Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090128

Applications of Adaptive Multi Step Differential Transform Method to Singular Perturbation Problems Arising in Science and Engineering

Essam R. El-Zahar[1,2,*]

[1] Department of Mathematics, College of Sciences and Humanities, Salman bin Abdulaziz University, P. O. Box 83, Alkharj, 11942, Saudi Arabia
[2] Department of Basic Engineering Science, Faculty of Engineering, Shebin El-Kom, Menofia University, Egypt

Received: 28 Mar. 2014, Revised: 28 Jun. 2014, Accepted: 29 Jun. 2014
Published online: 1 Jan. 2015

Abstract: In this paper, piecewise-analytical and numerical solutions of singular perturbation initial-value problems are obtained by an adaptive multi-step differential transform method (MsDTM). The principle of the method is introduced, and then applied to different types of practical problems arising in science and engineering. Analytical and numerical solutions are obtained using piecewise convergent series with easily computable components over a sequence of variable-length sub-intervals. Numerical results are compared to those obtained by the classical MsDTM and the Runge-Kutta method. The results demonstrate the reliability and efficiency of the method in solving the considered problems.

Keywords: Multi-step differential transformation Method; Variable step-size methods; Singular perturbation initial-value problems.

1 Introduction

Many mathematical problems arising from the real world cannot be solved completely by analytical means. One of the most important mathematical problems arising in applied science and engineering is Singular Perturbation Problems (SPPs), also known as stiff problems (Bender and Orszag [2], Johnson[15], Kumar and Parul[17]). SPPs, governing mathematical models, arise in many interesting fields of science and engineering, especially in automatic control, chemical and biochemical reactions, electrical circuits, fluid mechanics, solid state physics, atmospheric pollution, etc. A well known fact is that the solution of such problems has a multiscale character, i.e. there exist thin layers where the solution varies very rapidly, while away from the layers the solution behaves regularly and varies slowly. Therefore, the numerical treatment of SPPs presents some major computational difficulties. For a detailed discussion on the analytical and numerical treatment of such problems one may refer to the books of Doolan *et al.* [4], O'Malley [23], Roos *et al.* [25], Miller *et al.* [20] and Smith [27]. Recently, piecewise semi analytical- numerical methods, which do not require perturbation or linearization, are introduced for finding solutions of nonlinear problems. Multi-step Differential Transform Method (MsDTM) is one of the most effective, convenient and accurate methods for both weakly and strongly nonlinear problems. MsDTM does not require analytical integration or symbolic computations as other peer piecewise semi analytical-numerical methods. The method formulates the Taylor series in a totally different manner and provides the solution in terms of convergent series over a sequence of equal-length sub-intervals. Different applications of MsDTM can be found in (Odibat *et al.* [22], Keimanesh *et al.* [18], Gokdogan *et al.* [10], Yildirim *et al.* [28], Erturk *et al.* [8], El-Zahar [5] and Patra and Ray [24]). However, for some important classes of problems and for the sake of accuracy and efficiency, it is necessary to allow variable-length step-size to be used (Celik Kizilkan and Aydin [3], Habib and El-Zahar [12] and Gu *et al.* [11]). Therefore, two different algorithms of adaptive step-size MsDTM (AMsDTM) were presented in (Gokdogan *et al.*[9] and El-Zahar [6]) and succeeded in obtaining reliable approximate solutions for nonlinear problems. For singularly perturbed BVPs, the differential

* Corresponding author e-mail: essam_zahar2006@yahoo.com

transformation with asymptotic techniques are presented for obtaining approximate solutions of second and fourth orders BVPs by El-Zahar [6,7]. Variational iteration method is presented by (Zhao and Xiao [29], Zhao *et al.* [30]) for Singularly Perturbed IVPs (SPIVPs) with and without delays where the obtained sequence of iterates is based on the use of Lagrange multipliers. The aim of our study is to introduce the AMsDTM as an alternative to existing methods in solving SPIVPs. In this paper, the piecewise analytical and numerical solutions of SPIVPs are obtained by AMsDTM. The principle of the method is introduced and then applied directly without requiring linearization, perturbation, analytical integration or symbolic computations, to different types of practical problems arising in science and engineering. Analytical and numerical solutions are obtained using piecewise convergent series with easily computable components over a sequence of variable length sub-intervals. The current results are compared with those obtained by classical MsDTM and the fourth order Runge-Kutta method. The results demonstrate reliability and efficiency of the method in solving the considered problems.

2 Multi-Step Differential Transform Method

The basic definition and the fundamental theorems of the MsDTM are given in ([22],[18],[10],[28],[8],[5],[24]). For convenience of the reader, we present a review of the MsDTM. Consider the following IVP for systems of ODEs

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, ..., x_n), \\ x_2'(t) &= f_2(t, x_1, x_2, ..., x_n), \\ &\vdots \\ x_n'(t) &= f_n(t, x_1, x_2, ..., x_n), \end{aligned} \tag{1}$$

subject to the initial conditions

$$x_i(t_0) = c_i, \quad i = 1, 2, ..., n. \tag{2}$$

Let $[t_0, T]$ be the interval over which we want to find the solution of the initial value problem (1)-(2). In actual applications of the Differential Transform Method (DTM), the Nth-order approximate solution of the initial value problem (1)-(2) can be expressed by the finite series (Zhou[31], Jang *et al.*[14], Abdel-Halim Hassan[1])

$$x_i(t) = \sum_{k=0}^{N} X_i(k)(t - t_0)^k, \quad t \in [t_0, T], i = 1, 2, ..., n, \tag{3}$$

where

$$X_i(k) = \frac{1}{k!}\left[\frac{d^k x_i(t)}{dt^k}\right]_{t=t_0}, i = 1, 2, ..., n. \tag{4}$$

Equations (3) and (4) imply that the concept of differential transformation is derived from the Taylor series expansion. The following theorems can be deduced from (3) and (4)

Theorem 2.1. *If* $x(t) = \beta\,(u(t) \pm v(t))$, *then* $X(k) = \beta\,U(k) \pm \beta\,V(k)$.

Theorem 2.2. *If* $x(t) = u(t)\,v(t)$, *then* $X(k) = \sum_{\ell=0}^{k} U(\ell)\,V(k-\ell)$.

Theorem 2.3. *If* $x(t) = \frac{d^m u(t)}{dt^m}$, *then* $X(k) = \frac{(k+m)!}{k!} U(k+m)$.

Theorem 2.4. *If* $x(t) = (\beta + t)^m$, *then* $X(k) = H[m,k]\frac{m!}{k!|(m-k)|!}(\beta + t_0)^{m-k}$, *where* $H[m,k] = \begin{cases} 1 \;, & if \;\; m-k \geq 0 \\ 0 \;, & if \;\; m-k < 0 \end{cases}$.

Theorem 2.5. *If* $x(t) = e^{\lambda t}$, *then* $X(k) = \frac{\lambda^k}{k!} e^{\lambda t_0}$.

Theorem 2.6. *If* $x(t) = \sin(\omega t + \beta)$, *then* $X(k) = \frac{\omega^k}{k!}\sin(\omega t_0 + \beta + \frac{k\pi}{2})$.

Theorem 2.7. *If* $x(t) = \cos(\omega t + \beta)$, *then* $X(k) = \frac{\omega^k}{k!}\cos(\omega t_0 + \beta + \frac{k\pi}{2})$.

