



# Application of Zagarola/Smits scaling in turbulent boundary layers with pressure gradient

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## Abstract

The empirical velocity scale determined for zero pressure gradient turbulent boundary layers of Zagarola and Smits [1],  $U_\infty(\delta_*/\delta)$ , is derived for boundary layers with and without pressure gradient using similarity principles. This scaling is successful in removing the Reynolds number dependence of the outer mean velocity profiles. Even more interesting, it produces three profiles in turbulent boundary layers regardless of the strength of the pressure gradient: one for Adverse Pressure Gradient, APG, one for Favorable Pressure Gradient, FPG, and one for Zero Pressure Gradient, ZPG. These results have been shown to be consistent with Castillo and George [2] obtained by means of similarity analysis of the RANS equations.

## 1 Introduction

For more than 50 years researchers in the turbulence community have been trying to collapse (i.e. meaning profiles are described by a single curve) data of turbulent boundary layer, especially boundary layers with pressure gradient. In spite of the intense effort, attempts have not been quite successful. More recently, Castillo [3] and Castillo and George [2] showed using similarity analysis that the proper outer velocity scale in turbulent boundary layers is the free stream velocity,  $U_\infty$ . They were not able to completely collapse the data, however, with just,  $U_\infty$ . This was attributed to finite Reynolds number effects. Furthermore, Castillo [4] showed that the Reynolds number dependence observed in the flow was primarily due to the effect of the upstream conditions, and not to the local Reynolds number dependence as



suggested by George and Castillo [5] paper for the zero pressure gradient boundary layers. Even the classical scaling of the deficit velocity with the friction velocity,  $u_*$ , was not able to collapse the data as traditionally though it will, Castillo [3]. Zagarola and Smits [1] found an empirical velocity scale,  $U_\infty(\delta_*/\delta)$ , for the outer flow of the zero pressure gradient turbulent boundary layers. Recently Wosnik and George [6] showed that this scaling was consistent with the theory of George and Castillo [5] for zero pressure gradient boundary layers.

The purpose of this paper is to extend the Zagarola/Smits scaling for incompressible turbulent boundary layers with pressure gradient and to show that it removes all Reynolds number dependence observed in the overlap region. This is done especially for flows at separation and flows with strong dependence on the upstream conditions, for example, the “relax flow” data of Bradshaw and Ferris [7]. A relax flow is one where the conditions of the flow, like the pressure gradient, are suddenly changed to new conditions. Also, it will be shown that this scaling leads to only three basic velocity profiles: one for APG, one for FPG, and one for ZPG boundary layers. These results are consistent with results shown by Castillo and George [2]. Using strictly similarity analysis they demonstrated that only three profiles describing turbulent boundary layers exist. These conclusions hold regardless of the strength of the pressure gradient.

## 2 The mean velocity scale

In order to determine the outer velocity scale proposed by Zagarola and Smits [1], the similarity ideas proposed by George and Castillo [3] will be employed. The general concept is outlined below as:

- **Similarity Analysis:**

The outer mean velocity scale for turbulent boundary layers is determined from the displacement thickness equation and not chosen a priori or using dimensional analysis. George and Castillo [8] applied this concept to the RANS equations in order to determine the mean velocity, and Reynolds stresses scales in turbulent boundary layers.

- **Asymptotic Invariance Principle: AIP**

This principle means that in limit as the  $Re \rightarrow \infty$  the boundary layer equations become independent of  $Re$  and, therefore, any function or scale must also be independent of  $Re$  as well.

