Application of the X-FEM to the fracture of piezoelectric materials

Eric Béchet[1], Matthias Scherzer[2], Meinhard Kuna[2]

[1] LPMM– CNRS UMR 7554
Université de Metz - France
[2] IMFD – TU Bergakademie Freiberg - Germany

Outline

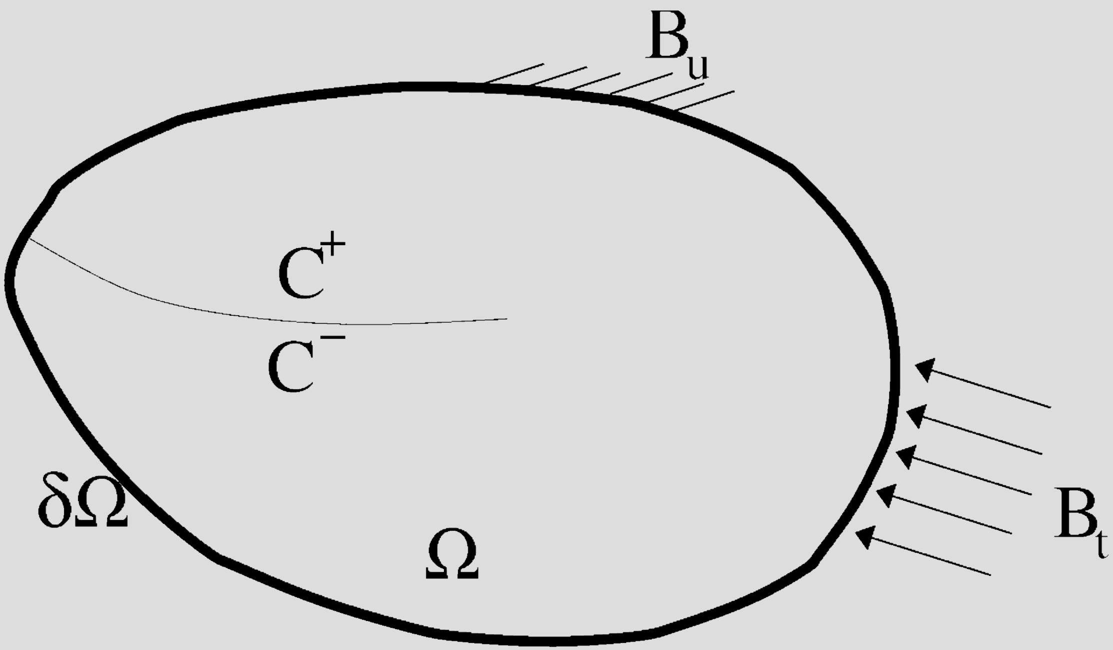
- Introduction
- Updated X-FEM formulation
  - New crack tip enrichment functions
  - Updated SIF computation scheme
  - Use of specific preconditionner
- Convergence study
- Conclusions

Introduction

- Goal
  - Propose a updated enrichment scheme for a cracked anisotropic piezoelectric media
  - Convergence study of the method
    - Energy error
    - SIFS and energy release rate
  - Development of a SIF evaluation scheme based on interaction integrals specific to piezoelectric materials
  - Numerical crack propagation using empirical laws

Introduction

- Physical model : linear piezoelectric media, electrically impermeable crack



$$
\begin{aligned}
\sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\
D_i &= e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k \\
\sigma_{ij,j} &= 0 \\
D_{j,j} &= 0 \\
\varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\
E_j &= -\varphi_{,i}
\end{aligned}
$$

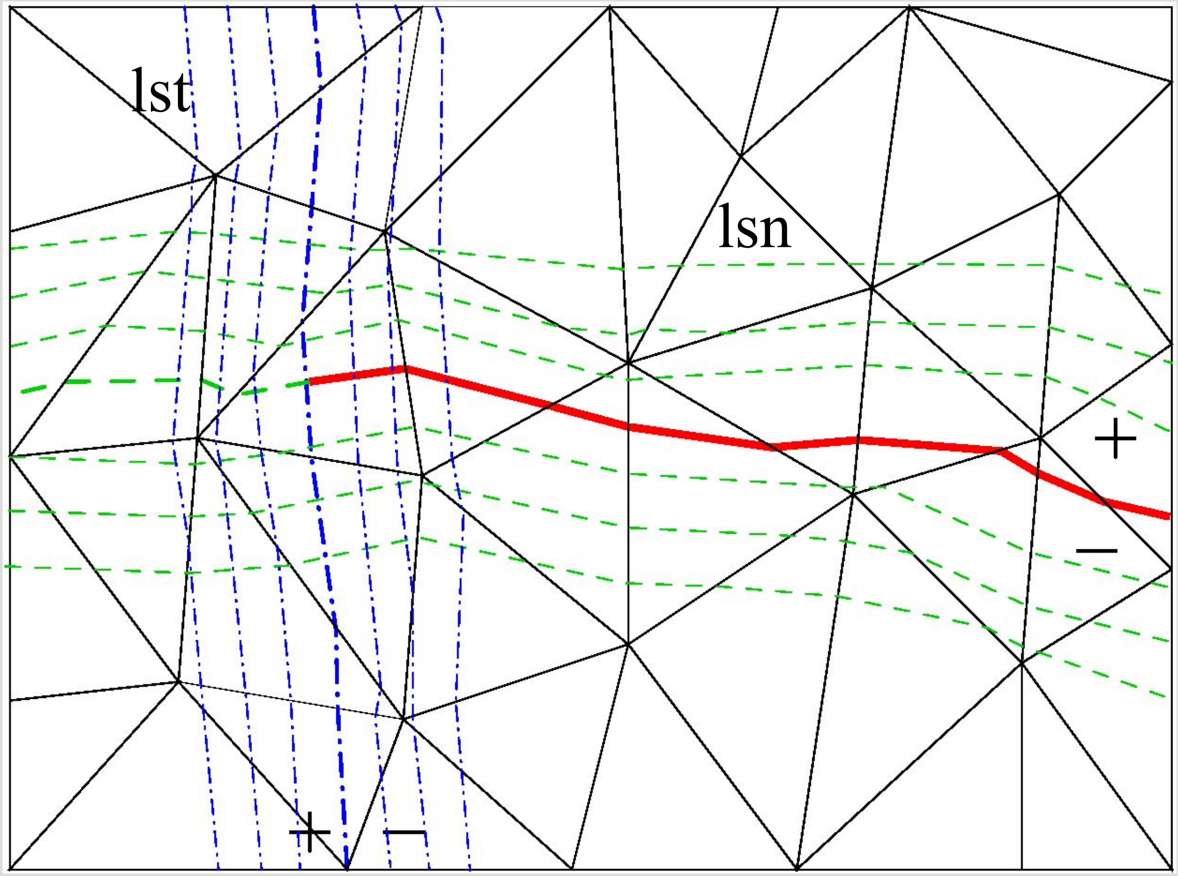
$$
\left.\begin{aligned}
\sigma_{ij}.n_i = 0 \\
D_i.n_i = 0
\end{aligned}\right. \quad \text{on } C^+ \cup C^-
$$

Introduction

- Numerical Model
  - Xfem field approximation
  - No remeshing
  - Interaction integrals used to compute the SIFs

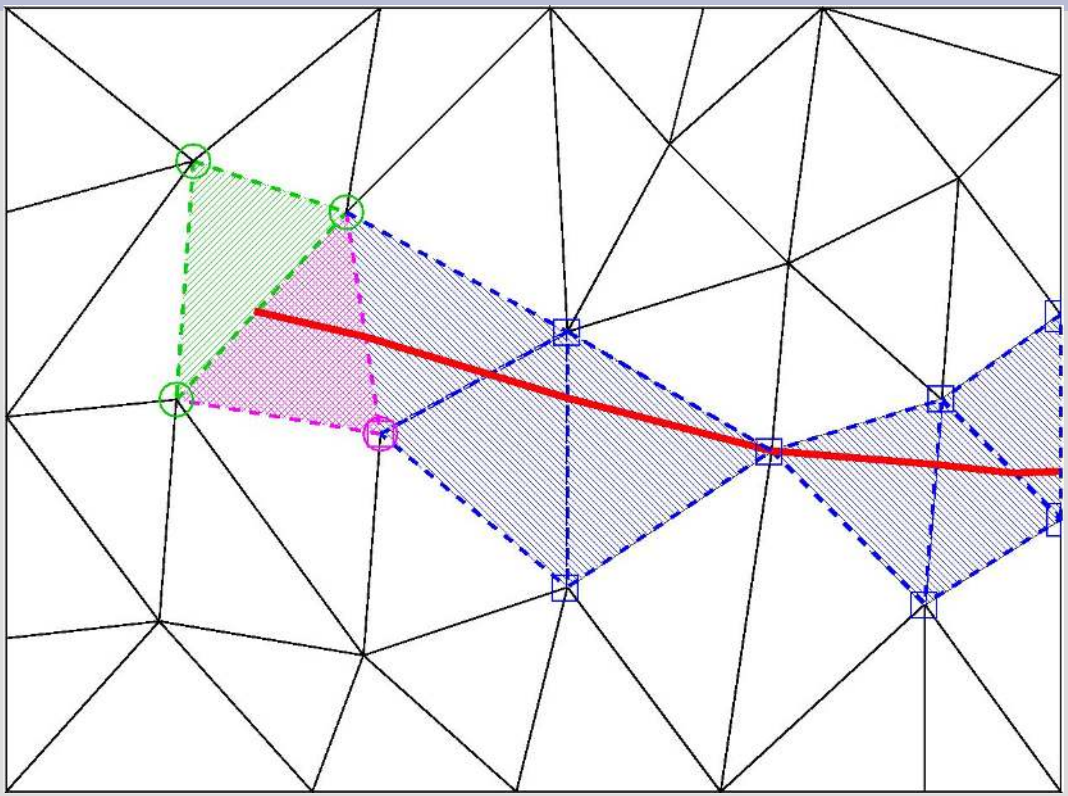
X-FEM

- Crack represented by level-sets
  - Local coodinates at the crack tip



X-FEM

- Local partition of unity enrichment
  - Singular functions around crack tip
  - Heaviside along crack surface
  - Remaining dofs unenriched



$$\mathbf{u}^h = \sum_{i\in R}\mathbf{N}_i a_i + \sum_{i\in R}\sum_{j=1\ldots n}\mathbf{N}_i g_j b_{ij} + \sum_{i\in H}\mathbf{N}_i h c_i$$

$$\varphi^h = \sum_{i\in R}N_i\alpha_i + \sum_{i\in R}\sum_{j=1\ldots n}N_i g_j \beta_{ij} + \sum_{i\in H}N_i h\gamma_i$$