Using some fundamental operations of DTM, we have the following recurrence relation:

$$(k+1)X_i(k+1) = F_i(k, X_1, X_2, ..., X_n), X_i(0) = c_i, i = 1, 2, ..., n, \tag{5}$$

where $F_i(k, X_1, X_2, ..., X_n)$is the differential transform of the function $f_i(t, x_1, x_2, ..., x_n)$, for $i = 1, 2, ..., n$.
The differential transform $X_i(k)$ of the unknown functions $x_i(t)$ can be obtained by solving the iterating algebraic system (5). In order to speed up the convergence rate and to improve the accuracy of resulting solutions, the entire interval $[t_0,\, T]$ is usually split into sub-intervals and the algorithm of MsDTM is applied as follows:
Assume that the interval $[t_0,\, T]$ is divided into M sub-intervals $[t_{m-1}\,,\, t_m]$, $m = 1, 2, ..., M$ of equal length step-size $h = (T - t_0)/M$ by using the nodes $t_m = t_0 + mh$. The main ideas of the MsDTM are as follows: First, we apply the DTM to the IVP (1)-(2) over the interval $[t_0, t_1]$, we will obtain the following approximate solution,

$$x_{i,1}(t) = \sum_{k=0}^{N} X_{i,1}(k)(t - t_0)^k \quad t \in [t_0, t_1], \tag{6}$$

using the initial conditions $x_i(t_0) = c_i$. For $m \geq 2$ and at each sub-interval $[t_{m-1}\,,\, t_m]$, we will use the initial conditions $x_{i,m}(t_{m-1}) = x_{i,m-1}(t_{m-1})$ and apply the DTM to the IVP (1)-(2) over the interval $[t_{m-1}\,,\, t_m]$. The process is repeated and generates a sequence of approximate solutions $x_{i,m}(t)$, $m = 1, ..., M$, $i = 1, 2, ..., n$ for the solutions $x_i(t)$,

$$x_{i,m}(t) = \sum_{k=0}^{N} X_{i,m}(k)(t - t_{m-1})^k \quad t \in [t_{m-1}, t_m], \tag{7}$$

Finally, the MsDTM assumes the following solution,

$$x_i(t) = \begin{cases} x_{i,1}(t) \quad , & t \in [t_0, t_1], \\ x_{i,2}(t) \quad , & t \in [t_1, t_2], \\ \vdots & \\ x_{i,M}(t) \quad , & t \in [t_{M-1}, t_M]. \end{cases} , i = 1, 2, ..., n. \quad (8)$$

3 Adaptive time step-size algorithm

While we apply MsDTM, we apply the following time step-size control algorithm presented by El-Zahar [6]

1. One gives the admissible local error $\delta > 0$, and chooses the order N of the MsDTM .

2. From calculations, the values $|X_{i,m}(N)|$, $i = 1, 2, ...n$, are known for every solution component i .

3. At the grid point t_m we calculate the value $\mathbf{E}_N = \max(|X_{i,m}(N)|)$, $i = 1, 2, ..., n$.

4. We select such step-size h_m for which $h_m = \tau \left(\frac{\delta}{\mathbf{E}_N}\right)^{1/N} \leq h_{\max}$ and $t_{m+1} = t_m + h_m$, where τ is a safety factor and $h_{\max}$ is the maximum allowed step-size .

Now, the present method is applied to obtain approximate analytical-numerical solutions of some important practical SPIVPs.

4 Applications to SPIVPs

In order to demonstrate the performance and efficiency of the present method in solving SPIVPs, we have applied it to four practical problems arising in various disciplines of science and engineering.

4.1 A diode oscillator with a current source

From the circuit diagram shown in Fig 1, the circuit equations can be constructed as follows (Johnson[15], Kumar and Parul[17])

$$i = C\frac{dV}{dt}, \;\; i_1 = (e^{\alpha V} - 1)I_s \, , i_2 = (1 - e^{-\alpha V})I_s \, , \quad (9)$$

and then Kirchhoff's law gives

$$C\frac{dV}{dt} + (e^{\alpha V} - e^{-\alpha V})I_s = I \sin \omega t \, , \quad (10)$$

which leads to the non-dimensional approximate equation ($x \propto e^{\alpha V}$)

$$\varepsilon \dot{x} \; = \; x \sin t - x^2 + \kappa, x(0) = a \; (1 > a > 0). \quad (11)$$

Typical values of the parameters are $\varepsilon = 0.03, \;\; \kappa = 10^{-5}$.

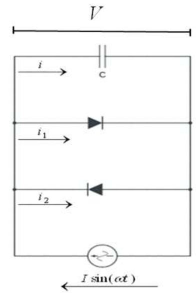

Fig 1: Circuit diagram for the diode oscillator with a current source

By using the fundamental operations of DTM, we obtained the following recurrence relation to (11):

$$X_m(k+1) = \frac{\sum\limits_{\ell=0}^{k} X_m(k-\ell)\frac{1}{\ell!}\sin(t_m + \frac{\ell\pi}{2}) - X_m(k-\ell)X_m(\ell) \; + H[0,k]\kappa}{(\varepsilon k + \varepsilon)}. \quad (12)$$

Solving the recurrence relation (12), the piecewise analytical solution of (11) for $a = 0.5$, $\varepsilon = 0.03$ and $t \in [0, 1]$, using AMsDTM with $N = 6$, $\delta = 0.001$, $\tau = 0.85$ and $h_{\max} = 0.2$ is given in Eq. (13).

$$x(t) \simeq \left\{ \begin{array}{l} 0.50000 - 8.33300t + 147.21667t^2 - 2499.87284t^3 + 42504.32619t^4 - 722256.25270t^5 \\ +12273105.78000t^6, \; t \in [0.00, 0.01770]. \;\; 0.50013 - 8.33041t + 146.00474t^2 - 2366.09663t^3 \\ +33807.93150t^4 - 356501.67870t^5 + 1951103.32400t^6, \; t \in [0.01770, \; 0.04174]. \;\; 0.49931 \\ -8.14601t + 131.07922t^2 - 1705.50379t^3 + 16526.61569t^4 - 101351.95220t^5 + 288748.0999t^6, \\ t \in [0.04174, 0.07481]. \;\; 0.49201 - 7.38469t + 98.18151t^2 - 931.12900t^3 + 6004.60280t^4 \\ -23043.87420t^5 + 39568.62653t^6, \; t \in [0.07481, \; 0.12085]. \;\; 0.46957 - 5.98433t + 61.56325t^2 \\ -412.26866t^3 + 1795.61565t^4 - 4510.16977t^5 + 4983.00556t^6, \; t \in [0.12085, \; 0.18589]. \\ 0.42729 - 4.29257t + 33.14124t^2 - 154.22318t^3 + 459.10519t^4 - 766.54274t^5 + 555.17846t^6, \\ t \in [0.18589, \; 0.27965]. \;\; 0.37498 - 2.87944t + 17.14826t^2 - 56.71364t^3 + 121.07922t^4 \\ -134.94633t^5 + 58.46625t^6, t \in [0.27965, \; 0.41609]. \;\; 0.29693 - 1.85672t + 11.82818t^2 \\ -43.14540t^3 + 104.78378t^4 - 129.39351t^5 + 61.42205t^6, t \in [0.41609, \; 0.55141]. \;\; -0.59046 \\ +10.40223t - 58.00728t^2 + 167.77051t^3 - 252.17056t^4 + 191.97064t^5 - 58.91925t^6, \\ t \in [0.55141, \; 0.68767]. \;\; 3.08113 - 26.40423t + 95.43070t^2 - 173.37622t^3 + 174.96493t^4 \\ -93.76382t^5 + 20.89383t^6, \; t \in [0.68767, \; 0.84963]. \;\; 1.68969 - 11.62824t + 36.83124t^2 \\ -56.26686t^3 + 47.64108t^4 - 21.47460t^5 + 4.02760t^6, \; t \in [0.84963, \; 1.0000] \, . \end{array} \right\} \quad (13)$$

Figure 2(a) shows the time-step length used by the AMsDTM for solving (11). We can observe from Fig 2 (a) that the given admissible local error $\delta = 0.001$, is achieved by AMsDTM using 11 time-step, $M = 11$, while MsDTM needs time-step size $h \leq 0.01770$ to achieve the given admissible local error and consequently needs at least 57 time-step. Figure 2(b) shows the obtained solutions of (11) using the AMsDTM ($N = 6$, $\delta = 0.001$), MsDTM ($N = 6$, $h = 1/M$), where $M = 11$ and the RK4 ($h = 0.0001$) at $\varepsilon = 0.03$. We can observe the high agreement between the AMsDTM solution and the RK4 solution, while MsDTM with 11 time-step results in a divergent solution. Figure 3 shows the high agreement between the AMsDTM solution and the RK4 solution at different values of the perturbation parameter ε, Fig 3 (a), and over a large interval, Fig 3 (b). The maximum absolute point wise differences between AMsDTM and RK4($h = 0.0001$) solutions at different values of δ and ε over the interval $[0, 1]$ are given in Table 1. We can observe that the admissible local error δ is achieved by the AMsDTM independent of the perturbation parameter ε.