- **Similarity Solution Form:**

The basic assumption is that it is possible to express any dependent variable, in this case the outer mean velocity,  $U$ , as a product of two functions,

$$U - U_\infty = U_{so}(x) f_{op}(\bar{y}, \delta^+; \Lambda; *) \quad (1)$$



where  $U_{so}$  is the outer velocity scale and depends on  $x$  only. Castillo and George [2] showed that for ZPG and PG flows the proper velocity scale in outer variables is the free stream velocity,  $U_{so} = U_{\infty}(x)$ . The arguments inside the similarity function  $f_{op}$  represent the outer similarity length scale,  $\bar{y} = y/\delta$ , the Reynolds number dependence,  $\delta^+ = \delta u_*/\nu$ , the pressure parameter,  $\Lambda$ , and any other possible dependence on the upstream conditions,  $*$ , respectively. The pressure parameter,  $\Lambda$  was determined via similarity analysis by Castillo [3], and Castillo and George [2] using the RANS equations and is given as,

$$\Lambda \equiv -\frac{\delta}{U_{\infty} d\delta/dx} \frac{dU_{\infty}}{dx} = constant \quad (2)$$

or equivalently

$$\Lambda \equiv \frac{\delta}{\rho U_{\infty}^2 d\delta/dx} \frac{dP_{\infty}}{dx} = constant \quad (3)$$

Furthermore, it is assumed that the function  $f_{op}$  could be expressed as a product of two functions. The first one,  $G(\delta^+; *)$ , contains the Reynolds number dependence and the upstream conditions, and the second one,  $f_{op\infty}(\bar{y}; \Lambda)$ , contains the similarity length scale,  $\bar{y}$ , and the pressure parameter. Thus, the function  $f_{op}$  is written as,

$$f_{op}(\bar{y}, \delta^+; \Lambda; *) = G(\delta^+; *) f_{op\infty}(\bar{y}; \Lambda) \quad (4)$$

where  $f_{op\infty}$  represents the asymptotic profile in the limit as  $Re \rightarrow \infty$ . As required by the AIP, this asymptotic profile must be independent of Reynolds number and its shape maybe different for ZPG, FPG, and APG turbulent flows. This decomposition of the profile has been used by Wosnik and George [6] in ZPG boundary layers to show that the George and Castillo [3] theory for ZPG is consistent with the Zagarola/Smits scaling. To determine the scale for the G function, eqn (4) is substituted into the displacement thickness equation,

$$[\delta_*] = [\delta G] \int_0^{\infty} f_{op\infty}(\bar{y}) d\bar{y} \quad (5)$$

Similarity exists only if the terms inside the square brackets vary together, in other words they must have the same  $x$ -dependence. This implies,

$$G \sim (\delta_*/\delta) \quad (6)$$

According to the Asymptotic Invariance Principle, AIP, the G function must be asymptotically independent of Reynolds number. Thus,  $G \rightarrow constant$ , so the outer velocity scale proposed by [3],  $U_{\infty}$  is the asymptotic outer velocity scale of the Zagarola/Smits scale,  $U_{\infty}(\delta_*/\delta)$ . The fact that the Zagarola/Smits scaling contains the Reynolds number dependence term  $\delta_*/\delta$  means that the boundary layer is indeed Reynolds number dependent as shown by [3].



### 3 Results

The goal is to show that the Zagarola/Smits scaling successfully removes the Reynolds number dependence of the flow and to show that three basic profiles are obtained in turbulent boundary layers, regardless of the strength of the pressure gradient.

#### 3.1 The outer mean velocity profiles

Figures 1 and 2 show the outer mean velocity profiles of three different zero pressure gradient turbulent boundary layers of Smith and Walker [9], Purtell et al. [10], and Österlund [11]. The Smith and Walker profiles vary in Reynolds number based on momentum thickness from about  $3,000 \leq R_\theta \leq 48,300$ . The data from Purtell et al. has a variation in Reynolds number from about  $460 \leq R_\theta \leq 5,100$ . This is one of the lowest Reynolds number data over a flat plate available today. Finally, the most recent data of Österlund varies in Reynolds number from about  $4,000 \leq R_\theta \leq 26,000$ .