Enrichement functions

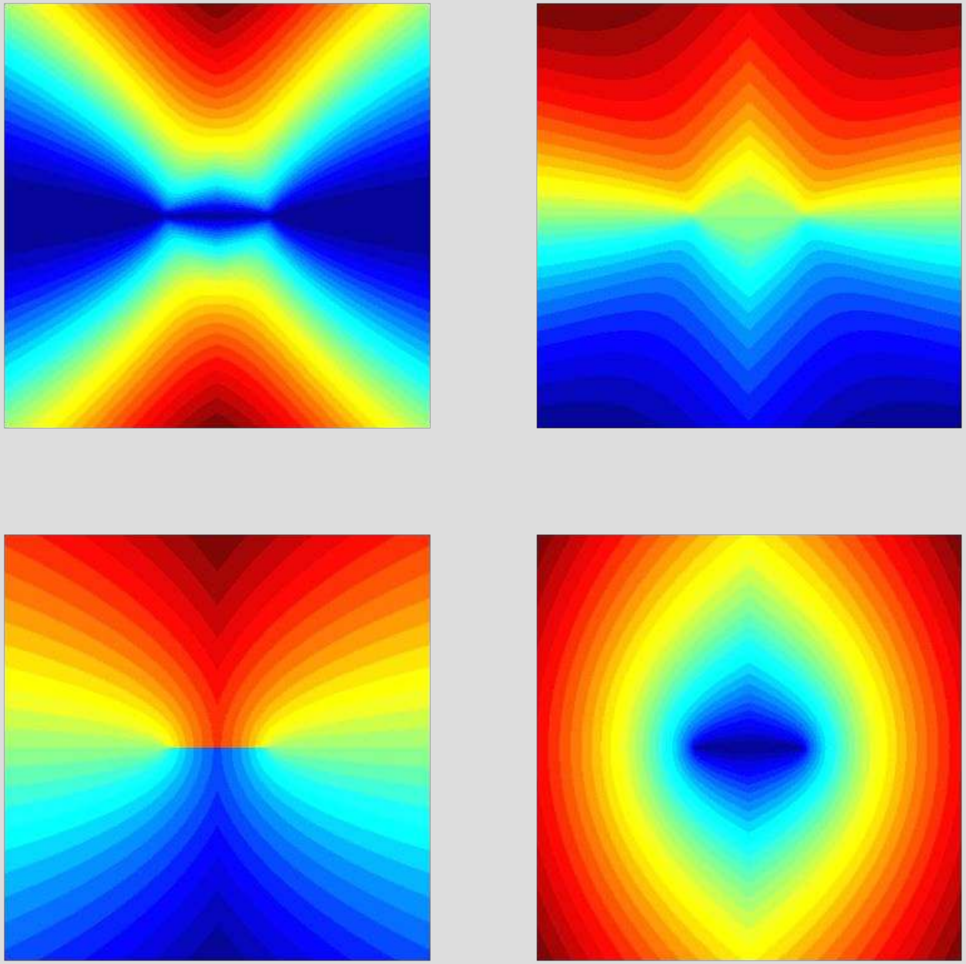
- Enrichment functions
  - Jump across the crack for displacements and potential :

$$h(\varphi) = \begin{cases} +1 & \text{if } \varphi \geq 0 \\ -1 & \text{if } \varphi < 0 \end{cases}$$

  - Crack tip for in a pure mechanical setting

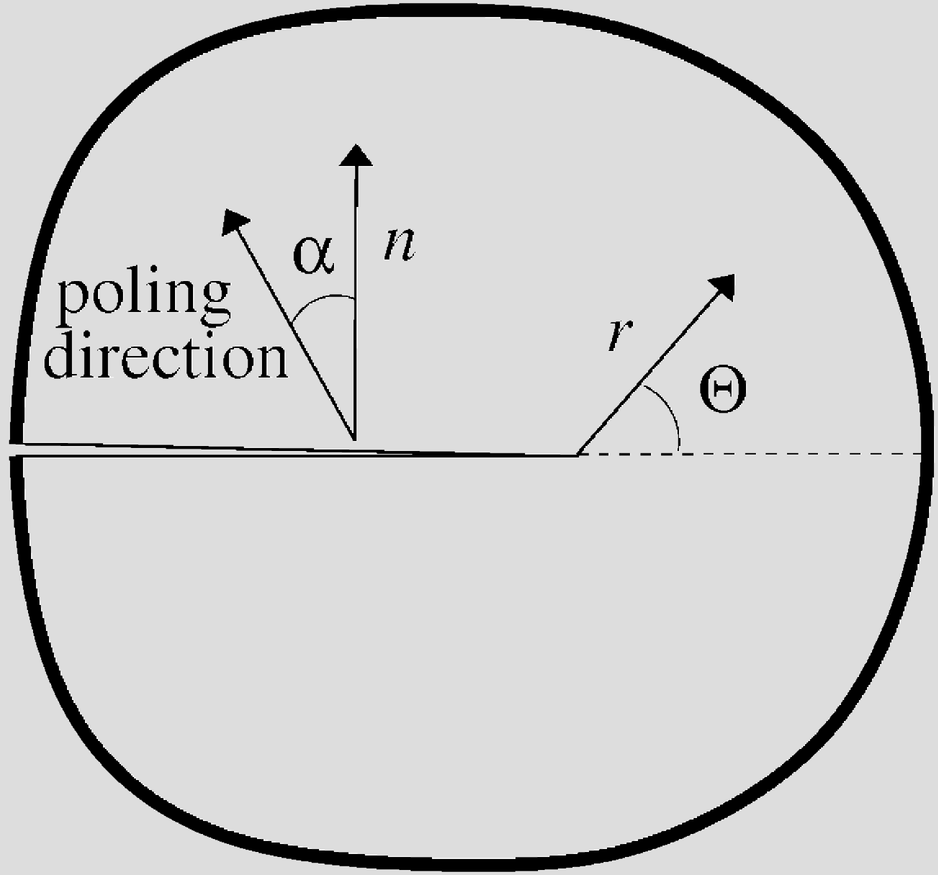
$$g_i(r,\theta) = \left\{ \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta \right\}$$

Enrichement functions



Enrichement functions

- Crack tip functions for a piezoelectrical setting
  - Must span the eigenfunction's space at the crack tip for displacements and potential
  - Depends on the material characteristics and the orientation
  - Depends on the permeability of the crack



$$g_i(r,\theta) = \left\{\sqrt{r}f_1(\theta), \sqrt{r}f_2(\theta), \sqrt{r}f_3(\theta), \sqrt{r}f_4(\theta), \sqrt{r}f_5(\theta), \sqrt{r}f_6(\theta)\right\}$$

Enrichement functions

$$f_i(\theta) \quad = \quad \phi(\omega(\theta,\alpha), a_{i,re}, a_{i,im})$$

$$= \quad \begin{cases} \rho(\omega, a_{i,re}, a_{i,im}) \cos \frac{\psi(\omega, a_{i,re}, a_{i,im})}{2} & \text{if } a_{i,im} > 0 \\ \rho(\omega, a_{i,re}, a_{i,im}) \sin \frac{\psi(\omega, a_{i,re}, a_{i,im})}{2} & \text{if } a_{i,im} \leq 0 \end{cases}$$