Table 2 presents a comparison of the processing time and the time-step M used in solving (11) by the AMsDTM and MsDTM at $N = 6$ to achieve the specified tolerance δ at different values of ε over the interval $[0, 1]$, where all

Table 1: Maximum absolute differences between AMsDTM and RK4 solutions for problem 4.1.

	$\Delta = \max\lvert \text{AMsDTM} - \text{RK4}_{0.0001} \rvert$	
ε	$\delta = 0.001$	$\delta = 0.0001$
0.030	2.4766e-004	5.7011e-005
0.010	2.4029e-004	5.4202e-005
0.005	2.4216e-004	2.5836e-005
0.001	2.2804e-004	8.8854e-005

Table 2: Comparison of processing time and time-step for problem 4.1 at different values of δ and ε.

		MsDTM		AMsDTM	
ε	δ	Time-step M	Processing time (s)	Time-step M	Processing time (s)
0.030	0.0010	57	0.512	11	0.062
	0.0001	83	0.512	15	0.063
0.010	0.0010	167	0.824	19	0.078
	0.0001	246	1.254	24	0.125
0.005	0.0010	333	1.989	33	0.140
	0.0001	489	2.015	37	0.187
0.001	0.0010	1659	4.012	137	0.657
	0.0001	2435	5.982	141	0.672

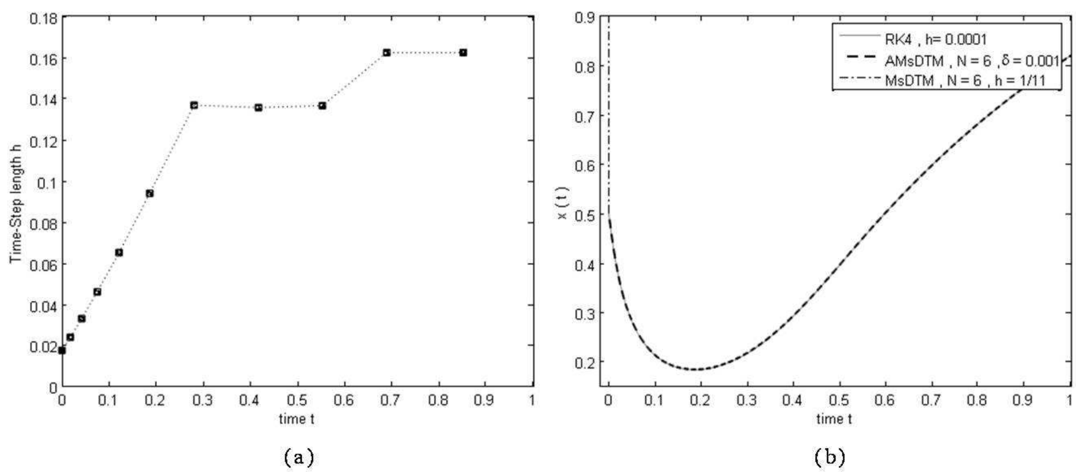

Fig 2: The time-step length h used by the AsDTM for solving problem 4.1 (**a**), and the approximate solution using AMsDTM, MsDTM and RK4 method (**b**) at $\varepsilon = 0.03$.

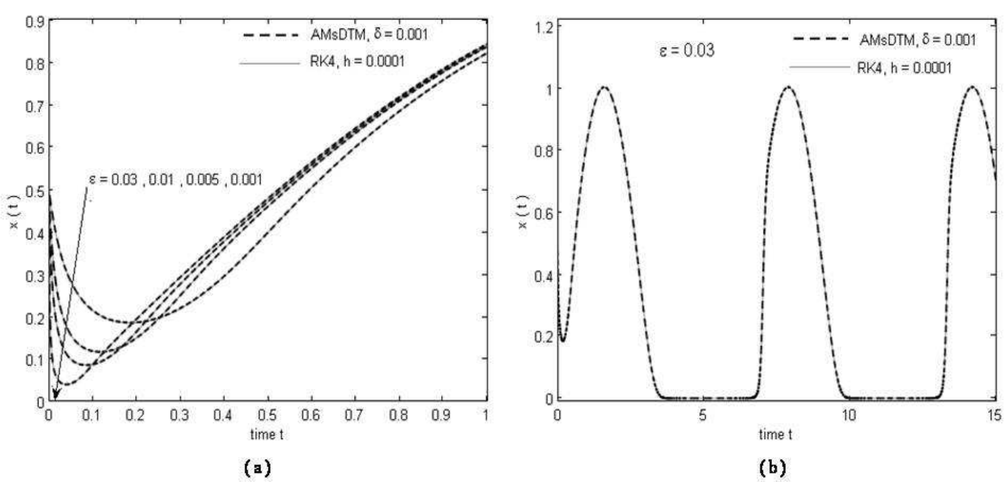

Fig 3: The solutions of problem 4.1 by AMsDTM and RK4 method (**a**) at different values of ε for $t \in [0,\ 1]$ and (**b**) at $\varepsilon = 0.03$ for $t \in [0,\ 15]$.

calculations are carried out by MAPLE 14 software in a PC with a Pentium 2 GHz and 512 MB of RAM.
We can observe that the AMsDTM is more effective than the MsDTM in approximating the solution of the diode oscillator SPIVP (11) with a minimum size of computations.

4.2 Thermal decomposition of Ozone

The kinetic steps involved for a dilute ozone-oxygen mixture are (Lapidus *et al.*[19], Miranker[21],Shampine *et al.*[26])

$$O_3 + O_2 \underset{k_2}{\overset{k_1}{\rightleftarrows}} O + 2O_2, O_3 + O \overset{k_3}{\rightarrow} 2O_2.$$

If the following dimensionless variables are defined $x = [O_3]/[O_3]_o, y = [O]/\varepsilon[O_3]_o, \kappa = 2k_2[O_2]_o/k_1, \varepsilon = k_1[O_2]_o/2k_3[O_3]_o$, and the time scale divided by $2/k_1[O_2]_o$, then the transient behavior is described by

$$\left.\begin{aligned} \frac{dx}{dt} &= -x - xy + \varepsilon\kappa y \\ \varepsilon\frac{dy}{dt} &= x - xy - \varepsilon\kappa y \end{aligned}\right\}, \tag{14}$$

with $x(0) = 1$, $y(0) = 0$. Typical values of the parameters are $\varepsilon = 1/98$, $\kappa = 3$ (Lapidus *et al.*[19]). The problem has a first-order asymptotic solution given by Ilea and Turnea[13]

$$\left.\begin{aligned} x(t) &= e^{-2t} + \varepsilon e^{-2t} - \varepsilon e^{-t/\varepsilon} + O(\varepsilon^2) \\ y(t) &= 1 - e^{-t/\varepsilon} - \varepsilon\kappa e^{-2t} + \varepsilon\left[\left(\frac{t}{\varepsilon} + \frac{t\kappa}{\varepsilon} + \kappa - 1\right)e^{-t/\varepsilon} + e^{-2t/\varepsilon}\right] + O(\varepsilon^2) \end{aligned}\right\}. \tag{15}$$