Figure 1 shows the corresponding velocity profiles plotted as a deficit and normalized by  $U_\infty$  and  $\delta_{99}$ . Note that the Smith and Walker data, top left, and the Österlund data, bottom figures, show a Reynolds number dependence. It can be observed that as the Reynolds number increases the profiles tend to approach an asymptotic profile. This observation is consistent with the Asymptotic Invariance Principle, AIP. However, the low Reynolds number data of Purtell et al. collapsed with the free stream velocity,  $U_\infty$ , in spite of the fact that it is in this range of Reynolds number ( $460 \leq R_\theta \leq 5,100$ ) where the boundary layers tend to exhibit a strong dependence on Reynolds number. The difference between the data of Purtell et al. and the other two relatively high Reynolds number data of Österlund and Smith and Walker is that Purtell et al. kept the upstream conditions fixed and the other experiments basically kept the streamwise position fixed and vary the upstream conditions, such as the wind tunnel speed.

The same data is shown in figure 2 but normalized by the outer velocity scale,  $U_{so} = U_\infty(\delta_*/\delta)$ , the Zagarola/Smits scaling, and the boundary layer thickness,  $\delta_{99}$ . Although it is very clear that this scaling removes all the Reynolds number dependence, Castillo [4] showed that when the same data are normalized by  $U_\infty$  but for fixed upstream conditions (for example, same wind tunnel speed,  $U_\infty$ , or same dimensions of tripping wire etc.), the variation in  $Re$  seen in these profiles of figure 1 is no longer observed and the Zagarola/Smits scaling is not necessary. In other words, Castillo [4] showed that the Reynolds number dependence in turbulent boundary layers is due primarily to the upstream conditions and not to the local Reynolds number dependence as originally suggested by George and Castillo [5] for



## ZPG turbulent boundary layers.

Figures 3 and 4 show the APG experimental data of Newman [12], Ludwig and Tillmann [13], Bradshaw and Ferris [7], and Clauser [14] normalized by  $U_\infty$  and  $U_\infty(\delta_*/\delta)$ , respectively. The data of Newman has a strong APG over an airfoil and the profiles eventually separate. The range of Reynolds number based on the momentum thickness for this data is between,  $5,510 \leq R_\theta \leq 26,800$ . The Ludwig and Tillmann experiment has also a very strong APG and eventually separation takes place from the diverging channel. The Reynolds number varies from about  $5,400 \leq R_\theta \leq 48,300$ . In the experimental data of Bradshaw and Ferris the conditions of the flow are suddenly changed from moderate adverse pressure gradient to zero pressure gradient and the profiles vary in Reynolds number from about  $8,600 \leq R_\theta \leq 22,600$ . This flow is called "relax flow", Coles and Hirst [15], because conditions in flow are suddenly changed. Finally, the experiment from Clauser is very unique and different from the previous APG data. This data varies in Reynolds number from 8,000 to 31,000. In this experiment a moderate APG data was design particularly to show that equilibrium flows (according to Clauser [4] definition) do not vary with streamwise direction when scaled with the friction velocity,  $u_*$ . However, notice that these profiles are not supposed to collapse with the free stream velocity,  $U_\infty$ , but they do collapse. Castillo [4] showed that when the upstream conditions are maintained constant the outer mean profiles collapse with just  $U_\infty$  as in Clauser's own data.

The mean velocity profiles are shown in figure 3 as a deficit and normalized by  $U_\infty$  and  $\delta_{99}$ . Clearly, the mean profiles show a dependence on Reynolds number and they tend to approach an asymptotic state as the Reynolds number increases. On the other hand, the same data are shown in figure 4 but normalized by  $U_{so} = U_\infty(\delta_*/\delta)$  and  $\delta_{99}$ . As before, the Zagarola/Smits scaling successfully removes all the Reynolds number dependence of the mean profiles, even for near separated flows and at separated turbulent flows. It is very important to understand that these profiles do not collapse at all if the data is normalized by the classical scaling,  $(U_\infty - U)/u_*$ , where  $u_*$  is the friction velocity. Castillo and George [2] include other data in their analysis, such as Clauser [14] and Bradshaw [16], and were able to collapse those mean profiles with just  $U_\infty$  better than using the tradition scaling.

