xu}(x_1, \eta_1)\bar{\Gamma}_i(\eta_1, \eta_2)h^*{}_{xu}(x_2, \eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2, \tag{5.2c}$$

$$\bar{\Gamma}_0(u_1, u_2) = \frac{1}{4\pi^2}\iint h_{uu}(u_1, \eta_1)\bar{\Gamma}_i(\eta_1, \eta_2)h^*{}_{uu}(u_2, \eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2. \tag{5.2d}$$

Unlike the *four* system representations [Eqs. (5.1) or (5.2)] described above, there is only *one* system representation when we describe the input and the output signals by their WDF's. Indeed, combining the system representations [Eqs. (5.2)] with the definitions (3.1) of the WDF results in the relation

$$F_0(x_0, u_0) = \frac{1}{2\pi}\iint K(x_0, u_0, x_i, u_i)F_i(x_i, u_i)\mathrm{d}x_i\mathrm{d}u_i, \tag{5.3}$$

in which the WDF's of the input and the output signals are related through a superposition integral. The function K is completely determined by the system and can be expressed in terms of the four system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu}. We find that

$$\begin{aligned} K(x_0, u_0, x_i, u_i) = \iint & h_{xx}(x_0 + \tfrac{1}{2}x'_0, x_i + \tfrac{1}{2}x'_i) \\ & \times h^*{}_{xx}(x_0 - \tfrac{1}{2}x'_0, x_i - \tfrac{1}{2}x'_i) \\ & \times \exp[-i(u_0x'_0 - u_ix'_i)]\mathrm{d}x'_0\mathrm{d}x'_i \end{aligned} \tag{5.4}$$

and similar expressions for the other system functions.[37] Relation (5.4) can be considered as the definition of a *double* WDF; hence the function K has all the properties of a WDF, for instance, the property of realness.

Let us think about the physical meaning of the function K. In a formal way, the function K is the response of the system in the space–frequency domain when the input signal is described by a product of two Dirac functions $F_i(x, u) = 2\pi\delta(x - x_i)\delta(u - u_i)$; only in a formal way, since an actual input signal yielding such a WDF does not exist. Nevertheless, such an input signal could be considered as a *single ray* entering the system at the position x_i with direction u_i. Hence the function K might be called the *ray-spread function* of the system.

It is not difficult to express the ray-spread function of a *cascade* of two systems in terms of the respective ray-spread functions K_1 and K_2. The ray-spread function of the overall system reads as

$$K(x_0, u_0, x_i, u_i) = \frac{1}{2\pi}\iint K_2(x_0, u_0, x, u)K_1(x, u, x_i, u_i)\mathrm{d}x\mathrm{d}u. \tag{5.5}$$

Some examples of ray-spread functions of elementary optical systems[38,39] might elucidate the concept of the ray-spread function.

(1) First, let us consider a thin *lens* with a focal distance f. Its point-spread function takes the well-known form $h_{xx}(x_0, x_i) = \exp[-i(k/2f)x_0^2]\delta(x_0 - x_i)$. Clearly, a thin lens is a modulator, whose modulation function is a quadratic-phase function. The corresponding ray-spread function takes the special form of two Dirac functions, $K(x_0, u_0, x_i, u_i) = 2\pi\delta(x_i - x_0)\delta(u_i - u_0 - kx_0/f)$, and the input–output relationship of a thin lens becomes very simple, $F_0(x, u) = F_i(x, u + kx/f)$. The ray-spread function represents exactly the geometric-optical behavior of a thin lens: If a single ray is incident upon a thin lens, it will leave the lens from the same position, but its direction will change according to the actual position; in any event, there is only one output ray.

(2) Our second example will be the dual of a thin lens, namely, a section of *free space* over a distance z in the Fresnel approximation. Its point-spread function reads as $h_{xx}(x_0, x_i) = (k/2\pi iz)^{1/2}\exp[i(k/2z)(x_0 - x_i)^2]$. Clearly, a section of free space is a shift-invariant system, and the point-spread function is again a quadratic-phase function. The corresponding ray-spread function again takes the special form of two Dirac functions, $K(x_0, u_0, x_i, u_i) = 2\pi\delta(x_i - x_0 + zu_0/k)\delta(u_i - u_0)$, and the input–output relationship of a section of free space becomes very simple, $F_0(x, u) = F_i(x - zu/k, u)$. The ray-spread function represents exactly the geometric-optical behavior of a section of free space: If a single ray propagates through free space, its direction will remain the same, but its position will change according to the actual direction; in any event, there is again only one output ray.