Applying AMsDTM on (14) results in the following recurrence relation:

$$\left.\begin{aligned} X_m(k+1) &= \left(-X_m(k) - \textstyle\sum_{\ell=0}^{k} X_m(\ell)\,Y_m(k-\ell) + \varepsilon\kappa Y_m(k)\right)/(k+1) \\ Y_m(k+1) &= \left(X_m(k) - \textstyle\sum_{\ell=0}^{k} X_m(\ell)\,Y_m(k-\ell) - \varepsilon\kappa Y_m(k)\right)/(\varepsilon k + \varepsilon) \end{aligned}\right\}. \tag{16}$$

The piecewise analytical solution of (14) for $t \in [0,\ 0.1]$, using AMsDTM ($N = 6$, $\delta = 0.001$) is given in Eqs.(17)-(18).

$$x(t) \simeq \left\{\begin{array}{ll} 1 - t - 47t^2 + 1663.3t^3 - 41696t^4 + 7.8859E5t^5 - 1.0491E7t^6, & t \in\ [0.0000,\ 0.00987] \\ 0.9868 - 1.5699(t-0.00987) - 15.902(t-0.00987)^2 + 607.43(t-0.00987)^3 & \\ -15668(t-0.00987)^4 + 3.2018E5(t-0.00987)^5 - 5.2701E6(t-0.00987)^6, & t \in [0.00987, 0.02095] \\ 0.96810 - 1.7642(t-0.02095) - 3.8834(t-0.02095)^2 + 193.29(t-0.02095)^3 & \\ -4993.8(t-0.02095)^4 + 1.0351E5(t-0.02095)^5 - 1.7902E6(t-0.02095)^6, & t \in [0.02095, 0.03423] \\ 0.9443 - 1.7992(t-0.03423) + 0.2995(t-0.03423)^2 + 49.344(t-0.03423)^3 & \\ -1275(t-0.03423)^4 + 2614(t-0.03423)^5 - 4.5212E5(t-0.03423)^6\ , & t \in [0.03423, 0.05093] \\ 0.9145 - 1.7640(t-0.05093) + 1.4609(t-0.05093)^2 + 8.6913(t-0.05093)^3 & \\ -241.63(t-0.05093)^4 + 4838.0(t-0.05093)^5 - 81870(t-0.05093)^6\ , & t \in [0.05093, 0.07313] \\ 0.8761 - 1.6927(t-0.07313) + 1.6437(t-0.07313)^2 + 0.3412(t-0.07313)^3 & \\ -34.133(t-0.07313)^4 + 667.12(t-0.07313)^5 - 10888(t-0.07313)^6\ , & t \in [0.0.07313, 0.10000] \end{array}\right\} \tag{17}$$

$$y(t) \simeq \left\{\begin{array}{ll} 98t - 4998t^2 + 1.6993E5t^3 - 4.2596E6t^4 + 8.0759E7t^5 - 1.0832E9t^6, & t \in\ [0.0000, 0.00987] \\ 0.60982 + 35.904(t-0.00987) - 1820(t-0.00987)^2 + 62129(t-0.00987)^3 - & \\ 1.5989E6(t-0.00987)^4 + 3.2682E7(t-0.00987)^5 - 5.3869E8(t-0.00987)^6, & t \in [0.00987, 0.02095] \\ 0.84915 + 11.764(t-0.02095) - 588.75(t-0.02095)^2 + 19867(t-0.02095)^3 - & \\ 5.0972E5(t-0.02095)^4 + 1.0560E7(t-0.02095)^5 - 1.8262E8(t-0.02095)^6, & t \in [0.02095, 0.03423] \\ 0.93557 + 3.1557(t-0.03423) - 156.43(t-0.03423)^2 + 5168.1t-0.03423)^3 - & \\ 1.3029E5(t-0.03423)^4 + 2.6681E6(t-0.03423)^5 - 4.6121E7(t-0.03423)^6, & t \in [0.03423, 0.05093] \\ 0.96105 + 0.60804(t-0.05093) - 31.526(t-0.05093)^2 + 1010.2(t-0.05093)^3 - & \\ 24769(t-0.05093)^4 + 4.9438E5(t-0.05093)^5 - 8.3596E6(t-0.05093)^6\ , & t \in [0.05093, 0.07313] \\ 0.96571 + 0.047123(t-0.07313) - 4.9378(t-0.07313)^2 + 150.71(t-0.07313)^3 - & \\ 3554.4(t-0.07313)^4 + 68305(t-0.07313)^5 - 1.1135E6(t-0.07313)^6\ , & t \in [0.07313, 0.1000] \end{array}\right\} \tag{18}$$

The solution of (14) using AMsDTM($N = 6$, $\delta = 0.001$), MsDTM ($N = 6$, $h = 1/M$), where $M = 20$, and the RK4 ($h = 0.001$) at $\varepsilon = 1/98$ and $\kappa = 3$ for $t \in [0,\ 1]$ is shown in Fig 4. As we can see the MsDTM with the same number of time steps used by AMsDTM , $M = 20$, results in a solution which is far away from the RK4 even at $t < 0.1$, while AMsDTM and RK4 solutions agree very well. Figures 5 and 6 show that the asymptotic solution, Eq.(15), have a small interval of convergence for the oxygen concentration solution, $y(t)$, and deviates much from RK4 solution, while the AMsDTM solution has a wide interval of convergence and agrees very well with RK4 solution at different values of the parameters ε and κ. We can observe that the singular perturbation method is not accurate for this problem when the parameters ε and κ are not enough small (Shampine *et al.*[26]). Results in Tables 3 and 4 confirm that the accuracy of the AMsDTM is independent of the perturbation parameter values and it is effective in approximating the solution of the thermal decomposition SPIVP (14) with a minimum size of computations.

4.3 Actuator control with high-gain feedback

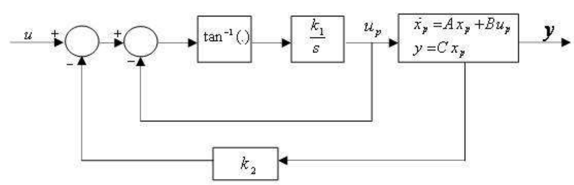

Fig 7: Actuator control with high gain feedback.

Consider the feedback control system shown in Fig 7 (Khalil[16]). The inner loop represents actuator control with high-gain feedback. The plant is a single-input single-output system represented by the state model$\{A\ , B, C\}$. The state equations of the closed loop system at $A = -1$, $B = 1$, $k_2 = 1/C$, $u = 1$ can be represented by the following SPIVP

$$\left.\begin{aligned} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= z \\ \varepsilon\frac{dz}{dt} &= \frac{-z}{1+y^2} + \varepsilon(4x - 2y + z) \end{aligned}\right\}, \tag{19}$$

where $\varepsilon = 1/k_1$, $x = x_p - 0.5$, $y = x_p + u_p - 1$ and $z = \dot{x}_p + \dot{u}_p$. Applying AMsDTM on (19), we obtain the following nonlinear recurrence relation:

$$\left.\begin{array}{l} X_m(k+1) = \left(-2X_m(k) + Y_m(k)\right)/(k+1)\ , Y_m(k+1) = Z_m(k)/(k+1)\ , \\ \varepsilon\sum_{\ell=0}^{k}\sum_{\gamma=0}^{k-\ell}(\ell+1)Z_m(\ell+1)Y_m(\gamma)\,Y_m(k-\ell-\gamma) = (\varepsilon-1)Z_m(k) + \varepsilon\left(4X_m(k) - 2Y_m(k)\right) \\ +\varepsilon\sum_{\ell=0}^{k}\left(\sum_{\gamma=0}^{k-\ell}4X_m(\ell)) - 2Y_m(\ell) + Z_m(\ell)\right)Y_m(\gamma)Y_m(k-\ell-\gamma) - \varepsilon(k+1)Z_m(k+1) \end{array}\right\}. \tag{20}$$