$$\omega = \theta - \alpha$$

$$\rho(\omega, a_{i,re}, a_{i,im}) =$$

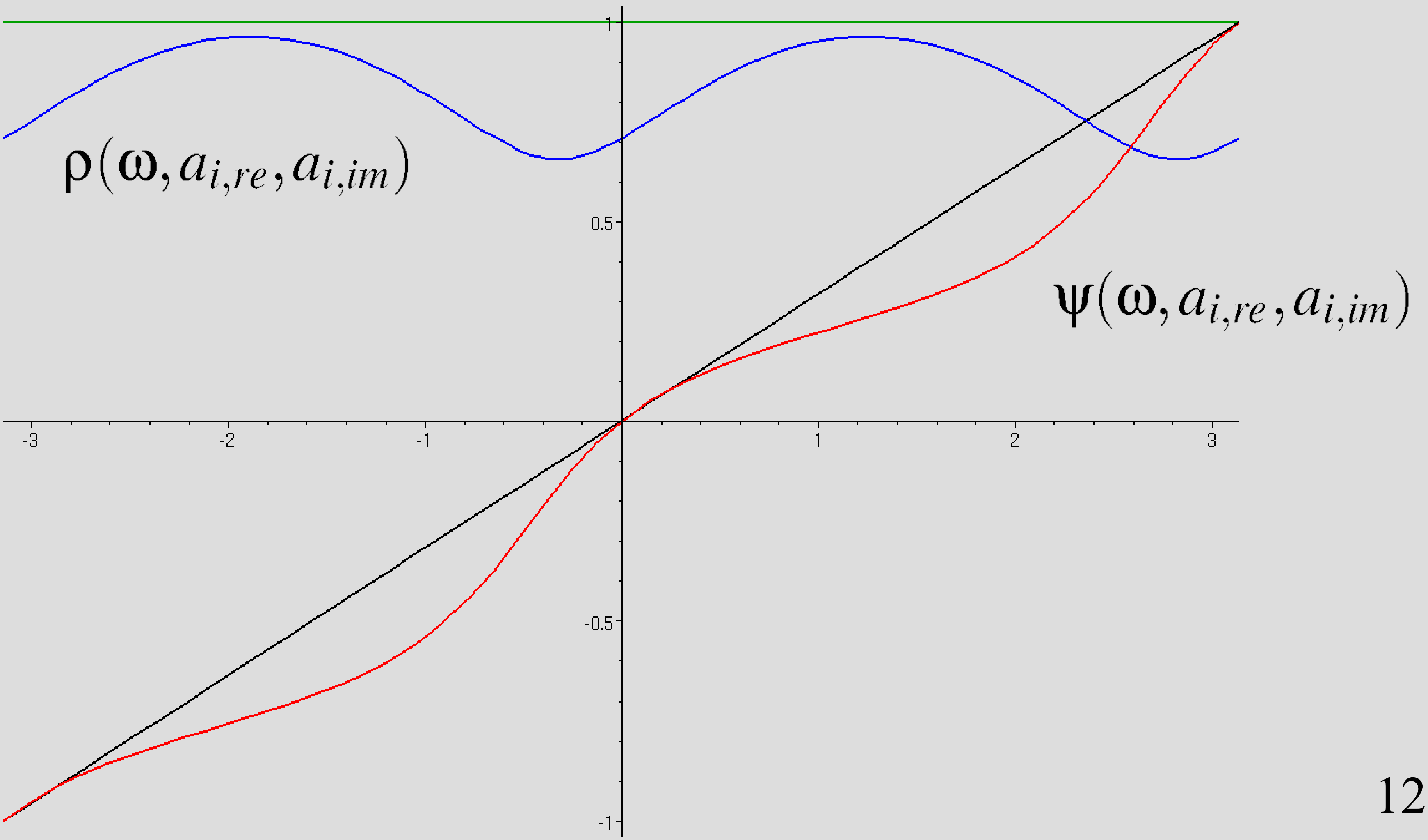
$$\frac{1}{\sqrt{2}} \sqrt[4]{a_{i,re}^2 + a_{i,im}^2 + a_{i,re} \sin 2\omega - \left(a_{i,re}^2 + a_{i,im}^2 - 1\right) \cos 2\omega}$$

$$\psi(\omega, a_{i,re}, a_{i,im}) = \frac{\pi}{2} + \pi \mathrm{int}\left(\frac{\omega}{\pi}\right)$$

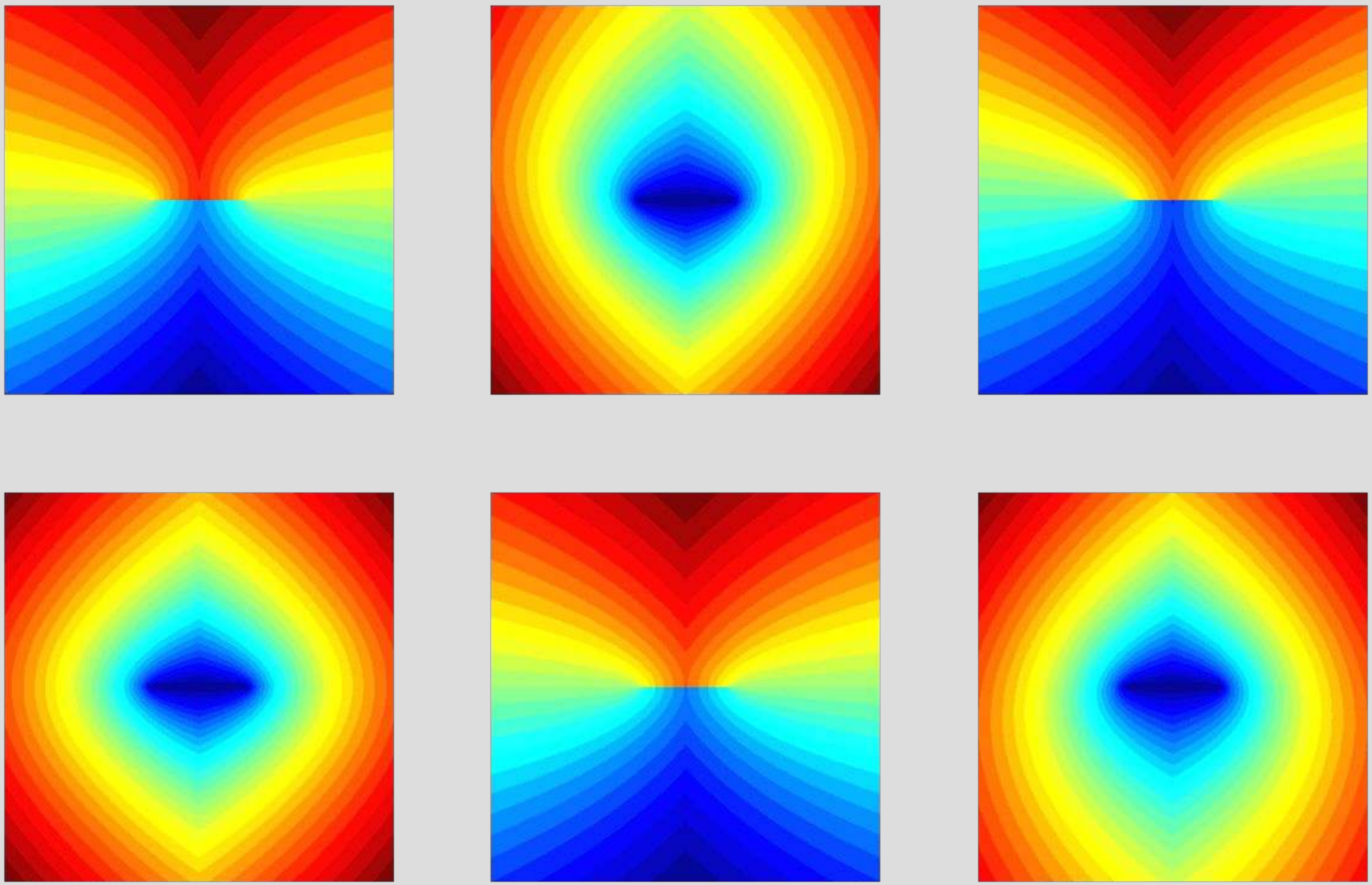
$$-\arctan\left(\frac{\cos\left(\omega - \pi\mathrm{int}\left(\frac{\omega}{\pi}\right)\right) + a_{i,re} \sin\left(\omega - \pi\mathrm{int}\left(\frac{\omega}{\pi}\right)\right)}{|a_{i,im}| \sin\left(\omega - \pi\mathrm{int}\left(\frac{\omega}{\pi}\right)\right)}\right)$$