(3) For a *Fourier transformer* whose point-spread function reads as $h_{xx}(x_0, x_i) = (\beta/2\pi i)^{1/2}\exp(-i\beta x_0x_i)$, the ray-spread function takes the form $K(x_0, u_0, x_i, u_i) = 2\pi\delta(x_i + u_0/\beta)\delta(u_i - \beta x_0)$, and the input–output relationship reduces to $F_0(x, u) = F_i(-u/\beta, \beta x)$. We conclude that the space and frequency domains are interchanged, as can be expected for a Fourier transformer.

(4) Let a *magnifier* be represented by a point-spread function $h_{xx}(x_0, x_i) = m^{1/2}\delta(mx_0 - x_i)$; then its ray-spread function will read as $K(x_0, u_0, x_i, u_i) = 2\pi\delta(x_i - mx_0)\delta(u_i - u_0/m)$, and the input–output relationship becomes $F_0(x, u) = F_i(mx, u/m)$. We note that the space and frequency domains are merely scaled, as can be expected for a magnifier.

(5) A thin lens, a section of free space in the Fresnel approximation, a Fourier transformer, and a magnifier are special cases of Luneburg's *first-order optical systems*,[40] which will be our final example. A first-order optical system can, of course, be characterized by its system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu}: They are all quadratic-phase functions. (Note that a Dirac function can be considered as a limiting case of such a quadratically varying function.) A system representation in terms of WDF's, however, is far more elegant. The ray-spread function of a first-order system takes the form of two Dirac functions,

$$K(x_0, u_0, x_i, u_i) = 2\pi\delta(x_i - Ax_0 - Bu_0)\delta(u_i - Cx_0 - Du_0), \tag{5.6}$$

and the input–output relationship reads very simply as

$$F_0(x, u) = F_i(Ax + Bu, Cx + Du). \tag{5.7}$$

From the ray-spread function [Eq. (5.6)], we conclude that a single input ray, entering the system at the position x_i with direction u_i, will yield a single output ray, leaving the system at the position x_0 with direction u_0. The input and output positions and directions are related by the matrix relationship

$$\begin{pmatrix} x_i \\ u_i \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \tag{5.8}$$

Relation (5.8) is a well-known geometric-optical matrix description of a first-order optical system[40]; the *ABCD* matrix in this relation is known as the *ray-transformation matrix*.[41] We remark that this *ABCD* matrix is *symplectic*[40–42]; for a 2 × 2 matrix, symplecticity can be expressed by the condition $AD - BC = 1$. We observe again a perfect resemblance to the geometric-optical behavior of a first-order optical system (see also, for instance, Ref. 43).

B. Transport Equations for the Wigner Distribution Function

In the previous subsection we studied, in example (2), the propagation of the WDF through free space by considering a section of free space as an optical system with an input plane and an output plane. It is possible, however, to find the propagation of the WDF through free space directly from the differential equation that the signal must satisfy. We therefore let the longitudinal variable z enter into the formulas and remark that the propagation of coherent light in free space (at least in the Fresnel approximation) is governed by the differential equation (see Ref. 6, p. 358)

$$-i\,\frac{\partial q}{\partial z} = \left(k + \frac{1}{2k}\,\frac{\partial^2}{\partial x^2}\right) q; \tag{5.9}$$

partially coherent light must satisfy the differential equation

$$-i\,\frac{\partial \Gamma}{\partial z} = \left[\left(k + \frac{1}{2k}\,\frac{\partial^2}{\partial {x_1}^2}\right) - \left(k + \frac{1}{2k}\,\frac{\partial^2}{\partial {x_2}^2}\right)\right]\Gamma. \tag{5.10}$$

The propagation of the WDF is now described by a *transport equation*,[44–49] which in this case takes the form

$$\frac{u}{k}\,\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} = 0. \tag{5.11}$$

[Relation (5.11) is a special case of the more general transport equations (5.14), which are studied below.] The transport equation (5.11) has the solution

$$F(x, u; z) = F\left(x - \frac{u}{k} z, u; 0\right), \tag{5.12}$$

which is equivalent to the result in the previous subsection.