Solving (20), the piecewise analytical solution of (19) for $x(0) = 1$, $y(0) = 0$, $z(0) = -2$, and $\varepsilon = 0.1$, using AMsDTM ($N = 6$, $\delta = 0.001$) is given in Eqs.(21)-(22). Table 5 shows the maximum absolute differences between AMsDTM ($N = 6$, $\delta = 0.001$) and RK4 ($h = 0.0001$) solutions and the time-step, M, used to achieve the

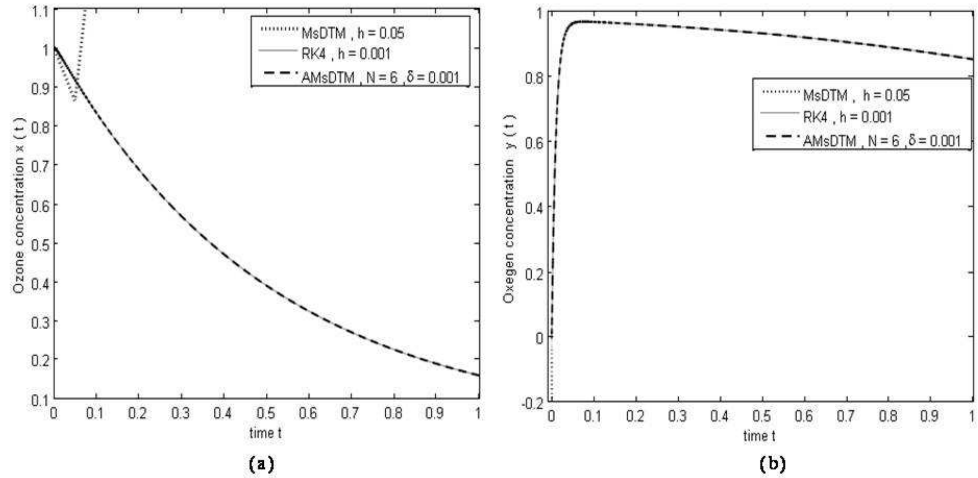

Fig 4: Solution of (**a**) ozone concentration, $x(t)$, and (**b**) oxygen concentration, $y(t)$ for Problem 4.2 obtained by AMsDTM, MsDTM and RK4 method at $\varepsilon = 1/98$ and $\kappa = 3$ for $t \in [0,\ 1]$.

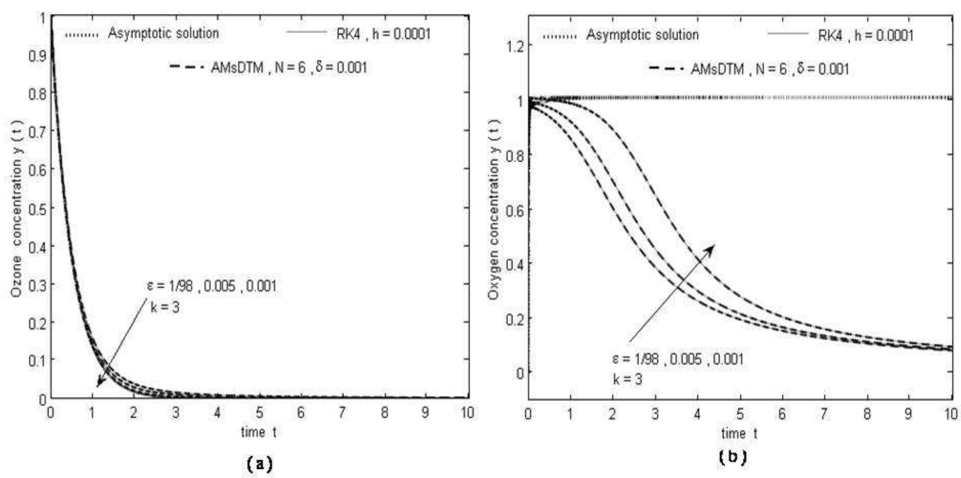

Fig 5: Solution comparison of (**a**) ozone concentration, $x(t)$, and (**b**) oxygen concentration, $y(t)$ for Problem 4.2 obtained by different methods at different values of ε for $t \in [0,\ 10]$.

Table 3: Maximum absolute differences between AMsDTM and RK4 solutions for problem 4.2.

$\Delta = \max\lvert \text{AMsDTM} - \text{RK4}_{0.0001} \rvert$				
$\kappa = 3$	$\delta = 0.001$		$\delta = 0.0001$	
ε	Δx	Δy	Δx	Δy
1/98	1.5981e-006	**1.5243e-004**	1.5960e-007	**1.5218e-005**
0.005	7.5944e-007	**1.4754e-004**	7.9307e-008	**1.4748e-005**
0.001	1.4439e-007	**1.3873e-004**	2.4041e-008	**1.3873e-005**

Table 4: Comparison of processing time and time-step for problem 4.2 at different values of δ and ε.

		MsDTM		AMsDTM	
ε	δ	Time-step M	Processing time (s)	Time-step M	Processing time (s)
1/98	0.0010	120	0.3111	19	0.0742
	0.0001	175	0.4012	22	0.0623
0.005	0.0010	247	0.6108	32	0.0930
	0.0001	362	1.0541	36	0.1091
0.001	0.0010	1241	4.2108	129	0.3284
	0.0001	1821	5.5540	133	0.3435

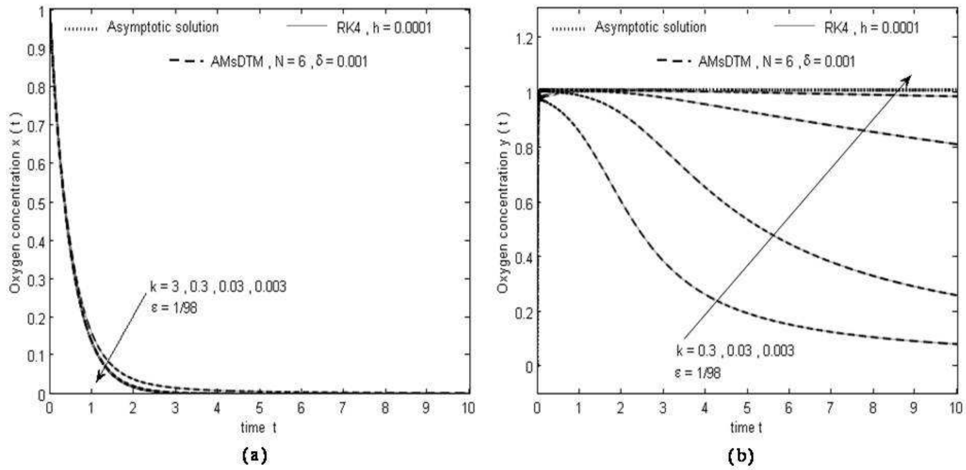

Fig 6: Solutions comparison of (**a**) ozone concentration, $x(t)$, and (**b**) oxygen concentration, $y(t)$ for Problem 4.2 obtained by different methods at $\varepsilon = 1/98$ and different values of κ for $t \in [0,\ 10]$.

Table 5: The time-step, M, and the maximum absolute differences between AMsDTM and RK4 solutions of problem 4.3.

	$\Delta = \max\lvert \text{AMsDTM} - \text{RK4}_{0.0001}\rvert$							
	$\delta = 0.001$				$\delta = 0.0001$			
ε	M	Δx	Δy	Δz	M	Δx	Δy	Δz
0.100	9	7.8509e-006	3.0889e-005	**1.7433e-004**	12	1.2709e-006	4.3168e-006	**3.1115e-005**
0.050	11	4.4637e-006	1.7980e-005	**1.3520e-004**	14	8.1371e-007	2.5351e-006	**3.0668e-005**
0.010	33	3.7743e-008	1.4625e-006	**1.3489e-004**	37	8.9847e-009	1.4530e-007	**1.3491e-005**
0.005	61	7.0163e-009	6.8916e-007	**1.3489e-004**	65	9.0666e-009	6.8828e-008	**1.3491e-005**

Table 6: Maximum absolute differences between MsDTM ($h = 1/M$) and RK4 solutions of problem 4.3.

