APPLICATIONS OF AFFINE ROOT SYSTEMS TO THE THEORY OF SYMMETRIC SPACES[1]

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Introduction. Let $(G;\ K_1,\ K_2)$ be a compact symmetric triad in the sense of [3], G simply connected. The natural action of K_1 on G/K_2 is of interest because it is variationally complete [5]. In [3] we introduced certain "affine root systems" in order to describe the orbits of this K_1-action, and in the piesent note we wish to announce the classification [4] of these systems and to indicate further applications to the theory of symmetric spaces.

1. **Preliminaries.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, ν an automorphism of $\mathfrak{g}$, and set $\mathfrak{g}_\nu = \{X \in \mathfrak{g} : \nu(X) = X\}$. The following is due essentially to de Siebenthal [7] (cf. also [4, §7]).

(1.1) PROPOSITION. *If $\mathfrak{h}_\nu \subset \mathfrak{g}_\nu$ is a Cartan subalgebra, there is a unique Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}_\nu \subset \mathfrak{h}$. There is a finite family $\mathfrak{a} = \{\zeta : \mathfrak{h}_\nu \to \boldsymbol{C}/i\boldsymbol{Z}\}$ of affine functionals and an orthogonal direct sum decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum \mathfrak{g}_\zeta, \qquad \zeta \in \mathfrak{a}$$

where $\dim(\mathfrak{g}_\zeta) = 1$ *and*

$$\nu \circ \exp(\operatorname{ad}(Z)) \mid \mathfrak{g}_\zeta = \exp(2\pi\zeta(Z)),$$

for all $Z \in \mathfrak{h}_\nu$ and $\zeta \in \mathfrak{a}$. $\zeta(0)$ is pure imaginary for all $\zeta \in \mathfrak{a}$.

$\mathfrak{h}_\nu = V \oplus iV$ where V is the real subspace on which the "linear parts" $\tilde{\omega} = \omega - \omega(0)$ of the elements $\omega \in \mathfrak{a}$ are real. One defines

$$\mathfrak{A} = \{\tilde{\omega} \mid V - i\omega(0) : \omega \in \mathfrak{a}\}$$

interpreted as a set of affine functionals $V \to \boldsymbol{R}/\boldsymbol{Z}$. This is the system defined by de Siebenthal.

$\mathfrak{g} = \mathfrak{g}_* \oplus i\mathfrak{g}_*$ where $\mathfrak{g}_*$ is the compact real form of $\mathfrak{g}$. Let s_1 and s_2 be involutive automorphisms of $\mathfrak{g}_*$, σ_1 and σ_2 the extensions of these to anti-involutions of $\mathfrak{g}$. There correspond symmetric subalgebras $\mathfrak{k}_1$, $\mathfrak{k}_2$ of $\mathfrak{g}_*$ and noncompact real forms $\mathfrak{g}_1$, $\mathfrak{g}_2$ of $\mathfrak{g}$.

Let $\mathfrak{m} \subset \mathfrak{g}_*$ be the simultaneous -1 eigenspace of s_1 and s_2. Set $\nu = \sigma_1\sigma_2$ and choose $\mathfrak{h}_\nu$ as in (1.1), but such that $\mathfrak{h}_\nu \cap (\mathfrak{m} \oplus i\mathfrak{m})$ is maxi-

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mal abelian in $\mathfrak{m}\oplus i\mathfrak{m}$. Let σ denote $\sigma_1|\mathfrak{g}_\nu=\sigma_2|\mathfrak{g}_\nu$. Note that $\sigma(V)=V$ and that σ induces a permutation σ_* of $\mathfrak{A}$. The pair $(\mathfrak{A}, \sigma_*)$ will be called the affine σ-system associated to $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)$ (or to $(\mathfrak{g}_*; \mathfrak{k}_1, \mathfrak{k}_2)$).

If we let V^- denote the $+1$ eigenspace of $\sigma|V$ and $\mathfrak{A}^-$ the set of nonconstant restrictions of elements of $\mathfrak{A}$ to V^-, we obtain the affine root system of [3].

2. **Equivalences and classification.** One defines *isomorphism* $(\mathfrak{A}, \sigma_*)\cong(\mathfrak{A}', \sigma_*')$ via linear isometries $\phi: V\to V'$ carrying $\mathfrak{A}'\to\mathfrak{A}$ and such that $\phi\circ\sigma=\sigma'\circ\phi$, and one similarly defines *affine equivalence* $(\mathfrak{A}, \sigma_*)\sim(\mathfrak{A}', \sigma_*')$ via affine isometries $\phi: V\to V'$ with $\phi\circ\sigma=\sigma'\circ\phi$. *Isomorphism* $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\cong(\mathfrak{g}; \mathfrak{g}_1', \mathfrak{g}_2')$ is defined via an automorphism θ of $\mathfrak{g}$ leaving $\mathfrak{g}_*$ invariant such that $\theta(\mathfrak{g}_j)=\mathfrak{g}_j'$, $j=1, 2$. *Affine equivalence* $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\sim(\mathfrak{g}; \mathfrak{g}_1', \mathfrak{g}_2')$ means that there are inner automorphisms ζ_1, ζ_2 of $\mathfrak{g}$ leaving $\mathfrak{g}_*$ invariant such that $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\cong(\mathfrak{g}; \zeta_1(\mathfrak{g}_1'), \zeta_2(\mathfrak{g}_2'))$.