Enrichement functions

- Modified functions

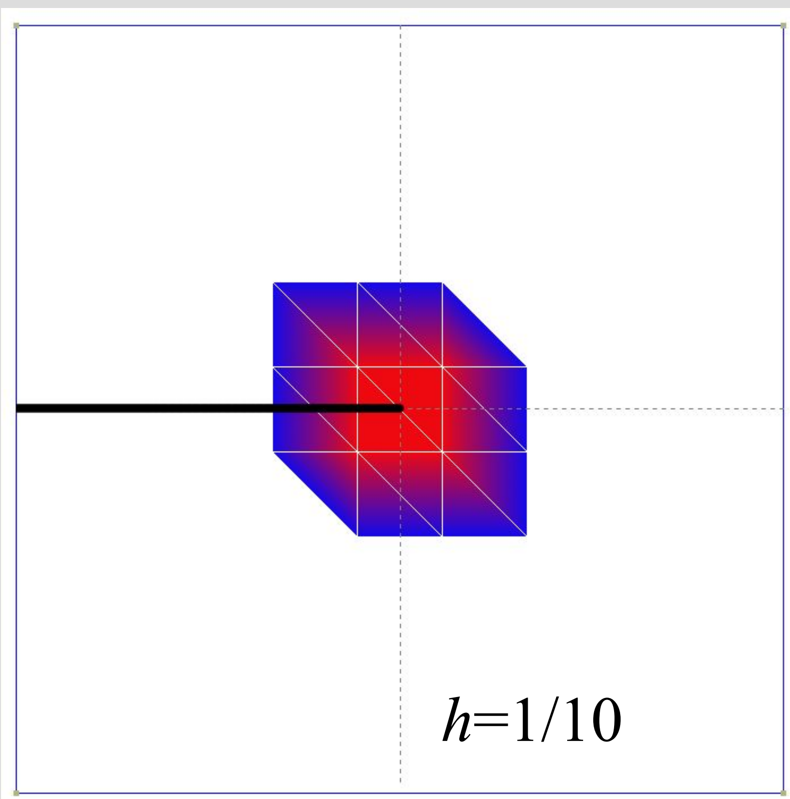


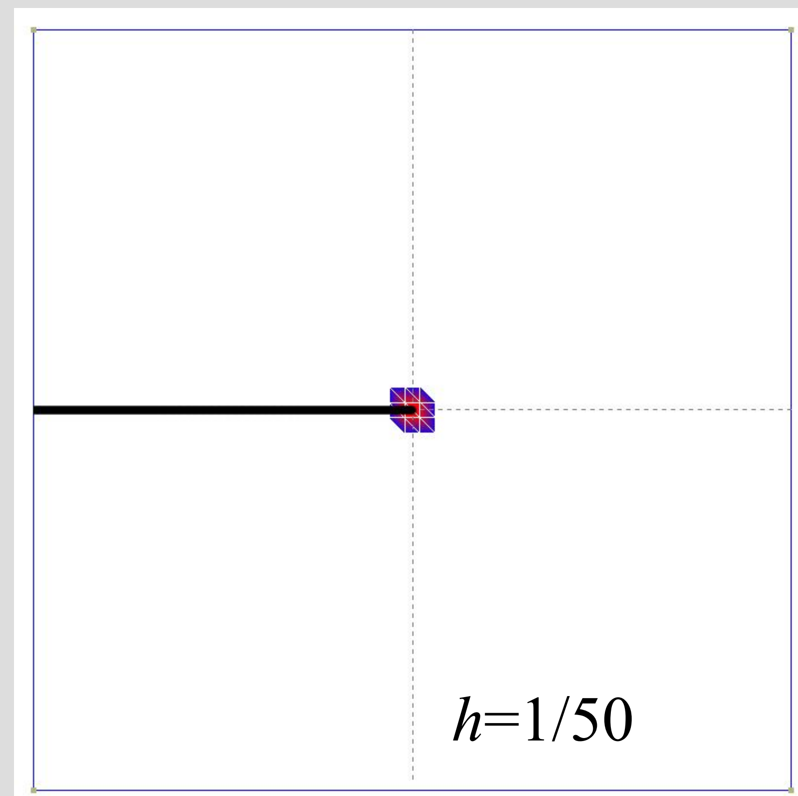
Enrichement functions



Updated enrichment scheme

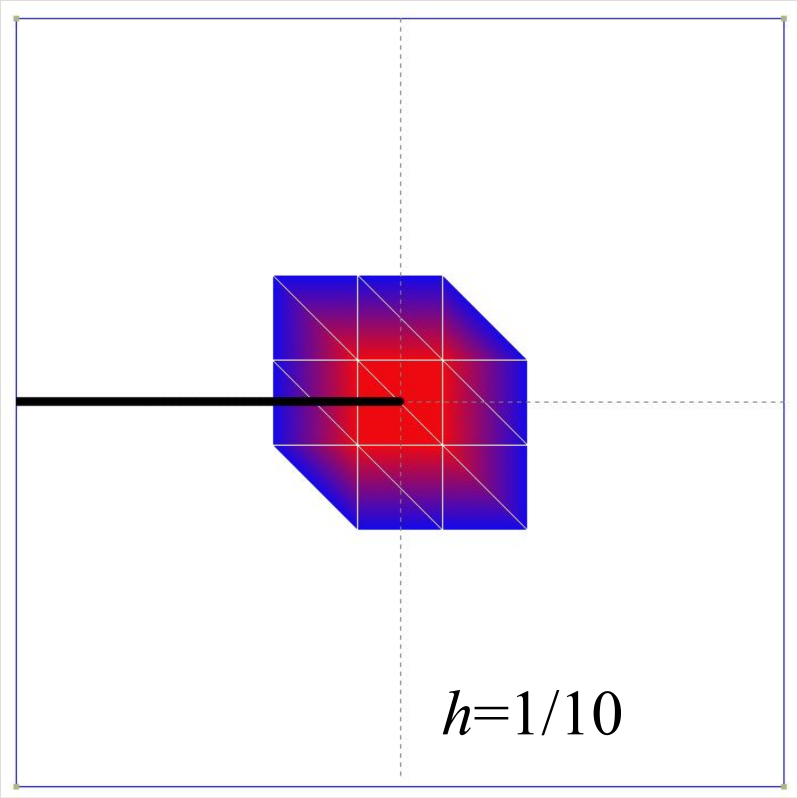
"topological" Enrichment

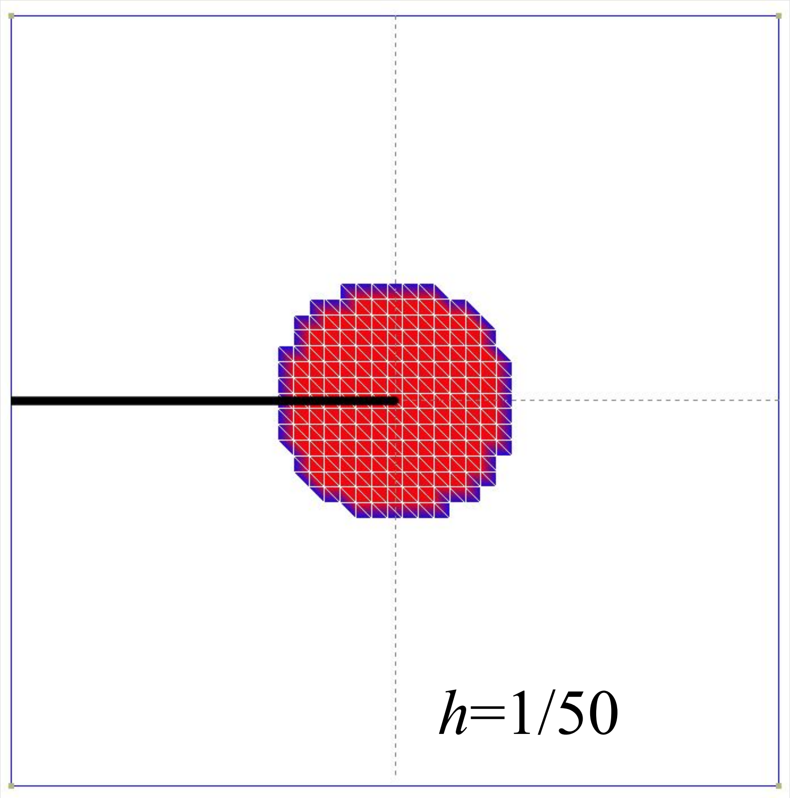




Updated enrichment scheme

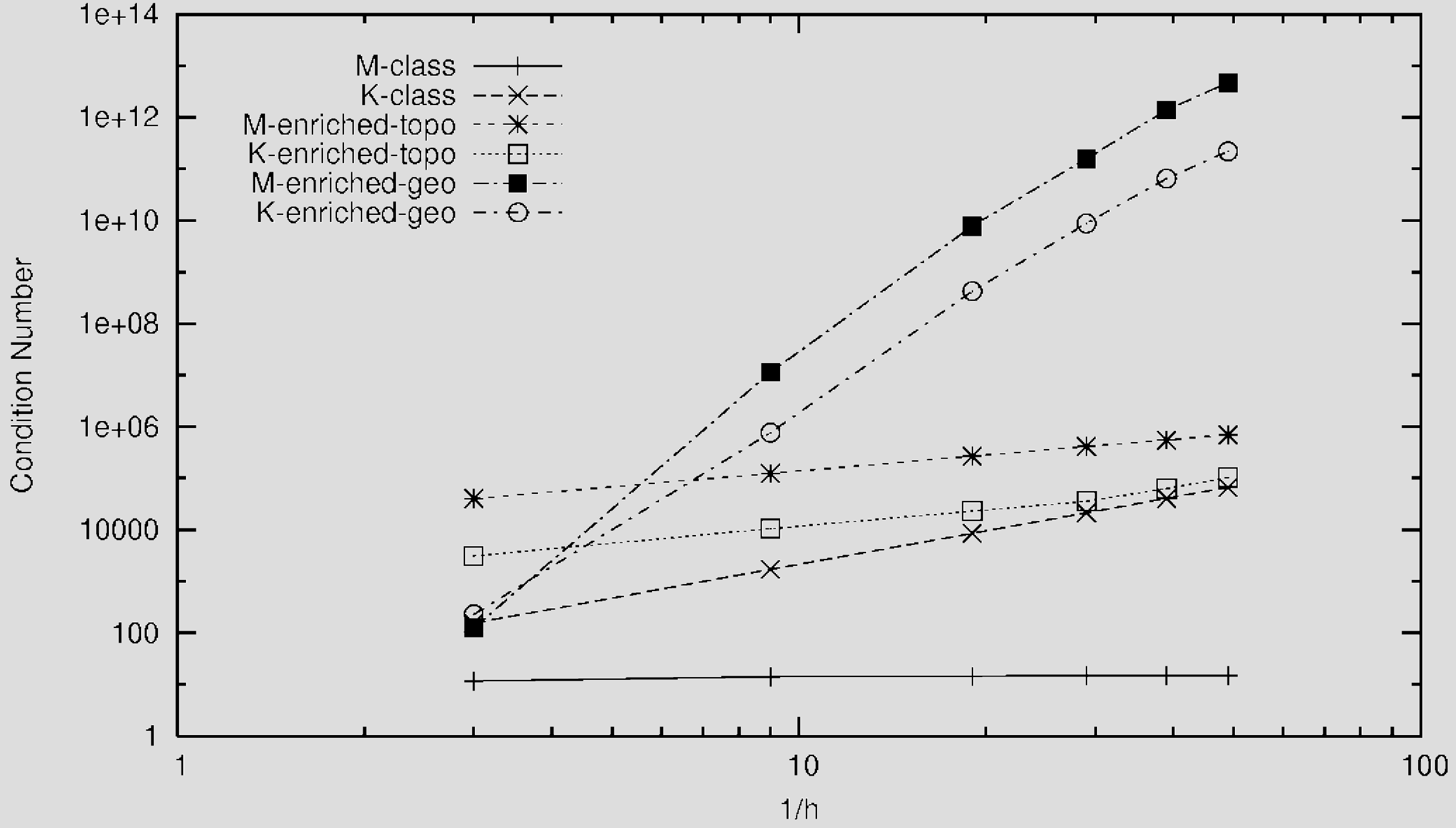
“Geometrical” Enrichment





Condition number

- The enrichment may lead to almost-singular matrices → difficult to use iterative solvers