The mild favorable pressure gradient data of Herring and Norbury [17], and the moderate FPG data of Ludwig and Tillmann [13] are shown in figure 5. The top figures are the mean profiles normalized with just  $U_\infty$ , while the bottom figures represent the same data normalized with  $U_\infty(\delta_*/\delta_{99})$ . The Zagarola/Smits scaling is not necessary for these data because the profiles collapse with just  $U_\infty$  reasonably well, however, it will be used later to prove that only three profiles exist in turbulent boundary layers.



### 3.2 The three basic velocity profiles

The purpose of this section is to use the Zagarola/Smits scaling to show that there are only three basic profiles needed to characterize turbulent boundary layers, and that this result is independent of the strength (weak, mild, strong) of the pressure gradient.

The first plot from the top of figure 6 shows some of the zero pressure gradient data of figures 1 and 2. Also note that all ZPG data collapse to a single curve. The second plot from the top of figure 6 represents some of the adverse pressure gradient data shown in figures 3 and 4. It is important to clarify that these profiles eventually separate and some of these profiles are from the relaxed flow data of Bradshaw and Ferris. Again, it is clear from figure 6 that all APG do collapse to a single curve, indicating that all APG have same profile shape. The third plot from the top of figure 6 is the favorable pressure gradient data, which clearly proves that all FPG data also collapse to one single curve. The fourth plot from the top of figure 6 shows the ZPG, APG, and FPG data together and clearly it illustrates that there are three basic profiles in turbulent boundary layers. The APG profile is distinctly different from the other two. The differences between the ZPG and FPG profiles are less obvious, but real nonetheless.

These results are consistent with Castillo and George [2] which showed that there are only three values of the pressure parameter,  $\Lambda$ : one for ZPG,  $\Lambda = 0$ , one for APG,  $\Lambda = 0.22$  and one for FPG,  $\Lambda = -1.915$ . Furthermore, they showed that the outer boundary layer equation is only dependent on this pressure parameter, and if there are only three values of  $\Lambda$  then there must be only three profiles, which has been shown here using the Zagarola/Smits scaling. Moreover, Castillo [4] was able to show that when all the upstream conditions of the flow are maintained fixed the outer mean profiles do not exhibit the Reynolds number dependence observed in most profiles shown here.

## 4 Conclusion

It is clear after careful analysis of the data for turbulent boundary layers that the flow does not scale with a single velocity scale as suggested in classical theory of Clauser [14] and Townsend [18]. In fact, from the similarity analysis of the outer boundary layer equations [3], it was shown that the free stream velocity,  $U_\infty$ , is the proper outer velocity scale. Furthermore, it was shown by [5] that the overlap region is governed by a two velocity scale. As a result, the overlap layer is indeed a Reynolds number dependent region. Moreover, when the Zagarola and Smits scaling,  $U_\infty(\delta_*/\delta_{99})$ , is used to normalize the outer mean profiles, the Reynolds number dependence is successfully removed. This result indicates that the boundary layer is in-



deed Reynolds number dependent. Even more interesting, is the fact that this scaling yields three basic velocity profiles, regardless of the strength of pressure gradient: one for ZPG, one for APG, and one for FPG. These results are consistent with the similarity analysis of [4] which achieved the same conclusion by using the RANS equations. Finally, the Zagarola/Smits scaling,  $U_\infty(\delta_*/\delta)$ , is asymptotically equivalent to the scaling determined by [2,5],  $U_\infty$ , for ZPG and PG boundary layers.