The differential equation (5.10) is a special case of the more general equation

$$-i\,\frac{\partial \Gamma}{\partial z} = \left[L\left(x_1, -i\,\frac{\partial}{\partial x_1}; z\right) - L^*\left(x_2, -i\,\frac{\partial}{\partial x_2}; z\right)\right]\Gamma, \tag{5.13}$$

where L is some explicit function of the space variables x and z and of the partial derivative of Γ contained in the operator $\partial/\partial x$. The transport equation that corresponds to this differential equation reads as

$$-\frac{\partial F}{\partial z} = 2\,\mathrm{Im}\left[L\left(x + \frac{i}{2}\,\frac{\partial}{\partial u}, u - \frac{i}{2}\,\frac{\partial}{\partial x}; z\right)\right]F, \tag{5.14a}$$

in which Im denotes the imaginary part; a derivation of this formula can be found in Appendix A. In the elegant, symbolic notation of Besieris and Tappert,[46] the transport equation (5.14a) takes the form

$$-\frac{\partial F}{\partial z} = 2\,\mathrm{Im}\left\{L(x, u; z)\exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\,\frac{\overrightarrow{\partial}}{\partial u} - \frac{\overleftarrow{\partial}}{\partial u}\,\frac{\overrightarrow{\partial}}{\partial x}\right)\right]\right\}F, \tag{5.14b}$$

where, depending on the directions of the arrows, the differential operators on the right-hand side operate on $L(x, u; z)$ or $F(x, u; z)$. In the *Liouville approximation* (or *geometric-optical approximation*) the transport equation (5.14b) reduces to

$$-\frac{\partial F}{\partial z} = 2\,\mathrm{Im}\left\{L(x, u; z)\left[1 + \frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\,\frac{\overrightarrow{\partial}}{\partial u} - \frac{\overleftarrow{\partial}}{\partial u}\,\frac{\overrightarrow{\partial}}{\partial x}\right)\right]\right\}F. \tag{5.15a}$$

Again, in the usual notation, the latter equation reads as

$$-\frac{\partial F}{\partial z} = 2\,(\mathrm{Im}\,L)F + \frac{\partial\,\mathrm{Re}\,L}{\partial x}\,\frac{\partial F}{\partial u} - \frac{\partial\,\mathrm{Re}\,L}{\partial u}\,\frac{\partial F}{\partial x}, \tag{5.15b}$$

in which Re denotes the real part. Relation (5.15b) is a first-order partial differential equation, which can be solved by the methods of characteristics[50]: Along a path described in a parameter notation by $x = x(s)$, $z = z(s)$, and $u = u(s)$ and defined by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{\partial\,\mathrm{Re}\,L}{\partial u}, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\partial\,\mathrm{Re}\,L}{\partial x}; \tag{5.16}$$

the *partial* differential equation (5.15b) reduces to the *ordinary* differential equation

$$-\frac{\mathrm{d}F}{\mathrm{d}s} = 2\,(\mathrm{Im}\,L)F. \tag{5.17}$$

In the special case that $L(x, u; z)$ is a *real* function of x, u, and z, Eq. (5.17) implies that, along the path defined by relations (5.16), the WDF has a *constant* value (see also, for instance, Ref. 51).

Let us consider some examples of transport equations.

(1) *In Free Space in the Fresnel Approximation.* The signal is governed by equation (5.10), and the function L reads as $L(x, u; z) = k - u^2/2k$. The corresponding transport equation (5.11) and its solution (5.12) have already been mentioned in the first paragraph of this subsection.

(2) *In Free Space (But Not Necessarily in the Fresnel Approximation).* A coherent signal must satisfy the Helmholtz equation, whereas the propagation of partially coherent light is governed by the differential equation

$$-i\,\frac{\partial \Gamma}{\partial z} = \left[\left(k^2 + \frac{\partial^2}{\partial {x_1}^2}\right)^{1/2} - \left(k^2 + \frac{\partial^2}{\partial {x_2}^2}\right)^{1/2}\right]\Gamma. \tag{5.18}$$

In this case, the function L reads as $L(x, u; z) = (k^2 - u^2)^{1/2}$. We can again derive a transport equation for the WDF; the exact transport equation is rather complicated, but in the Liouville approximation it takes the simple form

$$\frac{u}{k}\,\frac{\partial F}{\partial x} + \frac{(k^2 - u^2)^{1/2}}{k}\,\frac{\partial F}{\partial z} = 0. \tag{5.19}$$