Preconditionner

- Orthogonalize each subset of enriched dofs

$$\begin{bmatrix} & \vdots & \vdots & & \\ & a & b & \cdots & \\ & b & c & \cdots & \\ & & & \ddots & \end{bmatrix} \mathbf{u} = \mathbf{f} \rightarrow \begin{bmatrix} & \vdots & \vdots & & \\ & a & 0 & \cdots & \\ & 0 & c & \cdots & \\ & & & \ddots & \end{bmatrix} \tilde{\mathbf{u}} = \tilde{\mathbf{f}}$$

Preconditionner

- Cholesky decomposition & scaling for node *k* :

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \mathbf{A} = \mathbf{G}\mathbf{G}^T$$

$$\mathbf{D}_{ij} = \sqrt{\mathbf{A}_{ij}\delta_{ij}} \quad \text{(no summation)}$$

$$\mathbf{R} = \mathbf{G}^{-1}\mathbf{D}$$

"Assembly" of every submatrix R gives R*

$$\mathbf{R}^*\mathbf{K}\mathbf{R}^{*T}\tilde{\mathbf{u}} = \mathbf{R}^*\mathbf{f} \text{ with } \mathbf{u} = \mathbf{R}^{*T}\tilde{\mathbf{u}}$$

Preconditionner

- Trick for handling non positive definite systems (but blockwise positive definite)
  - If the matrix A belongs to the electrostatic part:
    - a, b and c are negative
    - we need to take the opposite matrix (which is positive definite) in order to generate the preconditionner

$$\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & K & \cdot & d^T \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & d & \cdot & -\varepsilon \end{array}$$

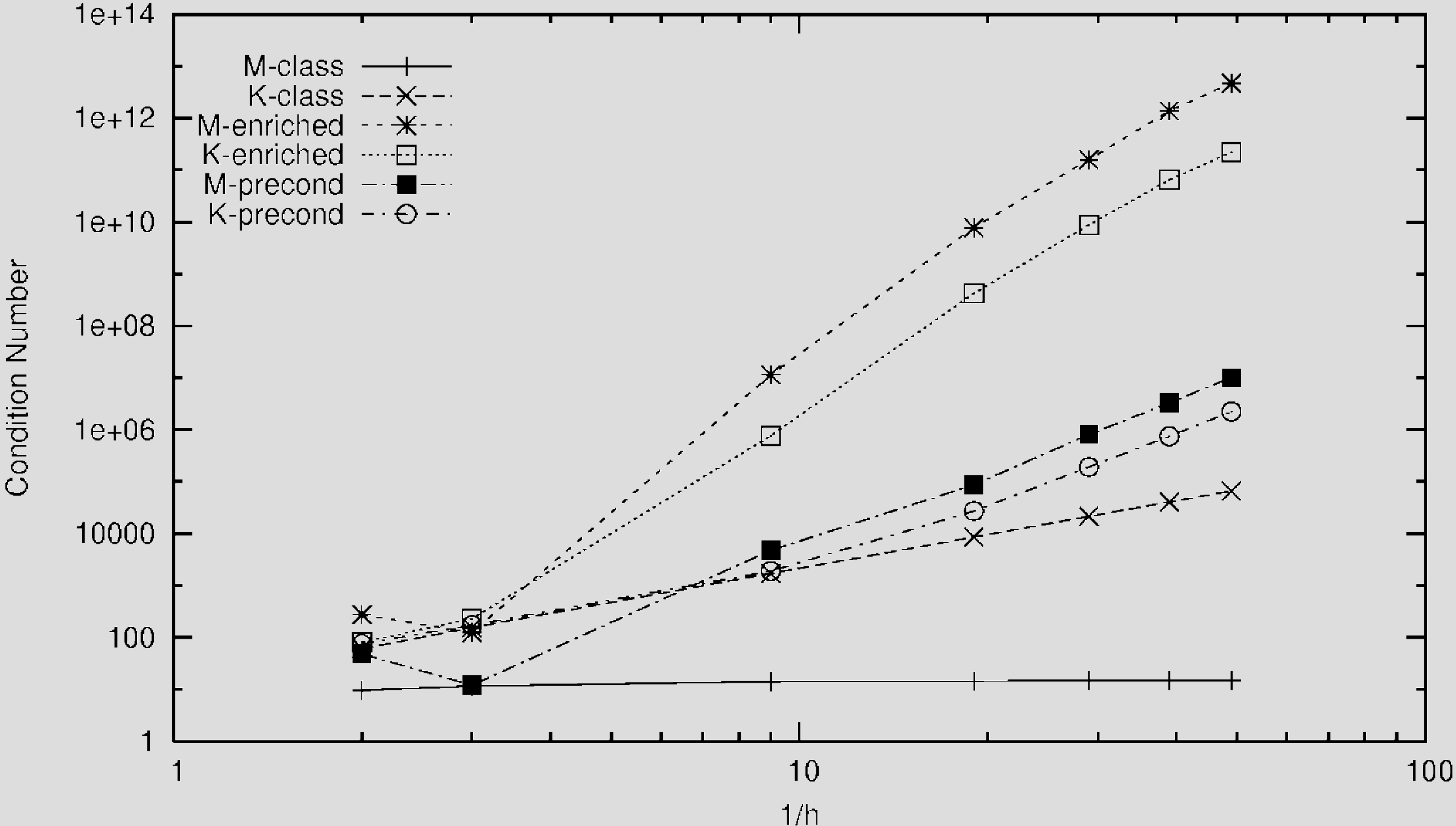
$$\mathbf{A} = \begin{bmatrix} -a & -b \\ -b & -c \end{bmatrix}$$

$$\mathbf{A} = \mathbf{G}\mathbf{G}^T$$

Preconditionner

Condition number of $\mathbf{R}^*\mathbf{K}\mathbf{R}^{*T}$ or $\mathbf{R}^*\mathbf{M}\mathbf{R}^{*T}$

(Geometrical enrichment)



Convergence study

- Exact solution – use of complex potentials
  - cf. H. Sosa, Plane problems in piezoelectric media with defects, Int. J. Sol. Struct. (1991)

$$\varepsilon_{xx} = 2\Re\left(\sum_{k=1}^{3}\left(a_{11}\tau_k^2 + a_{12} - b_{12}\kappa_k\right)\phi_k\left(z_k\right)\right)$$

$$\varepsilon_{xy} = 2\Re\left(\sum_{k=1}^{3}\left(a_{12}\tau_k^2 + a_{22} - b_{22}\kappa_k\right)\phi_k\left(z_k\right)\right)$$

$$\varepsilon_{xy} = \Re\left(\sum_{k=1}^{3}\left(-a_{33}\tau_k + b_{13}\tau_k\kappa_k\right)\phi_k\left(z_k\right)\right)$$

$$E_x = 2\Re\left(\sum_{k=1}^{3}\left(b_{13} + \delta_{11}\kappa_k\right)\tau_k\phi_k\left(z_k\right)\right)$$

$$E_y = -2\Re\left(\sum_{k=1}^{3}\left(b_{21}\tau_k^2 + b_{22} + \delta_{22}\kappa_k\right)\phi_k\left(z_k\right)\right)$$

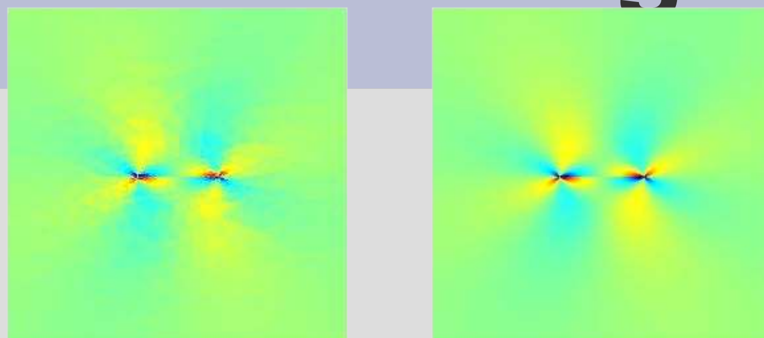
$$\kappa_k = -\frac{\left(b_{21} + b_{13}\right)\tau_k + b_{22}}{\delta_{11}\tau_k^2 + \delta_{22}}$$

$$z_k = x + \tau_k y$$

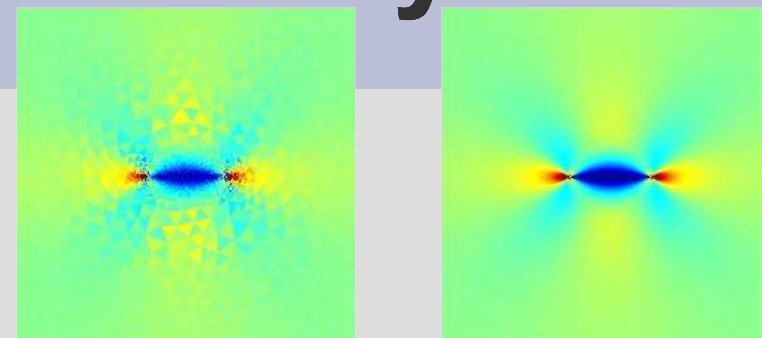
$$\phi_k\left(z_k\right) = \frac{A_k z_k}{\sqrt{z_k^2 - a^2}} + B_k$$

τ_k are roots of the characteristic equation.