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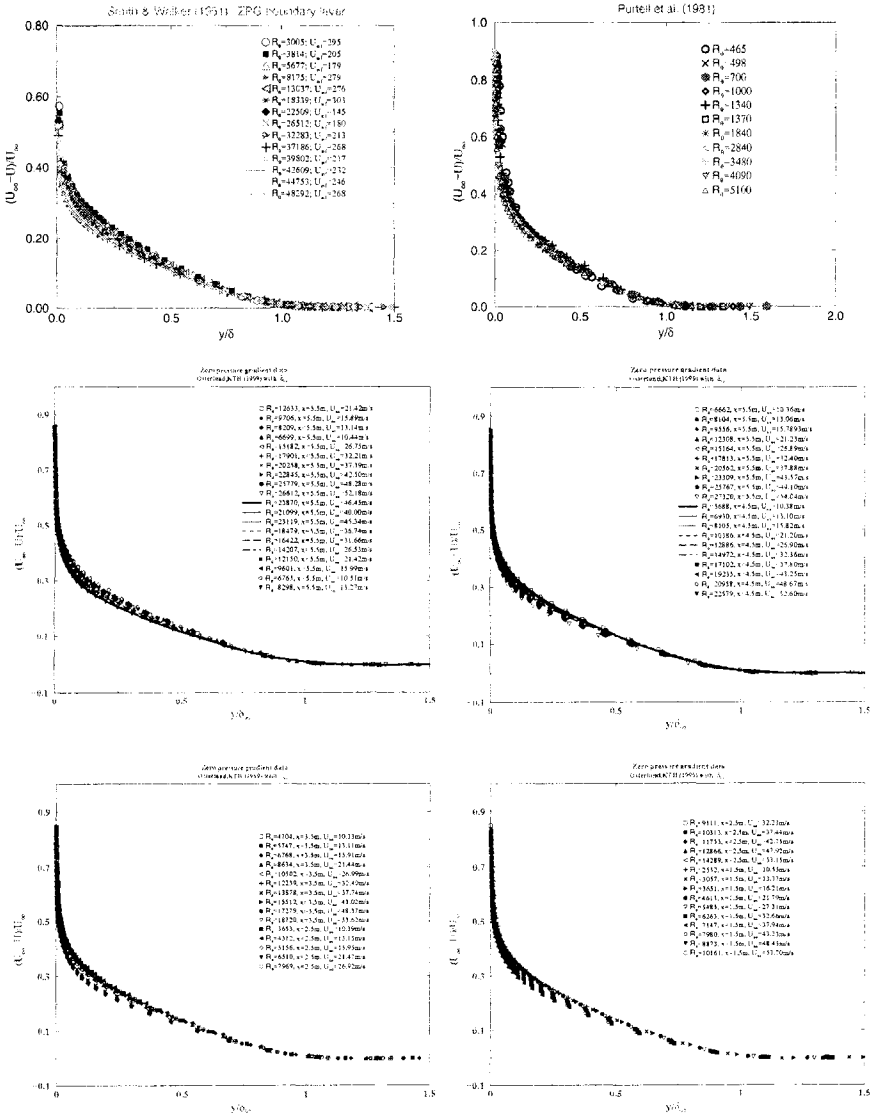


Figure 1: Mean velocity profiles in zero pressure gradient normalized by  $U_\infty$  and  $\delta_{99}$ . *Top left*, Smith and Walker [9], *top right*, Purtell et al. [10], and *bottom figures*, Österlund [11].