	$\Delta = \max\lvert \text{MsDTM} - \text{RK4}_{0.0001}\rvert$							
	$\delta = 0.001$				$\delta = 0.0001$			
ε	M	Δx	Δy	Δz	M	Δx	Δy	Δz
0.100	9	4.9904e-004	1.2996e-003	**1.6649e-002**	12	7.5551e-005	1.9472e-004	**2.3772e-003**
0.050	11	1.8349e-003	4.5794e-003	**1.0590e-001**	14	3.7810e-004	9.3720e-004	**2.1835e-002**
0.010	33	4.1775e-004	6.0716e-003	**5.0401e-001**	37	1.9854e-004	2.8058e-003	**2.3120e-001**
0.005	61	9.8934e-005	5.4915e-003	**1.0528e+000**	65	5.8954e-005	3.5866e-003	**6.8707e-001**

Table 7: Comparison of processing time and time-step for problem 4.3 at different values of δ and ε.

		MsDTM		AMsDTM	
ε	δ	Time-step M	Processing time (s)	Time-step M	Processing time (s)
0.100	0.0010	21	0.6178	10	0.5127
	0.0001	31	0.8891	12	0.5127
0.050	0.0010	31	0.8891	11	0.5127
	0.0001	45	1.3481	14	0.5943
0.010	0.0010	137	5.9841	33	1.9710
	0.0001	200	10.1290	37	2.0187
0.005	0.0010	277	16.5421	61	3.1492
	0.0001	407	22.0872	65	3.3120

specified tolerance δ at different values of ε .We can observe that decreasing ε results in increasing the time steps, where the AMsDTM needs more grid points inside the layer to achieve the specified tolerance δ and maintains the accuracy independent of ε. Table 6 shows the maximum absolute differences between RK4 ($h = 0.0001$) solution and MsDTM solution using the same number of time steps used by AMsDTM . The results in Tables 6 show that the number of the time steps used by AMsDTM is not enough for MsDTM to obtain accurate results for the fast solution component $z(t)$, compared to those obtained in Table 5. In addition, as ε decreases the accuracy of the fast solution component $z(t)$ decreases. As shown in Table 7, the AsDTM is still faster than MsDTM even though solving nonlinear relation (20) consumes much processing time. As we can see, in Fig 8, for $\varepsilon = 0.005$ the MsDTM solution is far away from the RK4 solution while AMsDTM solution agrees very well with RK4 solution.

$$x(t) \simeq \left\{ \begin{array}{ll} 1.0000 - 2.0000t + t^2 + 3.0000t^3 - 9.9167t^4 + 17.4833t^5 - 10.7750t^6, & t \in [0.0000,\ 0.0496] \\ 0.9970 - 1.8831t + 1.3199\,(t-0.0496)^2 + 1.4264\,(t-0.0496)^3 - 6.1688\,(t-0.0496)^4 & \\ \quad +12.4366\,(t-0.0496)^5 - 18.1518\,(t-0.0496)^6, & t \in [0.0496,\ 0.1173] \\ 0.9808 - 1.6912t + 1.4734\,(t-0.1173)^2 + 0.2234\,(t-0.1173)^3 - 3.0420\,(t-0.1173)^4 & \\ \quad +6.5573\,(t-0.1173)^5 - 10.5945\,(t-0.1173)^6, & t \in [0.1173,\ 0.2094] \\ 0.9369 - 1.4215t + 1.4218\,(t-0.2094)^2 - 0.4751\,(t-0.2094)^3 - 1.0394\,(t-0.2094)^4 & \\ \quad +2.7103\,(t-0.2094)^5 - 4.2911\,(t-.2094)^6, & t \in [0.2094, 0.3108] \\ 0.8668 - 1.1509t + 1.2357\,(t-0.3108)^2 - 0.6896\,(t-0.3108)^3 - 0.1603\,(t-0.3108)^4 & \\ \quad +1.0167\,(t-0.3108)^5 - 1.7037\,(t-0.3108)^6\ , & t \in [0.3108, 0.4341] \\ 0.7657 - 0.8779t + 0.9802\,(t-0.4341)^2 - .6637\,(t-0.4341)^3 + 0.1875\,(t-0.4341)^4 & \\ \quad +0.2498\,(t-0.4341)^5 - 0.5686\,(t-0.4341)^6\ , & t \in [0.4341, 0.5840] \\ 0.6384 - 0.6259t + 0.7121\,(t-0.5840)^2 - 0.5229\,(t-0.5840)^3 + 0.2486\,(t-0.5840)^4 & \\ \quad -0.0213\,(t-0.5840)^5 - 0.1273\,(t-0.5840)^6\ , & t \in [0.5840, 0.7673] \\ 0.4951 - 0.4117t + 0.4719\,(t-0.7673)^2 - 0.3574\,(t-0.7673)^3 + 0.1954\,(t-0.7673)^4 & \\ \quad -0.0713\,(t-.7673)^5 - 0.0006\,(t-0.7673)^6\ , & t \in [0.7673, 0.9673] \\ 0.3648 - 0.2601t + 0.2988\,(t-0.9673)^2 - 0.2283\,(t-0.9673)^3 + 0.1296\,(t-0.9673)^4 & \\ \quad -0.0565\,(t-0.9673)^5 + 0.0169\,(t-0.9673)^6\ , & t \in [0.9673, 1.0000] \end{array} \right\} \tag{21}$$

$$y(t) \simeq \left\{ \begin{array}{ll} -2.0000t + 11.0000t^2 - 33.6667t^3 + 67.5833t^4 - 29.6833t^5 - 578.4139t^{6.}, & t \in [0.0000,\ 0.0496] \\ -0.0200 - 1.1263t + 6.9191\,(t-0.0496)^2 - 21.8223\,(t-0.0496)^3 + 49.8456\,(t-0.0496)^4 & \\ \quad -84.0373\,(t-0.0496)^5 + 37.1866\,(t-0.0496)^6\ , & t \in [0.0496,\ 0.1173] \\ -0.0751 - 0.4355t + 3.6169\,(t-0.1173)^2 - 11.7213\,(t-0.1173)^3 + 26.7025\,(t-0.1173)^4 & \\ \quad -50.4524\,(t-0.1173)^5 + 81.9783\,(t-0.1173)^6\ , & t \in [0.1173,\ 0.2094] \\ -0.1433 + 0.0007t + 1.4183\,(t-0.2094)^2 - 5.1078\,(t-0.2094)^3 + 11.4729\,(t-0.2094)^4 & \\ \quad -20.3257\,(t-0.2094)^5 + 32.2227\,(t-0.2094)^6\ , & t \in [0.2094, 0.3108] \\ -0.1855 + 0.1696t + 0.4025\,(t-0.3108)^2 - 2.0204\,(t-0.3108)^3 + 4.7630\,(t-0.3108)^4 & \\ \quad -8.1890\,(t-0.3108)^5 + 11.6660\,(t-0.3108)^6\ , & t \in [0.3108, 0.4341] \\ -0.1975 + .2046t - 0.0307\,(t-0.4341)^2 - 0.5773\,(t-0.4341)^3 + 1.6239\,(t-0.4341)^4 & \\ \quad -2.9123\,(t-0.4341)^5 + 4.0673\,(t-0.4341)^6\ , & t \in [0.4341, 0.5840] \\ -0.1807 + 0.1724t - 0.1447\,(t-0.5840)^{2.} - 0.0516\,(t-0.5840)^3 + 0.3908\,(t-0.5840)^4 & \\ \quad -0.8065\,(t-0.5840)^5 + 1.1916\,(t-0.5840)^6\ , & t \in [0.5840, 0.7673] \\ -0.1456 + 0.1204t - 0.1284\,(t-0.7673)^2 + 0.0669\,(t-0.7673)^3 + 0.0341\,(t-0.7673)^4 & \\ \quad -0.1465\,(t-0.7673)^5 + 0.2484\,(t-0.7673)^6\ , & t \in [0.7673, 0.9673] \\ -0.1085 + 0.0773t - 0.0874\,(t-0.9673)^2 + 0.0619\,(t-0.9673)^3 - 0.0234\,(t-0.9673)^4 & \\ \quad -0.0119\,(t-0.9673)^5 + 0.0390\,(t-0.9673)^6\ , & t \in [0.9673, 1.0000] \end{array} \right\} \tag{22}$$