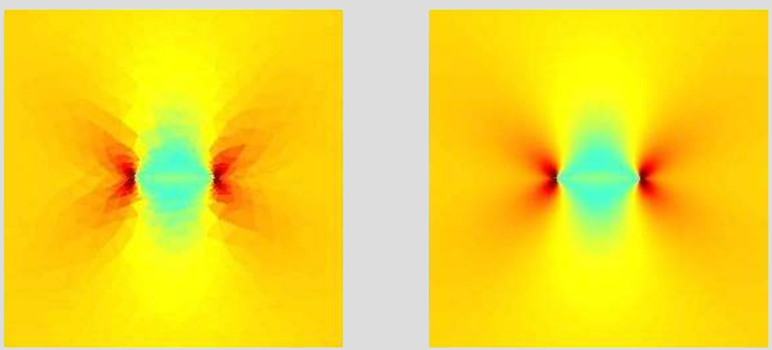
Convergence study



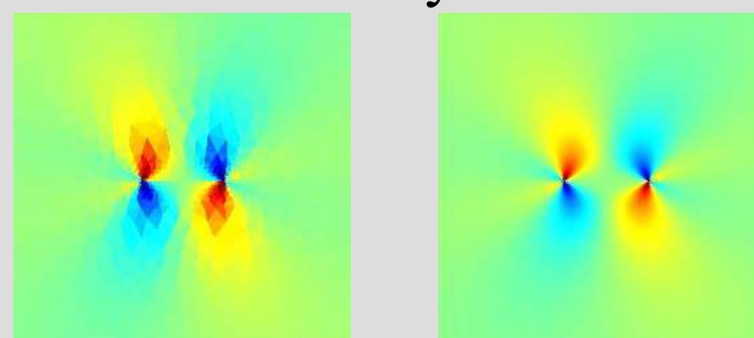
EDx



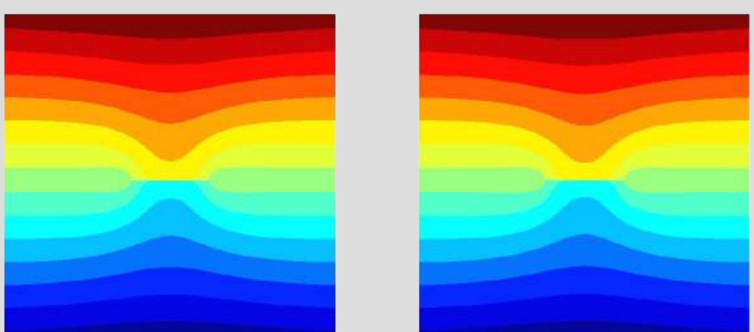
EDy



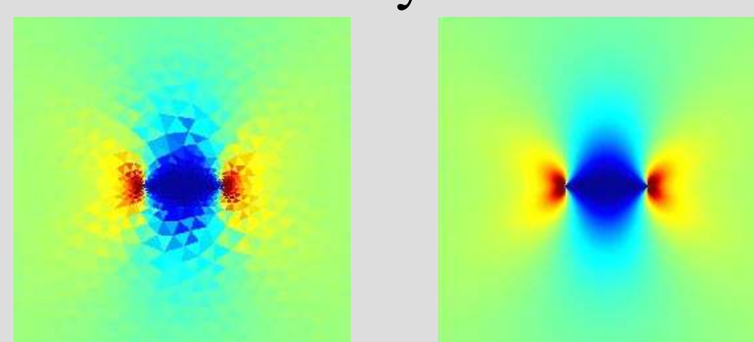
Sxx



Sxy

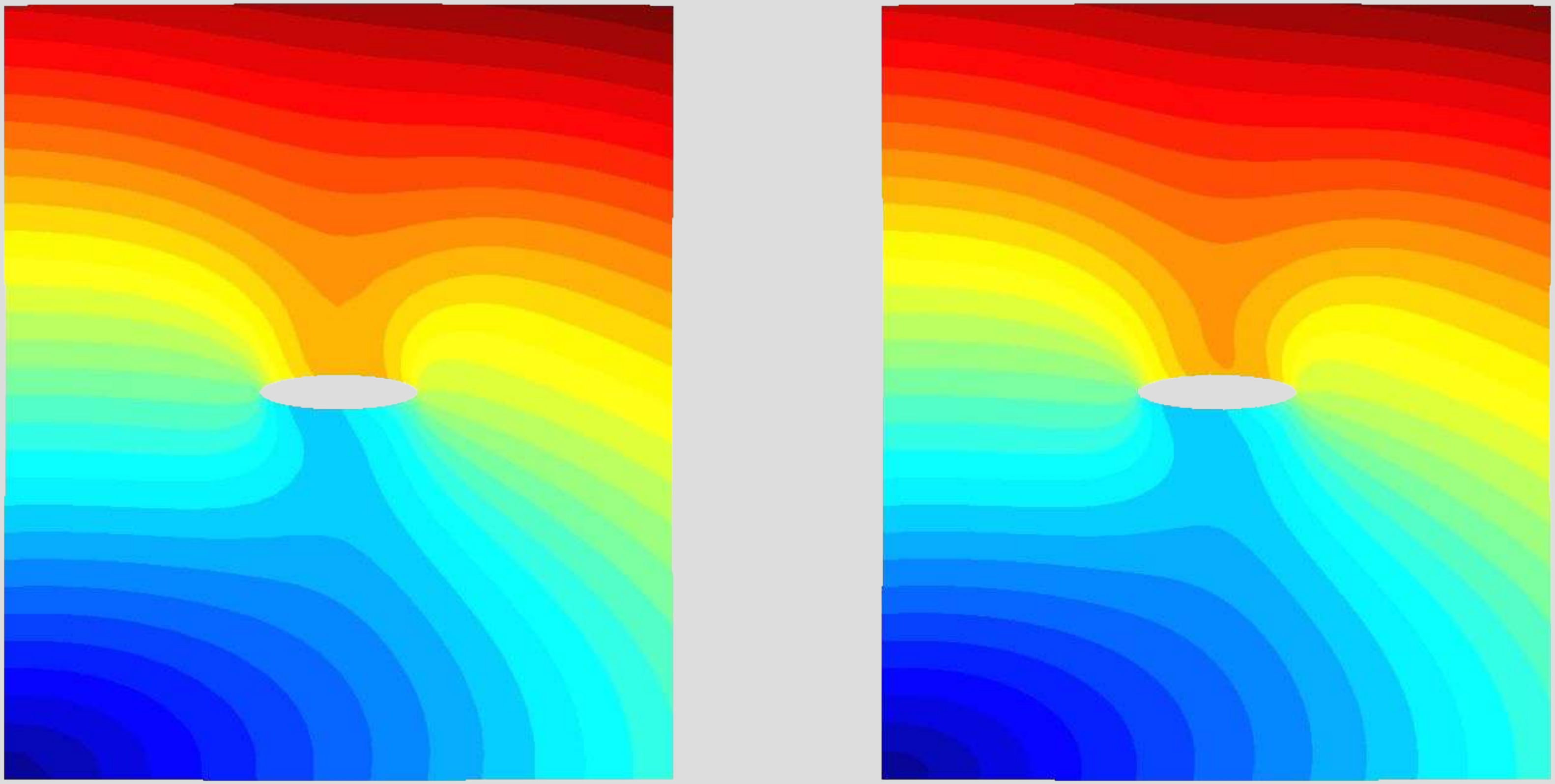


Potential



Syy

Convergence study



Displacement

Convergence study

- Energy norm with respect to the internal energy

$$U = \int_V \left(\frac{1}{2} \varepsilon_{ij} c_{ijkl} \varepsilon_{kl} + \frac{1}{2} E_i \varepsilon_{ij} E_j \right) dV$$

$$E_U =$$

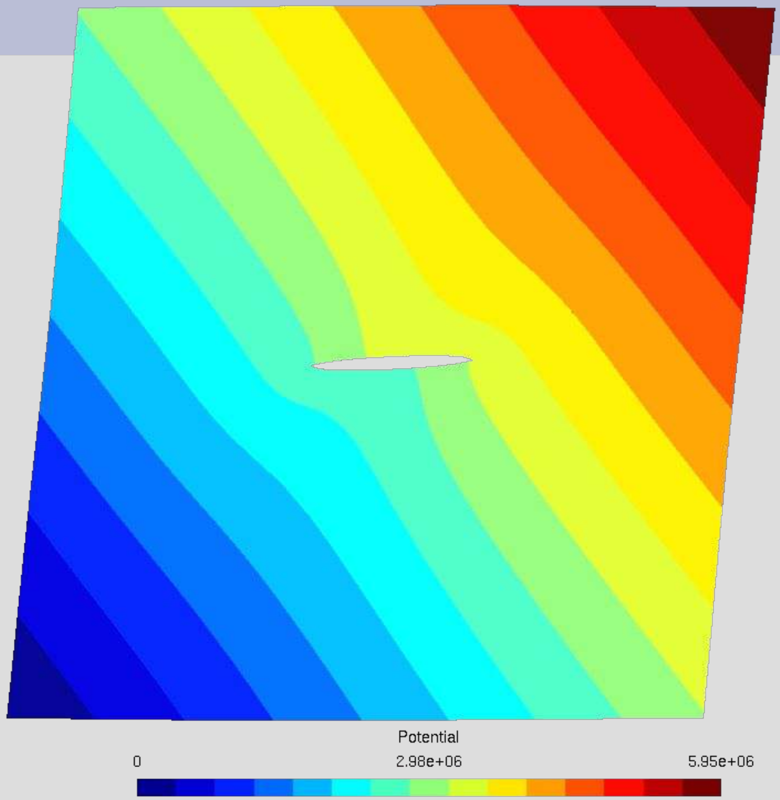
$$\int_V \sqrt{\frac{1}{2} \left(\varepsilon_{ij} - \varepsilon_{ij}^{ex} \right) c_{ijkl} \left(\varepsilon_{kl} - \varepsilon_{kl}^{ex} \right) + \frac{1}{2} \left(E_i - E_i^{ex} \right) \varepsilon_{ij} \left(E_j - E_j^{ex} \right)} dV$$

$$=$$

$$\int_V \sqrt{\frac{1}{2} \left(u_i - u_i^{ex} \right)_{,j} c_{ijkl} \left(u_k - u_k^{ex} \right)_{,l} + \frac{1}{2} \left(\varphi - \varphi^{ex} \right)_{,i} \varepsilon_{ij} \left(\varphi - \varphi^{ex} \right)_{,j}} dV$$