Displacement Thickness Scale:  $U_{so} = U_{\infty}(\delta^*/\delta_1)$

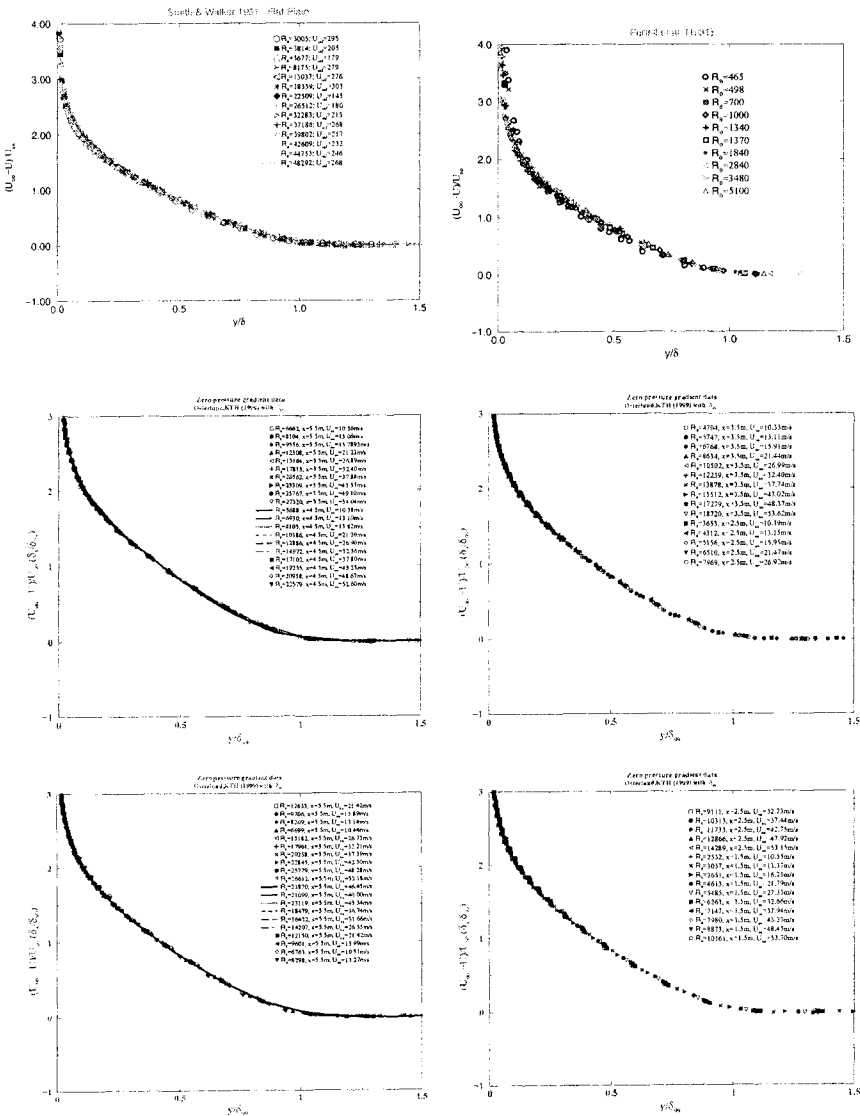


Figure 2: Mean velocity profiles in zero pressure gradient normalized by  $U_{so} = U_{\infty}(\delta^*/\delta_{99})$  and  $\delta_{99}$ . *Top left*, Smith and Walker [9], *top right*, Purtell et al. [10], and *bottom figures*, Österlund [11].