This transport equation can again be solved explicitly, and the solution reads as

$$F(x, u; z) = F\left[x - \frac{u}{(k^2 - u^2)^{1/2}} z, u; 0\right]. \tag{5.20}$$

The difference from the previous solution (5.12), in which we considered the Fresnel approximation, is that the sine u/k has been replaced by the tangent $u/(k^2 - u^2)^{1/2}$. Note that, in the Fresnel approximation, relations (5.18)–(5.20) reduce to relations (5.10)–(5.12), respectively. When we integrate the transport equation (5.19) over the frequency variable u and use definitions (3.8), we get the relation $\partial J_x/\partial x + \partial J_z/\partial z = 0$, which shows that the geometrical vector flux $\mathbf{J}$ has zero divergence.[25]

(3) *In a Weakly Inhomogeneous Medium.* The differential equation that the signal must satisfy again has the form of Eq. (5.18) but now with $k = k(x, z)$. The transport equation in the Liouville approximation now takes the form

$$\frac{u}{k}\frac{\partial F}{\partial x} + \frac{(k^2 - u^2)^{1/2}}{k}\frac{\partial F}{\partial z} + \frac{\partial k}{\partial x}\frac{\partial F}{\partial u} = 0, \tag{5.21}$$

which, in general, cannot be solved explicitly. With the method of characteristics we conclude that, along a path defined by

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{u}{k}, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{(k^2 - u^2)^{1/2}}{k}, \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\partial k}{\partial x}, \tag{5.22}$$

the WDF has a constant value. When we eliminate the frequency variable u from Eqs. (5.22), we are immediately led to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(k\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{\partial k}{\partial x}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(k\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial k}{\partial z}, \tag{5.23}$$

which are the equations for an optical light ray in geometrical optics.[52] We are thus led to the general conclusion that, in the Liouville approximation, the WDF has a constant value along the geometric-optical ray paths. Note that, in a homogeneous medium, i.e., $\partial k/\partial x = \partial k/\partial z = 0$, the transport equation (5.21) reduces to Eq. (5.19) and that the ray paths become straight lines.

6. APPLICATIONS

We have already considered a number of simple applications of the WDF in the previous sections of the paper. In fact, any example that we have considered represents such an application. In this section we study some more-advanced applications.

A. Uncertainty Principle and Informational Entropy

The ordinary uncertainty principle $2d_x d_u \geq 1$ tells us that the product of the effective widths of the intensity functions in the space and the frequency domain has a lower bound and that this lower bound is reached when the light is completely coherent and Gaussian.[6] We found the same uncertainty relation for partially coherent light in Subsection 4.C. In this subsection we derive a more sophisticated uncertainty principle,[53] by taking into account the overall degree of coherence of the light. As a matter of fact, what we expect from an uncertainty principle for partially coherent light is that the product of the effective widths still has a lower bound but that this lower bound depends on the overall degree of coherence of the light: The better the coherence, the smaller the lower bound. Hence we need a measure of the overall degree of coherence of the light. To define such a measure, we use the modal expansion (4.6) and choose Shannon's *informational entropy* defined by the expression (see, for instance, Ref. 27)

$$-\sum_{m=0}^{\infty}\left(\frac{\lambda_m}{\sum_{n=0}^{\infty}\lambda_n}\right)\ln\left(\frac{\lambda_m}{\sum_{n=0}^{\infty}\lambda_n}\right) \tag{6.1}$$

to measure the overall degree of coherence.

To find a more advanced uncertainty principle, we formulate the important relationship[54,55]

$$2d_x d_u \geq \frac{\sum_{m=0}^{\infty}\lambda_m(2m+1)}{\sum_{m=0}^{\infty}\lambda_m}, \tag{6.2}$$

proof of which can be found in Refs. 54 and 55. The equality sign in this relation holds if the eigenfunctions q_m are just the Hermite functions ψ_m. We now wish to solve the following problem: among all partially coherent wave fields with the same informational entropy, find the wave field that minimizes the product $d_x d_u$. To solve this problem, we have to find that distribution of eigenvalues λ_m for which the right-hand side of relation (6.2) takes its minimum value, under the constraints that the eigenvalues are nonnegative and that the informational entropy is constant. Using standard variation techniques, it is not difficult to show that the minimum of this right-hand side occurs when the eigenvalues are proportional to expression (4.4) and that this minimum takes the value $1/2\sigma$; the quantity σ in expression (4.4) is related to the informational entropy through the formula