Convergence study

- Energy norm
  - comparison with standard crack tip enrichment
  - Infinite body with embedded crack
  - inclined material axes (30°)
  - PZT4 orthotropic material



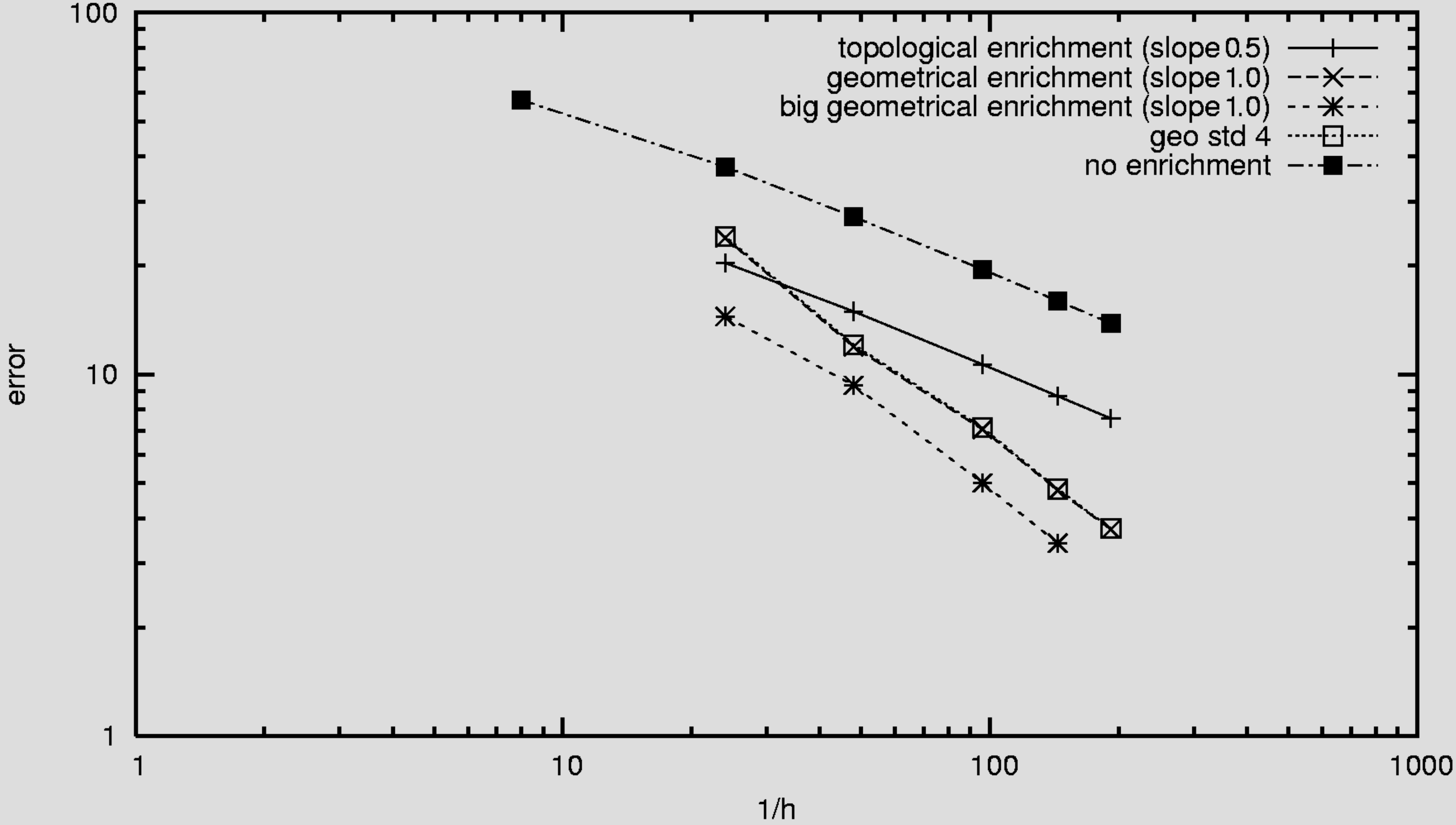
$$c = 10^{10}\begin{bmatrix} 14.02 & 7.892 & 7.565 & 0 & 0 & 0 \\ 7.892 & 14.02 & 7.565 & 0 & 0 & 0 \\ 7.565 & 7.565 & 11.58 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.527 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.527 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.064 \end{bmatrix}$$

$$e = 10^{0}\begin{bmatrix} 0 & 0 & 0 & 0 & 13.0 & 0 \\ 0 & 0 & 0 & 13.0 & 0 & 0 \\ -5.268 & -5.268 & 15.44 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon = 10^{-9}\begin{bmatrix} 6.68 & 0 & 0 \\ 0 & 6.68 & 0 \\ 0 & 0 & 5.523 \end{bmatrix}$$

Energy error

- Exact vs finite element fields



Energy error

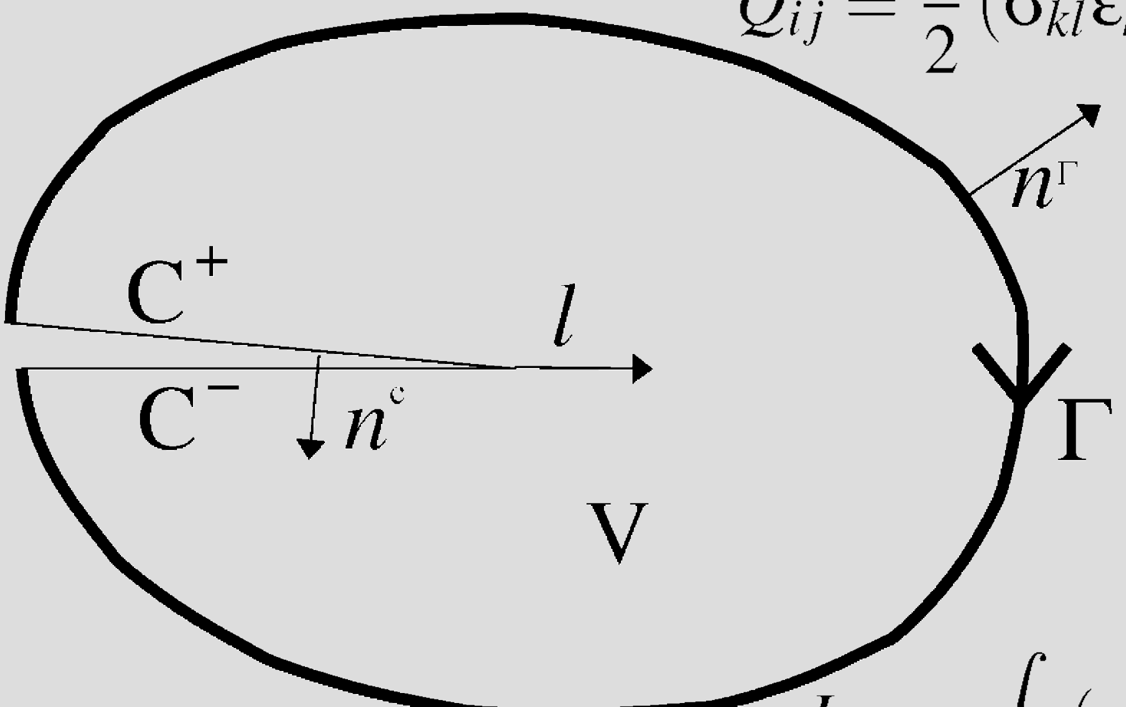
- The “classical” enrichment gives almost the same results as the specific enrichment, with less computational overhead.
- It is not clear whether different material laws (e.g. “more” anisotropic) lead to different results

$$g_i(r,\theta) = \left\{ \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta \right\}$$

SIFs computation

- Contour integrals on Γ –> Domain integrals over V

$$Q_{ij} = \frac{1}{2}\left(\sigma_{kl}\varepsilon_{kl} - D_k E_k\right)\delta_{ij} - \sigma_{kj}u_{k,i} - D_j\varphi_{,i}$$



$$q_i = \Lambda l_i$$

$$J = -\int_{\Gamma} q_i n_j Q_{ij}\, d\Gamma$$

$$J = -\int_V \left(q_i Q_{ij}\right)_{,j} dV + \int_{C^+ \cup C^-} q_i Q_{ij} n_j\, dS$$