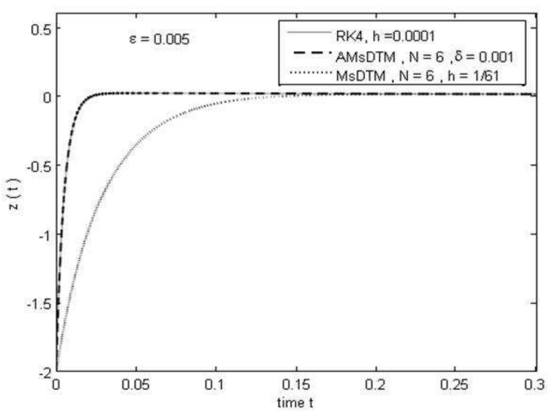

Fig 8: Fast solution component $z(t)$ in problem 4.3 using different methods at $\varepsilon = 0.005$.

4.4 A Quarter-Car suspension system

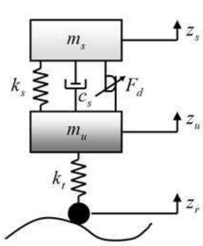

Fig 9: A quarter-car suspension system..

The dynamics equations of the suspension system shown in Fig 9 can be represented by (Khalil[16]).

$$\left.\begin{array}{l} \frac{dx}{dt_r} = y - w, \\ \frac{dy}{dt_r} = -x - \beta(y-w) + u, \\ \varepsilon\frac{dz}{dt_r} = w - v, \\ \varepsilon\frac{dw}{dt_r} = \alpha x - \alpha\beta(w-y) - z - \alpha u, \end{array}\right\}, \tag{23}$$

where $\varepsilon = \sqrt{\frac{k_s m_u}{k_t m_s}}$, $t_r = t\sqrt{k_s/m_s}$, $x = (z_s - z_u)/\ell$, $y = \dot{z}_s/\ell\sqrt{m_s/k_s}$, $z = (z_u - z_r)/\varepsilon\ell$, $w = \dot{z}_u/\ell\sqrt{m_s/k_s}$, $\alpha = \sqrt{\frac{k_s m_s}{k_t m_u}}$, $\beta = \frac{c_s}{\sqrt{k_s m_s}}$, $u = \frac{F_d}{k_s \ell}$, $v = \frac{\dot{z}_r}{\ell\sqrt{m_s/k_s}}$ and m_s, m_u, k_s , k_t , and c_s denote the mass, stiffness and the damping rate of the sprung and unsprung elements, respectively. The problem (21) was solved at $\alpha = 1.2$, $\beta = 0.5$, $\varepsilon = 0.01$, $u = v = 1$ and $x(0) = y(0) = z(0) = w(0) = 0.1$, using AMsDTM with $N = 6$ and $\delta = 0.001$ by solving the following recurrence relation

$$\left.\begin{array}{l} X_m(k+1) = (Y_m(k) - W_m(k))\,/\,(k+1)\,, \\ Y_m(k+1) = (-X_m(k) - \beta\,(Y_m(k) - W_m(k)) + u\delta(k))\,/\,(k+1) \\ Z_m(k+1) = (W_m(k) - v\delta(k))\,/\,(\varepsilon k + \varepsilon) \\ W_m(k+1) = (\alpha X_m(k) - \alpha\beta\,(W_m(k) - Y_m(k)) - Z_m(k) - \alpha u\delta(k))\,/\,(\varepsilon k + \varepsilon) \end{array}\right\}. \tag{24}$$

The results are shown in Fig 10 and Fig 11. Fig 10 shows the AMsDTM solution of (23) and the exact one. We can observe how the AMsDTM solution captures the fast variation of the boundary layer solution and approximates the exact solution very well over a long time interval. Fig 11 shows how the the error of AMsDTM solution($N = 6$, $\delta = 0.001$) is very small over the initial layer while the error of MsDTM solution ($N = 6$, $h = 1/M$) is very large for the same number of time steps, $M = 43$.

5 Conclusions

In this paper, we have applied the AMsDTM to SPIVPs and obtained their piecewise-analytical and numerical solutions. The validity of the method has been successful by applying it directly, without requiring linearization, perturbation, analytical integration or symbolic computations to four practical problems arising in modeling a diode oscillator with a current source, thermal decomposition of ozone, actuator control with high-gain feedback and a quarter-car suspension system. Numerical results are presented in figures and tables at different values of the tolerance δ and the perturbation parameter ε. The results show that the accuracy of the method is independent of the perturbation parameter ε and the method works successfully in handling the SPIVPs with a minimum size of computations and a wide interval of convergence. The results show that the proposed method is an accurate and efficient method compared to classical MsDTM in solving the considered problem. This emphasizes the fact that the present method is applicable to many other nonlinear real problems arising in different disciplines of science or engineering and it is reliable and

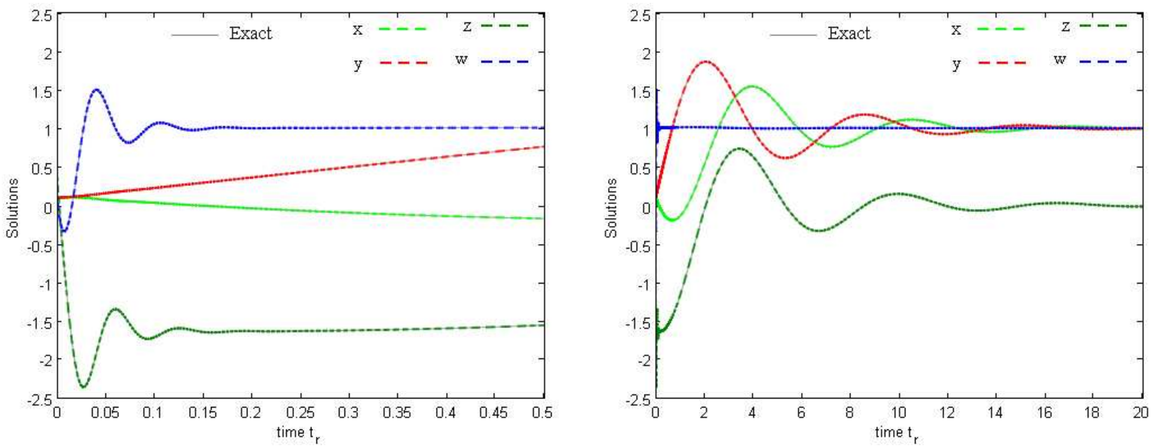

Fig 10: The approximate solution of (23) obtained by AMsDTM and the exact one at $\varepsilon = 0.01$.