$$\exp\left[\sum_{m=0}^{\infty}\left(\frac{\lambda_m}{\sum_{n=0}^{\infty}\lambda_n}\right)\ln\left(\frac{\lambda_m}{\sum_{n=0}^{\infty}\lambda_n}\right)\right] = \frac{2\sigma}{\sqrt{1-\sigma^2}}\left(\frac{1-\sigma}{1+\sigma}\right)^{1/2\sigma}. \tag{6.3}$$

Note that the case $\sigma = 1$ corresponds to the case of complete coherence for which there is only one nonvanishing eigenvalue and for which the informational entropy is zero.

If the eigenvalues are given by Eq. (4.4) and, moreover, if the eigenfunctions q_m are the Hermite functions ψ_m, then the corresponding power spectrum of the partially coherent light is Gaussian and can be expressed in the form of Eq. (3.3). For such Gaussian light, we have $(2\pi/\rho^2)d_x^2 = (\rho^2/2\pi)d_u^2 = 1/2\sigma$, and thus $2d_x d_u = 1/\sigma$. We conclude that an uncertainty principle for partially coherent light reads as

$$2d_x d_u \geq \frac{1}{\sigma}, \tag{6.4}$$

where σ is related to the informational entropy through Eq. (6.3) and where the equality sign holds if the light is Gaussian (but not necessarily coherent).

We can, of course, choose other quantities to measure the overall degree of coherence of the light[54,55]; however, an overall degree of coherence based on the informational entropy has certain advantages.[53] All the quantities that measure the overall degree of coherence of the light, which are based on the eigenvalues λ_m (and *not* on the eigenfunctions q_m), have an interesting property: Since a *lossless* system[38,39] does not alter the eigenvalues of the power spectrum,[9] an overall degree of coherence that is based on these eigenvalues remains invariant when the light propagates through such a system. In particular, the informational entropy, as defined by expression (6.1), is preserved in lossless systems.

B. Gaussian Beams and First-Order Optical Systems

If a Gaussian signal whose WDF has the form (3.4) is the input signal of a first-order optical system described by a symplectic ray-transformation matrix, according to relation (5.8), then the output WDF has the form

$$F(x, u) = 2\sigma \exp\left\{-\sigma\left[\frac{2\pi}{\rho^2}(Ax + Bu)^2 + \frac{\rho^2}{2\pi}(Cx + Du)^2\right]\right\}. \tag{6.5}$$

Since the ray-transformation matrix is symplectic, which implies the property $AD - BC = 1$, this output WDF can be expressed in the form

$$F(x, u) = 2\sigma \exp\left\{-\sigma\left[\left(\beta + \frac{\alpha^2}{\beta}\right)x^2 + 2\frac{\alpha}{\beta}xu + \frac{1}{\beta}u^2\right]\right\} \quad (\beta > 0), \tag{6.6}$$

where the parameters α and β are related to A, B, C, D, and ρ through the formulas

$$\frac{1}{\beta} = \frac{2\pi}{\rho^2}B^2 + \frac{\rho^2}{2\pi}D^2,$$

$$\frac{\alpha}{\beta} = \frac{2\pi}{\rho^2}AB + \frac{\rho^2}{2\pi}CD. \tag{6.7}$$

The WDF of the form (6.6) is the WDF of a cross section through a *Gaussian beam*.[41] When this beam propagates through first-order optical systems, the parameters α and β change, but the general form [Eq. (6.6)] of the WDF is preserved. To be more specific, if a Gaussian beam with input parameters α_i and β_i forms the input of a first-order optical system with a ray-transformation matrix as in relation (5.8), then the Gaussian beam at the output of the system has parameters α_0 and β_0 that are related to α_i and β_i by the relations

$$\beta_i/\beta_0 = (B\alpha_i + D)^2 + B^2\beta_i^2,$$

$$\alpha_0\frac{\beta_i}{\beta_0} = (A\alpha_i + C)(B\alpha_i + D) + AB\beta_i^2. \tag{6.8}$$

Relations (6.8) can be combined into one relation that has the *bilinear* form[42]

$$\alpha_0 + i\beta_0 = \frac{A(\alpha_i + i\beta_i) + C}{B(\alpha_i + i\beta_i) + D}. \tag{6.9}$$

The third beam parameter σ, which is related to the informational entropy of the beam by means of Eq. (6.3), does not change when the beam propagates through first-order optical systems; this is obvious, since a first-order optical system is lossless and the informational entropy in such a system is preserved.