(2.1) THEOREM. *Let* $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)$ *and* $(\mathfrak{g}; \mathfrak{g}_1', \mathfrak{g}_2')$ *have respective affine* σ*-systems* $(\mathfrak{A}, \sigma_*)$ *and* $(\mathfrak{A}', \sigma_*')$. *Then* $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\cong(\mathfrak{g}; \mathfrak{g}_1', \mathfrak{g}_2')\Rightarrow(\mathfrak{A}, \sigma_*)\cong(\mathfrak{A}', \sigma_*')\Rightarrow(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\cong(\mathfrak{g}; \mathfrak{g}_{w(1)}', \mathfrak{g}_{w(2)}')$ *for a suitable permutation* w *of* $\{1, 2\}$. *Likewise,* $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\sim(\mathfrak{g}; \mathfrak{g}_1', \mathfrak{g}_2')\Rightarrow(\mathfrak{A}, \sigma_*)\sim(\mathfrak{A}', \sigma_*')\Rightarrow(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)\sim(\mathfrak{g}; \mathfrak{g}_{w(1)}', \mathfrak{g}_{w(2)}')$.

The affine σ-systems for all triads $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)$ have been classified up to affine equivalence [4].

3. **Topological applications.** Consider the action of K_1 on G/K_2 as in the introduction. Let $T\subset G/K_2$ be the flat geodesic torus described in [3] and [6]. Then T meets orthogonally every K_1-orbit and V^- identifies in a natural way with the universal covering of T. The system $\mathfrak{A}^-$ describes the singular set in T relative to the K_1-action [3] and enables us to apply the theory of [2]. If $N\subset G/K_2$ is a K_1-orbit, Theorem 3.1 of [3] shows that the space $\Omega(G/K_2; x, N)$ of paths on G/K_2 from the point x to the submanifold N has no torsion in homology iff a certain "regularity" condition [3, p. 236] is satisfied by $\mathfrak{A}^-$. As a result of [4] we can list up to affine equivalence (and a permutation of $\{1, 2\}$) the triads $(\mathfrak{g}_*; \mathfrak{k}_1, \mathfrak{k}_2)$ for which $\mathfrak{A}^-$ is regular. For $\mathfrak{g}_*$ simple these are given in the following list.

Type A. $(A_r; A_q\times A_{r-q-1}\times R, A_k\times A_{r-k-1}\times R)$, $(A_{2r-1}; D_r, A_{2r-2}\times R)$, $(A_{2r}; B_r, A_{2r-1}\times R)$, $(A_{2r-1}; C_r, C_r)$, $(A_{2r-1}; C_r, D_r)$, $(A_{2r-1}; C_r, A_q\times A_{2r-q-2}\times R)$, $(A_{2r-1}; D_r, A_1\times A_{2r-3}\times R)$.

Type B. $(B_r; D_r, D_r)$, $(B_r; D_r, B_q\times D_{r-q})$.

Type C. $(C_r; C_q\times C_{r-q}, C_k\times C_{r-k})$, $(C_r; C_q\times C_{r-q}, A_{r-1}\times R)$.

Type D. $(D_r; B_{r-1}, B_{r-1})$, $(D_r; A_{r-1}\times R, A_{r-1}\times R)$, $(D_r; D_{r-1}\times R, D_k\times D_{r-k})$ where $r>k\geqq 1$, $(D_{2r+k}; D_r\times D_{r+k}, A_{r-1}\times R)$ where $k\geqq 0$,

$(D_r; B_{r-1}, D_k \times D_{r-k})$ where $r > k \geqq 1$, $(D_r; A_{r-1} \times R, B_k \times B_{r-k-1})$ where $r > k \geqq 1$, $(D_4; B_3, \omega(B_3))$, $(D_4; B_3, \omega(B_1 \times B_2))$. Here ω is the triality automorphism of D_4; B_3 and $B_1 \times B_2$ are standardly imbedded in D_4.

Type E. $(E_6; D_5 \times R, D_5 \times R)$, $(E_6; F_4, F_4)$, $(E_6; F_4, C_4)$, $(E_6; D_5 \times R, A_5 \times A_1)$, $(E_6; F_4, D_5 \times R)$, $(E_6; F_4, A_5 \times A_1)$, $(E_7; E_6 \times R, E_6 \times R)$, $(E_7; A_7, E_6 \times R)$, $(E_7; E_6 \times R, D_6 \times A_1)$.