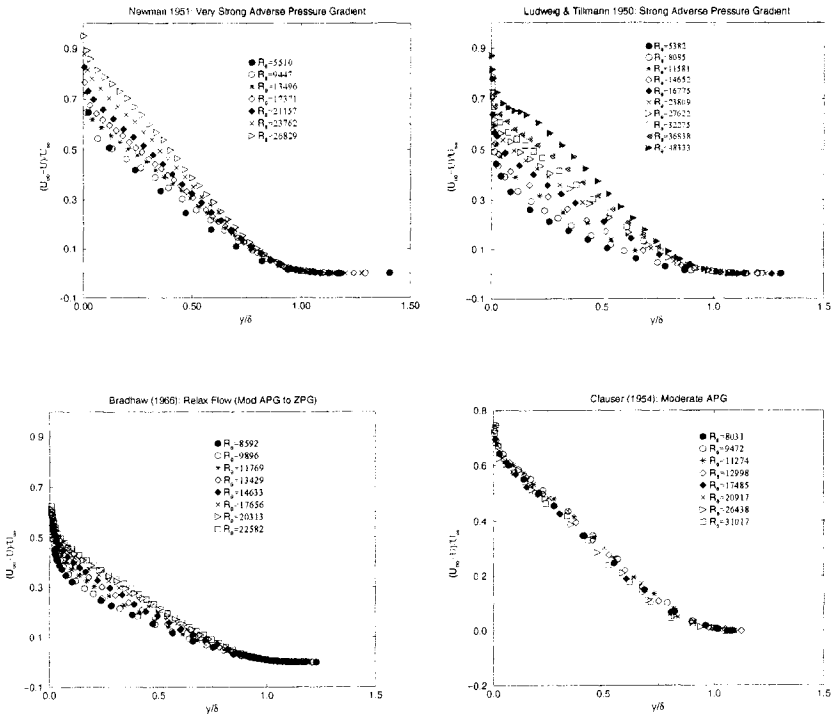


Figure 3: Mean velocity profiles in adverse pressure gradient boundary layers normalized by  $U_\infty$  and  $\delta_{99}$ . *Top left* is the strong APG data of Newman [12], *top right* is the strong APG data of Ludwig and Tillmann [13], at the *bottom left* is the relax flow data of Bradshaw and Ferris [7], and at the *bottom right* is the moderate APG data of Clauser [14].

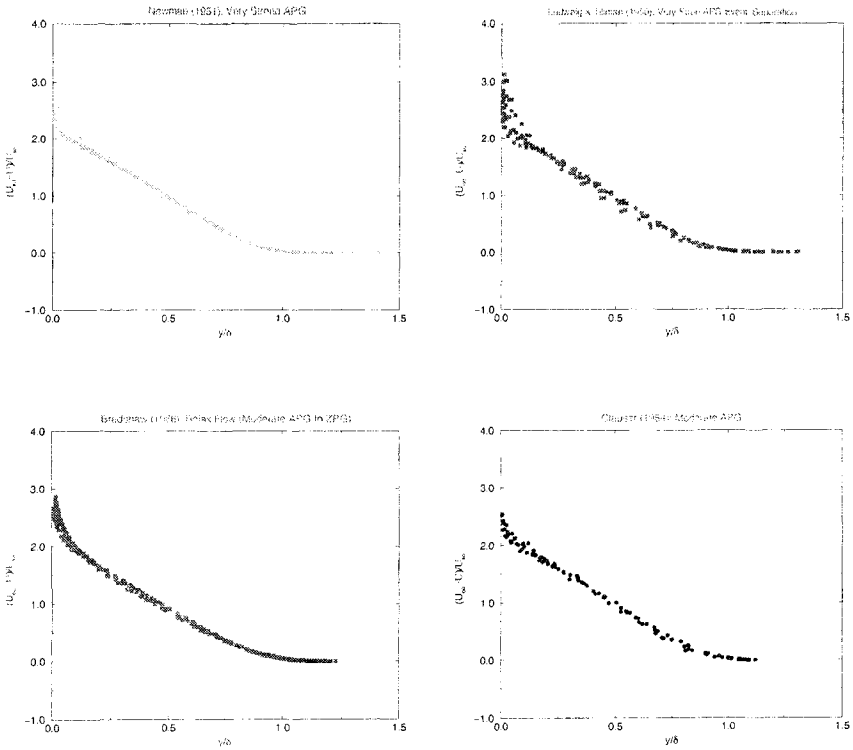


Figure 4: Mean velocity profiles in adverse pressure gradient boundary layers normalized by  $U_{so} = U_{\infty}(\delta_*/\delta)$  and  $\delta_{99}$ . *Top left* is the strong APG data of Newman [12], *top right* is the strong APG data of Ludwig and Tillmann [13], at the *bottom left* is the “relax flow” data of Bradshaw and Ferris [7], and finally at the *bottom right* is the moderate APG data of Clauser [14].