C. Geometric-Optical Systems

Let us start by studying a *modulator* described by the coherent input–output relationship $q_0(x) = m(x)q_i(x)$; for partially coherent light, the input–output relationship reads as $\Gamma_0(x_1, x_2) = m(x_1)\Gamma_i(x_1, x_2)m^*(x_2)$. The input and output WDF's are related by the relationship

$$F_0(x, u_0) = \frac{1}{2\pi}\int F_i(x, u_i)\mathrm{d}u_i \int m(x + \tfrac{1}{2}x')m^*(x - \tfrac{1}{2}x') \times \exp[-i(u_0 - u_i)x']\mathrm{d}x'. \tag{6.10}$$

This input–output relationship can be written in two distinct forms. On the one hand, we can represent it in a differential format reading as follows:

$$F_0(x, u) = m\left(x + \frac{i}{2}\frac{\partial}{\partial u}\right)m^*\left(x - \frac{i}{2}\frac{\partial}{\partial u}\right)F_i(x, u). \tag{6.11a}$$

On the other hand, we can represent it in an integral format that reads as follows:

$$F_0(x, u_0) = \frac{1}{2\pi}\int f_m(x, u_0 - u_i)F_i(x, u_i)\mathrm{d}u_i, \tag{6.11b}$$

where $f_m(x, u)$ is the coherent WDF [as defined by Eq. (3.5)] of the modulation function $m(x)$. Which of these two forms is superior depends on the problem.

We now confine ourselves to the case of a *pure phase-modulation* function $m(x) = \exp[i\gamma(x)]$. We then get

$$m(x + \tfrac{1}{2}x')m^*(x - \tfrac{1}{2}x') = \exp\left[i\sum_{k=0}^{\infty}\frac{2}{(2k+1)!}\gamma^{(2k+1)}(x)(x'/2)^{2k+1}\right], \tag{6.12}$$

where the expression $\gamma^{(n)}(x)$ is the nth derivative of $\gamma(x)$. If we consider only the first-order derivative in Eq. (6.12), we arrive at the following expressions:

$$m\left(x + \frac{i}{2}\frac{\partial}{\partial u}\right)m^*\left(x - \frac{i}{2}\frac{\partial}{\partial u}\right) \simeq \exp\left(-\frac{\mathrm{d}\gamma}{\mathrm{d}x}\frac{\partial}{\partial u}\right), \tag{6.13a}$$

$$f_m(x, u) \simeq 2\pi\delta\left(u - \frac{\mathrm{d}\gamma}{\mathrm{d}x}\right), \tag{6.13b}$$

and the input–output relationship of the pure phase modulator becomes

$$F_0(x, u) \simeq F_i\left(x, u - \frac{\mathrm{d}\gamma}{\mathrm{d}x}\right), \tag{6.14}$$

which is a mere coordinate transformation. We conclude that a single input ray yields a single output ray.

The ideas described above have been applied to the design of optical coordinate transformers[56] by Jiao *et al.* and to the theory of aberrations[57] by Lohmann *et al.* Now, if the first-order approximation is not sufficiently accurate, i.e., if we have to take into account higher-order derivatives in Eq. (6.12), the WDF allows us to overcome this problem. Indeed, we still have the exact input–output relationships Eqs. (6.11), and we can take into account as many derivatives in Eq. (6.12) as necessary. We thus end up with a more general differential form[58] than expression (6.13a) or a more general integral form[59] than expression (6.13b). The latter case, for instance, will yield an Airy function instead of a Dirac function, when we take not only the first but also the third derivative into account.