Type F. $(F_4; B_4, B_4)$, $(F_4; B_4, C_3 \times A_1)$.

4. **Commuting involutions.** Following Hermann [6] one asks whether there is an inner automorphism ζ of $\mathfrak{g}$ leaving $\mathfrak{g}_*$ invariant such that $\zeta\sigma_1\zeta^{-1}$ commutes with σ_2. Using (1.1) and (2.1) one can prove the answer is affirmative iff $(\mathfrak{A}, \sigma_*) \sim (\mathfrak{A}', \sigma_*')$ where $\phi \in \mathfrak{A}'$ implies $\phi(0) = 0$ or $\frac{1}{2}$.

As Hermann has shown [6, Proposition 2.1], the existence of totally geodesic K_1-orbits in G/K_2 is completely bound up with the solutions ζ to this problem. The system $(\mathfrak{A}, \sigma_*)$ somewhat clarifies this situation as we now indicate.

Let $p: V^- \to T$ be the natural covering map. Supposing that the commuting involutions problem has a solution, we lose no generality in assuming $\sigma_1\sigma_2 = \sigma_2\sigma_1$ (hence $s_1s_2 = s_2s_1$). Then if Λ is the lattice $\{X \in V^- : \phi(X) = 0 \text{ or } \frac{1}{2}, \text{ all } \phi \in \mathfrak{A}\}$, we have the following.

(4.1) PROPOSITION. *$\Sigma = p(\Lambda)$ is the subset of T consisting of the points whose K_1-orbits are totally geodesic in G/K_2.*

The assumption $s_1s_2 = s_2s_1$ implies that s_1 defines an involutive isometry (again called s_1) of G/K_2. This situation is quite general.

(4.2) PROPOSITION. *Let G be simply connected. Then every involutive isometry of G/K_2 having nonempty fixed point set is conjugate (in the isometry group) to one produced by an involutive automorphism s_1 of G commuting with s_2.*

We explicitly identify the fixed point set of the involution s_1 in G/K_2. For each $\phi \in \mathfrak{A}^-$, let $\bar{\phi}$ be the linear part as in §1 and define $h_\phi \in V^-$ by $h_\phi \perp \operatorname{Ker}(\bar{\phi})$ and $\bar{\phi}(h_\phi) = 2$. The lattice Λ_e spanned by these vectors h_ϕ is exactly $p^{-1}(\{K_2\})$.

(4.3) THEOREM. *Again assume G simply connected and $s_1s_2 = s_2s_1$. Let $\Lambda_* = \frac{1}{2}\Lambda_e$ and $\Sigma_* = p(\Lambda_*)$. Then $\Sigma_* \subset \Sigma$ and the fixed point set of s_1 in G/K_2 is exactly the union of the K_1-orbits of the elements of Σ_*.*

5. **Pseudo-Riemannian symmetric spaces.** The explicit solutions of the commuting involutions problem make possible a classification of the isomorphism classes of those $(\mathfrak{g}; \mathfrak{g}_1, \mathfrak{g}_2)$ for which $\sigma_1\sigma_2 = \sigma_2\sigma_1$. For

each of these $(\mathfrak{g}_1, \mathfrak{g}_1 \cap \mathfrak{g}_2)$ and $(\mathfrak{g}_2, \mathfrak{g}_1 \cap \mathfrak{g}_2)$ are dual pseudo-Riemannian symmetric pairs [1]. All pseudo-Riemannian pairs may be obtained in this way; hence [4] contains implicitly the classification [1].

In the following, $\mathfrak{R} = \{\phi \in \mathfrak{A} : \phi(0) = 0\}$ and $\mathfrak{R}^- = \{\phi \in \mathfrak{A}^- : \phi(0) = 0\}$. These are identified as subsets of the dual spaces V^* and $(V^-)^*$ respectively. For other terminology in the theorem below, cf. [1].

(5.1) THEOREM. *Let $\mathfrak{g}$ be simple, $\sigma_1\sigma_2 = \sigma_2\sigma_1$. The corresponding dual symmetric pairs are either both reducible or both irreducible. They are reducible iff $\mathfrak{R}^-$ spans a subspace of $(V^-)^*$ of codimension one, and in this case the dual pairs are mutually isomorphic. They are irreducible iff $\mathfrak{R}^-$ spans $(V^-)^*$. The dual symmetric pairs are either both complex symmetric or both fail to be so. They are complex symmetric iff $\mathfrak{R}$ spans a subspace of V^* of codimension one and $\mathfrak{R}^-$ spans $(V^-)^*$. In this case the dual pairs are actually semikählerian.*

These facts are proven without classification.

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