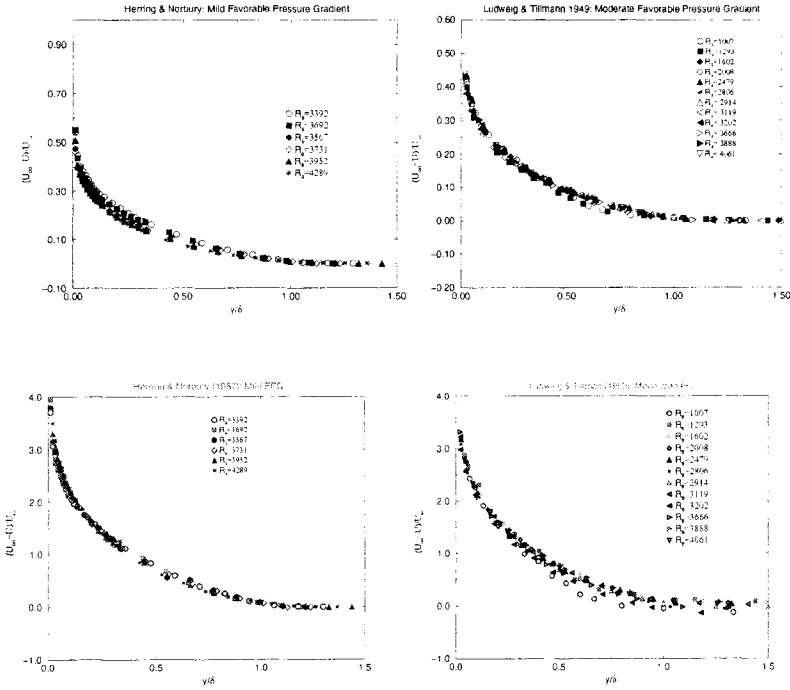


Figure 5: Mean velocity profiles in favorable pressure gradient boundary layers. The upper figures are the data of Herring and Norbury [17] at mild FPG, and Ludwig and Tillman [13] at moderate FPG respectively. These profiles are normalized by  $U_\infty$  and  $\delta_{99}$ , while the bottom figures are normalized by  $U_{s0} = U_\infty(\delta_*/\delta)$  and  $\delta_{99}$ .

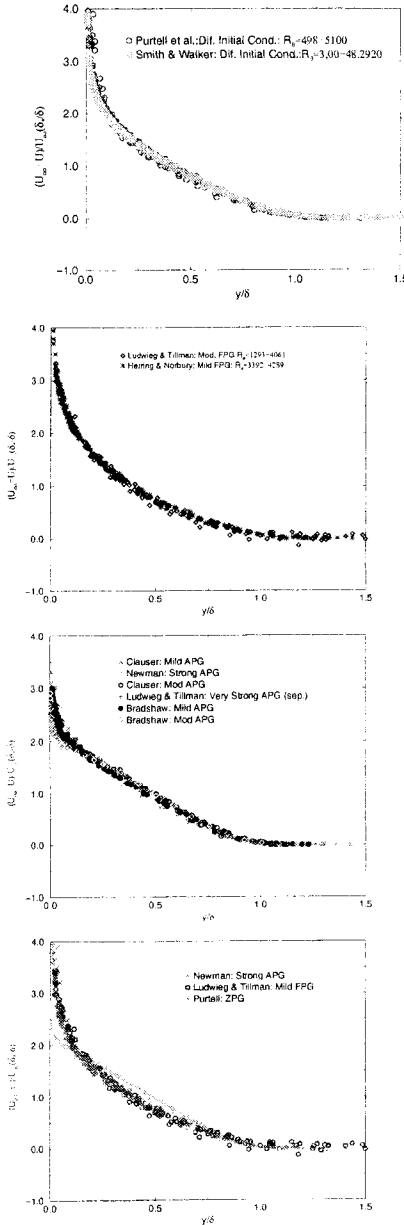


Figure 6: The three basic velocity profiles in turbulent boundary layers normalized by  $U_\infty(\delta^*/\delta_{99})$  and  $\delta_{99}$ . *First top figure* is the ZPG profiles, *second figure* is the FPG profiles and, *third figure* is the APG, and the *fourth figure* is the combined plot of the APG ( $\bullet$ ), FPG ( $\circ$ ) and ZPG ( $\ast$ ).