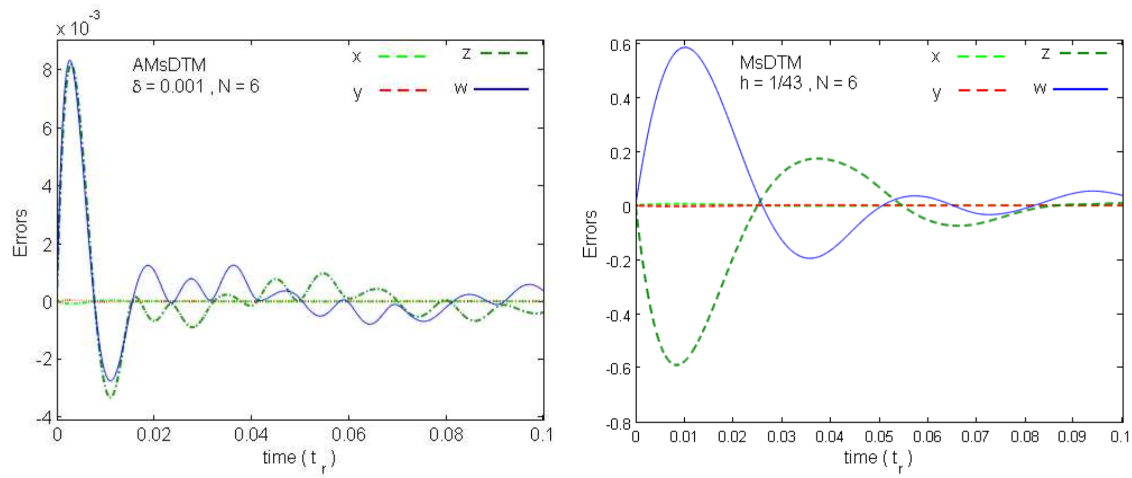

Fig 11: The error of AMsDTM and MsDTM solutions of (23) at $\varepsilon = 0.01$.

promising when compared with the existing methods.

Acknowledgement

This project was supported by the Deanship of Scientific Research at Salman bin Abdulaziz University under the research project # 21-T-33.
The author would like to thank Dr. Ehab. A. El-Sayed, Salman bin Abdulaziz University, for his supportive comments and suggestions.

References

[1] Abdel-Halim Hassan, I.H., 2002, Different applications for the differential transformation in the differential equations, Appl. Math. Comput., **129**, 183–201.

[2] Bender, C.M, and S.A. Orszag, 1999, *Advanced Mathematical Methods for Scientists and Engineers, Asymptotic Methods and Perturbation Theory*, Springer-Verlag New York, Inc.

[3] Celik Kizilkan, G. and K. Aydin, 2006, A new variable step size algorithm for Cauchy problem. Applied. Math. Comput., **183**, 878-884.

[4] Doolan, E.P., J.J.H. Miller, and W.H.A. Schilders, 1980, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, Ireland.

[5] El-Zahar, E.R., 2012, Approximate analytical solutions for singularly perturbed boundary value problems by multi-step differential transform method., J. Applied Sci., **12**, 2026-2034.

[6] El-Zahar, E.R, 2012, An Adaptive Step-Size Taylor Series Based Method and Application to Nonlinear Biochemical Reaction Model, Trends. Appl. Sci. Res., **7**, 901-912.

[7] El-Zahar, E.R., 2013, Approximate analytical solutions of singularly perturbed fourth order boundary value problems using differential transform method, J. KING SAUD. UNIV. SCI., **25**, 257–265.

[8] Erturk, V.S., Z.M. Odibat and S.Momani, 2012, The Multi-Step Differential Transform Method and its Application to Determine the Solutions of Non-Linear Oscillators, Adv. Appl. Math. Mech., **4**, 422-438.
[9] Gokdogan, A., M. Merdan and A. Yildirim, 2012, Adaptive multi-step differential transformation method to solving nonlinear differential equations, Math. Comput. Modelling., **55**, 761-769.
[10] Gokdogan, A., M. Merdan and A. Yildirim, 2012, A multistage differential transformation method for approximate solution of Hantavirus infection model, Commun. Nonlinear .Sci. Numer. Simul., **17**, 1–8
[11] Gu, W., W. Liu and K. Zhang, 2011, Variable step-size prediction-correction homotopy method for tracking high-dimension Hopf bifurcations. Int. J. Elect. Power Energy Syst., **33**, 1229-1235.
[12] Habib, H.M. and E.R. El-Zahar, 2008, Variable step size initial value algorithm for singular perturbation problems using locally exact integration, Appl. Math. Comput., **200**, 330–340.
[13] Ilea, M., and M. Turnea, 2011, Stiff Equations with Applications in Thermal Decomposition of Ozone, Annals of the University Dunarea De Jos of Galati: Fascicle II, Mathematics, Physics, Theoretical Mechanics, **34**, 191-202.
[14] Jang, M.J., C.L. Chen and Y.C. Liy, 2000, On solving the initial-value problems using the differential transformation method. Appl. Math. Comput., **115**, 145–160.
[15] Johnson, R.S., 2005, *Singular Perturbation Theory: Mathematical and Analytical Techniques with Applications to Engineering*, Springer-Verlag New York, Inc.
[16] Khalil, H.K., 2002, *Nonlinear Systems*, Upper Saddle River, NJ: Prentice-Hall.
[17] Kumar, M., and Parul, 2011, Methods for solving singular perturbation problems arising in science and engineering, Math. Comput. Modell., **54**, 556–575
[18] Keimanesh, M., M.M. Rashidi, A.J. Chamkha and R. Jafari, 2011, Study of a third grade non-Newtonian fluid flow between two parallel plates using the multi-step differential transform method, Comput. Math. Appl., **62**, 2871-2891.
[19] Lapidus, L., R.C. Aiken, and Y.A. Liu,1974, *The occurrence and numerical solution of physical and chemical systems having widely varying time constants*, R.A. Willoughby (Ed.), Stiff Differential Systems, Plenum Press, New York , 187–200.
[20] Miller, J.J.H., E. O'Riordan, and G.I. Shishkin, 1996, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore.
[21] Miranker, W.L., 1981, *Numerical Methods for Stiff Equations and Singular Perturbation Problems*, Reidel, Dordrecht.
[22] Odibat, Z.M., C. Bertelle, M.A. Aziz-Alaoui and G.H.E. Duchamp, 2010, A multi-step differential transform method and application to non-chaotic or chaotic systems, Comput. Math. Appl., **59**, 1462–1472.
[23] O'Malley, R.E., 1991, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer Verlag, New York.
[24] Patra, A. and S. Saha Ray,2013, Multistep Differential Transform Method for Numerical Solution of Classical Neutron Point Kinetic Equation, Comput. Math. Model., **24**, 604-615.
[25] Roos, H.G., Stynes, M. and Tobiska, L., 1996, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer, Berlin.
[26] Shampine, L.F., I. Gladwell and S. Thompson , 2003, *Solving ODEs with MATLAB*, Cambridge University Press.
[27] Smith, D.R, 2009, *Singular-Perturbation Theory an Introduction with Applications*, Cambridge University Press, Cambridge.
[28] Yildirim, A., A. Gokdogan and M. Merdan, 2012, Chaotic systems via multi-step differential transformation method. Can. J. Phys., **90**, 391-406.
[29] Zhao,Y. , Xiao, A., 2010, Variational iteration method for singular perturbation initial value problems, Comput. Phys. Commun., **181**, 947–956.
[30] Zhao, Y. , Xiao, A., Li, L., Zhang, C., 2013, Variational iteration method for singular perturbation initial value problems with delays, Math. Probl. Eng., in press.
[31] Zhou, J.K., 1986, *Differential Transformation and its Applications for Electrical Circuits*, Wuhan, China: Huazhong University Press.

Essam R. Zahar received Ph.D and employed as an Assistant Professor in Basic Engineering Sciences Department at the Faculty of Engineering, Shebin El-Kom, Menofia University, Egypt. Now, he is an Associate Professor at Salman bin Abdulaziz University, KSA. His research interests are in the areas of Engineering Mathematics and Numerical methods for Differential Equations. He has published research articles in reputed international journals of mathematical and engineering sciences. He is referee of many mathematical journals.