From expression (6.14) we concluded that a single input ray yields a single output ray. This may also happen in more general—not just modulation-type—systems; we call such systems *geometric-optical systems*. These systems have the simple input–output relationship

$$F_0(x, u) \simeq F_i[g_x(x, u), g_u(x, u)], \tag{6.15}$$

where the $\simeq$ sign becomes an $=$ sign in the case of linear functions g_x and g_u, i.e., in the case of Luneburg's first-order optical systems, which we have considered in Subsection 5.A. There appears to be a close relationship to the description of such geometric-optical systems by means of the *Hamilton characteristics*.[37]

Instead of the black-box approach of a geometric-optical system, which leads to the input–output relationship (6.15), we can also consider the system as a continuous medium and formulate a transport equation, as we did in Subsection 5.B. For geometric-optical systems, this transport equation takes the form of a first-order partial differential equation,[60] which can be solved by the method of characteristics. In Subsection 5.B we reached the general conclusion that these characteristics represent the geometric-optical ray paths and that along these ray paths the WDF has a constant value.

The use of the transport equation is not restricted to deterministic media; Bremmer[47] has applied it to stochastic media. Neither is the transport equation restricted to the scalar treatment of wave fields; Bugnolo and Bremmer[61] have applied it to study the propagation of vectorial wave fields. In the vectorial case, the concept of the WDF leads to a Hermitian matrix rather than to a scalar function and permits the description of nonisotropic media as well.

We have already considered some examples of geometric-optical systems in Section 5; two more-advanced examples are studied in Subsections 6.D and 6.E. Other examples are described by Ojeda-Castañeda and Sicre.[62]

D. Flux Transport through Free Space

Let us consider, in the $z = 0$ plane, a quasi-homogeneous planar Lambertian source[12,13] whose positional intensity is uniform in the x interval $(-x_{\max}, +x_{\max})$ and vanishes outside that interval and whose radiant intensity has the directional dependence $\cos\theta$ in the θ interval $(-\theta_{\max}, +\theta_{\max})$ and vanishes outside that interval; as usual, θ is the observation angle with respect to the z axis. The WDF of such a source is given by

$$F(x, u) = \frac{\pi}{2x_{\max} k \sin\theta_{\max}} \times \mathrm{rect}\left(\frac{x}{2x_{\max}}\right)\mathrm{rect}\left(\frac{u}{2k\sin\theta_{\max}}\right)\frac{k}{(k^2-u^2)^{1/2}}, \tag{6.16}$$

where $\mathrm{rect}(t) = 1$ for $-\frac{1}{2} < t \le \frac{1}{2}$ and $\mathrm{rect}(t) = 0$ elsewhere; for convenience, we have normalized the total radiant flux[12] to unity

$$\frac{1}{2\pi}\iint F(x, u)\,\frac{(k^2-u^2)^{1/2}}{k}\,\mathrm{d}x\mathrm{d}u = 1. \tag{6.17}$$

We wish to determine the radiant flux through an aperture with width $2x_{\max}$, parallel to the source plane, and symmetrically located around the z axis at a distance z_0 from the source plane. In the geometric-optical approximation, the WDF at the $z = z_0$ plane reads as $F[x + z_0 u/(k^2 - u^2)^{1/2}, u]$, and the radiant flux through the aperture follows readily from the integral

$$\frac{1}{2\pi}\int_{-x_{\max}}^{x_{\max}} \mathrm{d}x \int \mathrm{d}u F\left[x + \frac{u}{(k^2-u^2)^{1/2}}z_0, u\right]\frac{(k^2-u^2)^{1/2}}{k} = \frac{\sin\gamma - \dfrac{z_0}{2x_{\max}}(1-\cos\gamma)}{\sin\theta_{\max}}, \tag{6.18}$$

where $\gamma = \min[\theta_{\max}, \arctan(2x_{\max}/z_0)]$.

Similar techniques can be applied in the more general case when the source and the aperture have different widths and when the optical axes of the source and the aperture planes are translated or even rotated with respect to each other. Such problems arise, for instance, when two optical fibers are not ideally connected to each other and we want to determine the energy transfer from one fiber to the other.[63]

E. Rotationally Symmetric Fiber

As our last example, let us consider—by way of exception—a two-dimensional example. In an *optical fiber* that extends along the z axis, the signal depends, at a certain z value, on the two transverse space variables x and y; its WDF depends on these space variables and on the two frequency variables u and v. The transport equation in the fiber now has the form