Interaction integrals

- Same procedure used to compute interaction integrals (no crack loading) :

$$
\begin{aligned}
I \quad = \quad & \int_V q_{i,j} \left(\sigma_{kj} u^{aux}_{k,i} + \sigma^{aux}_{kj} u_{k,i} + D_j \varphi^{aux}_{,i} + D^{aux}_j \varphi_{,i} \right) dV \\
& + \int_V q_i \left(\sigma_{kj} u^{aux}_{k,ij} + \sigma^{aux}_{kj,j} u_{k,i} + D_j \varphi^{aux}_{,ij} + D^{aux}_{j,j} \varphi_{,i} \right) dV \\
& - \int_V q_{i,j} \left(\sigma_{kl} \varepsilon^{aux}_{kl} - D_k E^{aux}_k \right) \delta_{ij} dV \\
& - \int_V q_i \left(\sigma_{kl} \varepsilon^{aux}_{kl,i} - D_k E^{aux}_{k,i} \right) dV
\end{aligned}
$$

$$
I = \int_\Gamma G^{aux} \Lambda d\Gamma \qquad G^{aux} = Y_{MN} K_M K^{aux}_N \qquad Y_{MN} K_M K^{aux}_N = \frac{I}{\int_\Gamma \Lambda d\Gamma}
$$

Interaction integrals

- Relation between *G* and the *K* factors
  - Simpler case of the isotropic elasticity well known

$$Y_{MN}K_MK_N^{aux} = \{K_I\, K_{II}\, K_{III}\}\begin{bmatrix} \frac{2(1-\nu^2)}{E} & 0 & 0 \\ 0 & \frac{2(1-\nu^2)}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix}\begin{Bmatrix} K_I^{aux} \\ K_{II}^{aux} \\ K_{III}^{aux} \end{Bmatrix}$$

  - The Irwin matrix depends on the material orientation and is not explicitely known for piezos.

$$G^{aux1,aux2} = Y_{MN}K_M^{aux1}K_N^{aux2}$$

$$I^{aux1,aux2} = \int_\Gamma G^{aux1,aux2}\Lambda d\Gamma \qquad Y_{MN}K_M^{aux1}K_N^{aux2} = \frac{I^{aux1,aux2}}{\int_\Gamma \Lambda d\Gamma}$$

Interaction integrals

- By using the eigenfunction set, every term in the Irwin matrix can be determined
  - for instance : $\left\{K_I^{aux1} = 1\, K_{II}^{aux1} = 0\, K_{III}^{aux1} = 0\, K_{IV}^{aux1} = 0\right\}$

    $\left\{K_I^{aux2} = 0\, K_{II}^{aux2} = 1\, K_{III}^{aux2} = 0\, K_{IV}^{aux2} = 0\right\}$

$$
\begin{aligned}
\frac{I^{aux1,aux2}}{\int_\Gamma \Lambda d\Gamma} &= Y_{MN} K_M^{aux1} K_N^{aux2} \\
&= Y_{12} = Y_{21}
\end{aligned}
$$

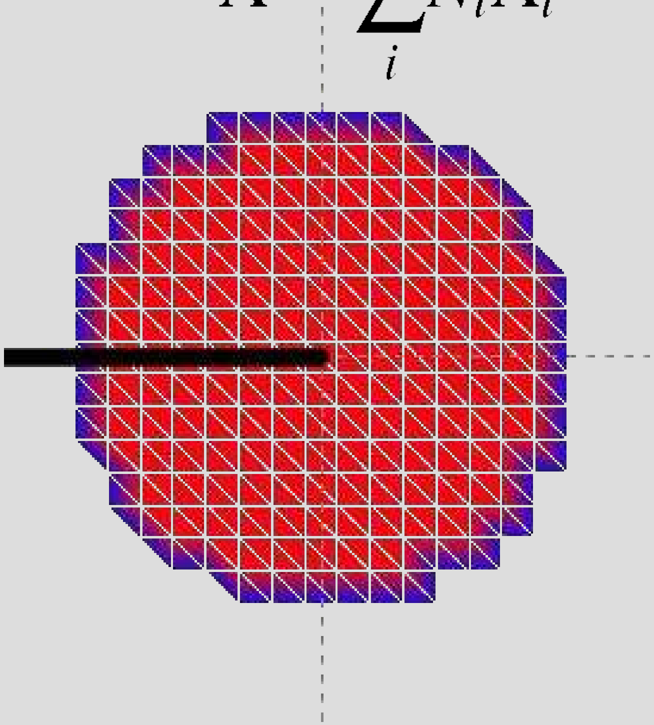
  - No need of finite element support because the Irwin matrix is intrinsic (for a given material orientation)

$$
\begin{aligned}
I^{aux1,aux2} &= \int_\Gamma q_i n_j \left(\sigma_{kj}^{aux1} u_{k,i}^{aux2} + \sigma_{kj}^{aux2} u_{k,i}^{aux1} + D_j^{aux1} \varphi_{,i}^{aux2} + D_j^{aux2} \varphi_{,i}^{aux1}\right) d\Gamma \\
&- \int_\Gamma q_i n_j \frac{1}{2} \left(\sigma_{kl}^{aux1} \varepsilon_{kl}^{aux2} + \sigma_{kl}^{aux2} \varepsilon_{kl}^{aux1} - D_k^{aux1} E_k^{aux2} - D_k^{aux2} E_k^{aux1}\right) \delta_{ij} d\Gamma
\end{aligned}
$$

Choice of Λ

The field Λ describes the geometry of the integration domain *S*.

$$\Lambda = \sum_i N_i \Lambda_i \qquad \Lambda_i = \begin{cases} 1 & \text{if } support(N_i) \subset S \\ 0 & \text{otherwise} \end{cases}$$

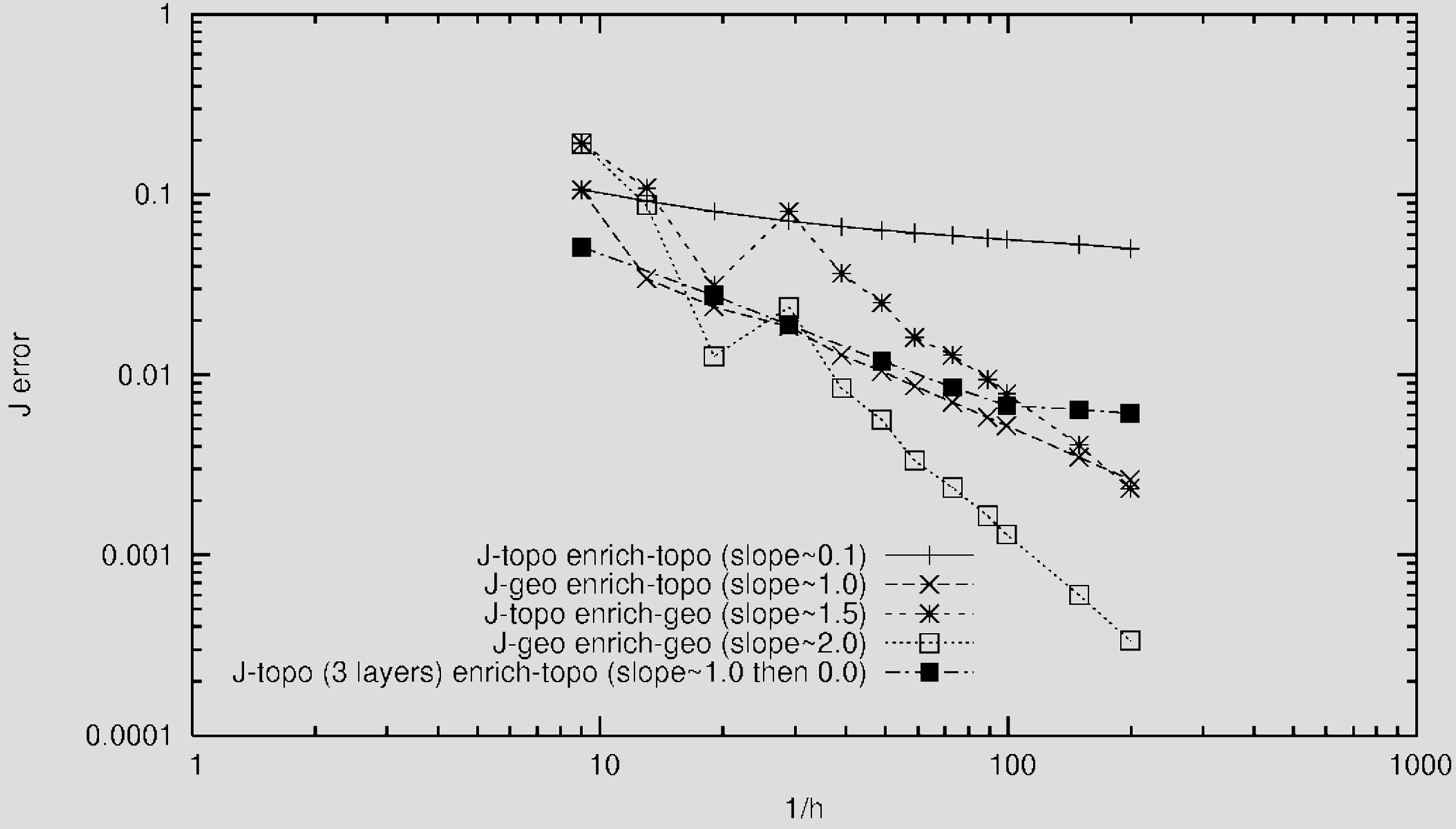


Two choices of integration domain with regard to *h*:

- Topological
- Geometrical

J error

- Exact J vs computed J-integral



Conclusion & future work

- Application of the X-Fem for piezos
- The convergence study shows that the four classical enrichment functions are enough
- Use of equivalement volume integrals to compute the electromechanical J-integral
- Interaction integrals will be used to extract K factors
- Auxiliary fields can also be used to compute the local Irwin matrix

Conclusion & future work

- Systematic investigations of J and the K factors's accuracy
- Propagation laws
- Investigation for a electrically permeable crack
- 3D extensions (esp. for the eigenfunctions needed for K extraction)