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + [k^2 - (u^2+v^2)]^{1/2}\frac{\partial F}{\partial z} + k\frac{\partial k}{\partial x}\frac{\partial F}{\partial u} + k\frac{\partial k}{\partial y}\frac{\partial F}{\partial v} = 0, \tag{6.19}$$

which is the two-dimensional analog of the transport equation (5.21). We now assume that the index of refraction has a *rotationally symmetric* profile; hence $k = k[(x^2+y^2)^{1/2}]$. When we apply the coordinate transformation $x = r\cos\phi$, $y = r\sin\phi$, $h = vx - uy$, $k^2 = u^2 + v^2 + w^2$, we arrive at the transport equation

$$\left(k^2 - w^2 - \frac{h^2}{r^2}\right)^{1/2}\frac{\partial F}{\partial r} + \frac{h}{r^2}\frac{\partial F}{\partial\phi} + w\frac{\partial F}{\partial z} = 0. \tag{6.20}$$

We remark that the derivatives of the WDF with respect to the *ray invariants* h and w do not enter the transport equation (6.20). From the definition of the characteristics

$$w\frac{\mathrm{d}h}{\mathrm{d}z} = 0, \qquad w\frac{\mathrm{d}w}{\mathrm{d}z} = 0, \qquad w\frac{\mathrm{d}r}{\mathrm{d}z} = \left(k^2 - w^2 - \frac{h^2}{r^2}\right)^{1/2}, \qquad w\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{h}{r^2}, \tag{6.21}$$

we conclude that $\mathrm{d}h/\mathrm{d}z = \mathrm{d}w/\mathrm{d}z = 0$, and h and w are, indeed, invariant along a ray.

APPENDIX A: DERIVATION OF THE TRANSPORT EQUATION

We start with the differential equation (5.13)

$$\left[i\frac{\partial}{\partial z}+L\left(x_1,-i\frac{\partial}{\partial x_1};z\right)-L^*\left(x_2,-i\frac{\partial}{\partial x_2};z\right)\right]\times\Gamma(x_1,x_2;z)=0. \quad \text{(A1)}$$

Expressing the power spectrum $\Gamma(x_1, x_2; z)$ in terms of the WDF $F(x, u; z)$ through an inverse Fourier transformation yields

$$\frac{1}{2\pi}\int\left[i\frac{\partial}{\partial z}+L\left(x_1,-i\frac{\partial}{\partial x_1};z\right)-L^*\left(x_2,-i\frac{\partial}{\partial x_2};z\right)\right]\times F\left(\frac{x_1+x_2}{2},u_0;z\right)\exp\left[iu_0(x_1-x_2)\right]\mathrm{d}u_0=0, \quad \text{(A2)}$$

which can be expressed as

$$\frac{1}{2\pi}\int\left[i\frac{\partial}{\partial z}+L\left(x+\tfrac{1}{2}x',u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)-L^*\left(x-\tfrac{1}{2}x',u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)\right]\times F(x,u_0;z)\exp(iu_0x')\mathrm{d}u_0=0. \quad \text{(A3)}$$

Multiplication of relation (A3) by $\exp(-iux')$ and writing the integral over x' yields

$$\frac{1}{2\pi}\iint\left[i\frac{\partial}{\partial z}+L\left(x+\tfrac{1}{2}x',u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)-L^*\left(x-\tfrac{1}{2}x',u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)\right]\times F(x,u_0;z)\exp[i(u_0-u)x']\mathrm{d}u_0\mathrm{d}x'=0, \quad \text{(A4)}$$

which can be expressed as

$$\frac{1}{2\pi}\iint\left[i\frac{\partial}{\partial z}+L\left(x+\frac{i}{2}\frac{\partial}{\partial u},u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)-L^*\left(x+\frac{i}{2}\frac{\partial}{\partial u},u_0-\frac{i}{2}\frac{\partial}{\partial x};z\right)\right]\times F(x,u_0;z)\exp[i(u_0-u)x']\mathrm{d}u_0\mathrm{d}x'=0. \quad \text{(A5)}$$

Carrying out the integrations in relation (A5) leads to

$$\frac{\partial F}{\partial z}+2\,\mathrm{Im}\left[L\left(x+\frac{i}{2}\frac{\partial}{\partial u},u-\frac{i}{2}\frac{\partial}{\partial x};z\right)\right]F=0, \quad \text{(A6)}$$

which is equivalent to the desired result [Eq. (5.14a)].

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