Math. Program., Ser. B 97: 177–207 (2003)

Digital Object Identifier (DOI) 10.1007/s10107-003-0442-2

Zhi-Quan Luo

Applications of convex optimization in signal processing and digital communication*

Received: January 22, 2003 / Accepted: April 29, 2003
Published online: May 28, 2003 –

Abstract. In the last two decades, the mathematical programming community has witnessed some spectacular advances in interior point methods and robust optimization. These advances have recently started to significantly impact various fields of applied sciences and engineering where computational efficiency is essential. This paper focuses on two such fields: digital signal processing and communication. In the past, the widely used optimization methods in both fields had been the gradient descent or least squares methods, both of which are known to suffer from the usual headaches of stepsize selection, algorithm initialization and local minima. With the recent advances in conic and robust optimization, the opportunity is ripe to use the newly developed interior point optimization techniques and highly efficient software tools to help advance the fields of signal processing and digital communication. This paper surveys recent successes of applying interior point and robust optimization to solve some core problems in these two fields. The successful applications considered in this paper include adaptive filtering, robust beamforming, design and analysis of multi-user communication system, channel equalization, decoding and detection. Throughout, our emphasis is on how to exploit the hidden convexity, convex reformulation of semi-infinite constraints, analysis of convergence, complexity and performance, as well as efficient practical implementation.

1. Introduction

Over the last two decades, there have been significant advances in the research of interior point methods [61, 92] and conic optimization [85]. Powerful optimization models and efficient algorithmic tools as well as software [58] have been produced. Recently these advances have begun to significantly impact various applied science and engineering fields, such as mechanical structure design [11], VLSI circuit design [18, 77], systems and control [13], discrete optimization [35, 36, 91], statistics and probability [12, 31], where efficient optimization is essential. The goal of this paper is to review some existing as well as new optimization models and tools for convex conic and robust optimization, and to survey recent successes in applying these modern optimization tools to solve several core problems in signal processing and communication.

The application of modern optimization methods to signal processing and digital communication is well motivated. In the past thirty years, the work-horse algorithms in the field of digital signal processing and communication have been the gradient descent

Z-Q. Luo: Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, L8S 4K1, Canada, e-mail: luozq@mcmaster.ca

New address after April 1, 2003: Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455, USA

* This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant No. OPG0090391, and by the Canada Research Chair program.

and the least squares algorithms. While these algorithms have served their purpose well, they suffer from slow convergence and sensitivity to the algorithm initialization and stepsize selection, especially when applied to ill-conditioned or nonconvex problem formulations. This is unfortunate, since many design and implementation problems in signal processing and digital communication naturally lead to nonconvex optimization formulations, the solution of which by the gradient descent algorithm usually works poorly. Examples of such problems include channel equalization [26,57], maximum likelihood detection [56], pulse shaping filter design [23, 24], sidelobe control in digital beamforming [80], to name just a few. The main problem with applying the least squares or the gradient descent algorithms directly to the nonconvex formulations is slow convergence and local minima. One powerful way to avoid these problems is to derive an exact convex reformulation or a convex relaxation of the original nonconvex formulation. Once a convex reformulation or relaxation is obtained, we can be guaranteed of finding the globally optimal design efficiently without the usual headaches of stepsize selection, algorithm initialization and local minima.

Another challenging problem in signal processing and communication is the robustness of the obtained solution against data perturbation and implementation errors. These errors can be caused by measurement noise, small sample size, model mismatch or the finite arithmetic of digital hardware. The pervasive nature of these errors in signal processing and communication underscores the need and importance of robust solutions. Without properly modelling the required robustness, the "optimal designs" obtained from noisy data can be useless, especially when truncated for implementation on digital hardware. Unfortunately, due to the lack of tractable mathematical models and optimization techniques, most of the previous research in signal processing and communication do not address the robustness issue directly nor satisfactorily.

In recent years, the mathematical programming community has witnessed extraordinary advances in interior point methods [61], Semidefinite Programming (SDP) [76], Second Order Cone programming (SOCP) [50], and robust optimization techniques [8]. These new advances and the related software tools [70, 74, 86] have allowed us to model many problems which were previously considered intractable as convex programs and solve them much more efficiently than the classical gradient descent algorithm. Boyd and Vandenberghe [76] were among the first to successfully apply the interior point optimization methods to the real world engineering problems including robust control (see the book by Boyd, El-Ghaoui, Feron and Balakrishnan [13]). Previously, the control community had mainly been looking for closed form solutions for their problems. With the availability of efficient numerical tools (such as those based on interior point methods), many challenging control problems can now be efficiently solved numerically with high accuracy (not in closed form). This new approach has significantly broadened the class of control problems that are considered solvable. These exciting studies show the great potential of convex reformulation of classical engineering design problems and their subsequent solution by interior point methods.

Equally exciting are the recent applications of convex optimization techniques in signal processing and communication. Successful examples of this kind include detection and estimation [5,27,47,69], circuit design [77], channel equalization [26,45,57], filter design [4, 23, 24, 75, 87], digital beamforming [46, 80], and communication system design [52, 56, 82, 89, 94]. In the case where the underlying signal processing and communication problems defy a convex reformulation, a high quality convex relaxation solution may be sought (see the quasi-maximum likelihood detection example in Section 4). These examples illustrate the great impact of modern optimization techniques on the fields of signal processing and digital communication. In fact, the impact is felt in both directions: the process of formulating practical signal processing and communications problems also helps to inspire optimizers to further strengthen and generalize their existing theory and methods. For example, the existing robust optimization theory of Ben-Tal and Nemirovski [8] explicitly models the uncertainty in the input data and seeks a solution which is robust against input data perturbations. In practice there is an additional source of perturbation caused by truncation error when the obtained solution is to be implemented on digital hardware. Without properly modelling this implementation error, the obtained solution may not be sufficiently robust (see the robust magnitude filter design problem in Section 3). The work of Ben-Tal and Nemirovski [8–10] does consider implementation error, but instead models it as a form of input data perturbation. It turns out that we can develop more general robust optimization models [55] which explicitly account for both the data perturbation and the implementation error. Such extensions (under ellipsoidal uncertainty set) can be derived easily using the so called S-procedure [90] or using the results of [9, 10, 55].

Here and throughout, vectors are in lower letters, and matrices are in capital letters. The vector (or matrix) transpose is expressed by superscript ‘ T ’, while the Hermitian matrix transpose is denoted by superscript ‘ H ’. The set of n by n symmetric matrices is denoted by $\mathcal{S}^n$; and the set of n by n real positive semidefinite matrices is denoted by $\mathcal{S}^n_+$. The set of n by n Hermitian matrices is denoted by $\mathcal{H}^n$; and the set of n by n Hermitian positive semidefinite matrices is denoted by $\mathcal{H}^n_+$. The second order cone $\{(t, x) \in \mathbb{R}^n \mid t \geq \|x\|\}$ where ‘$\|\cdot\|$’ represents the Euclidean norm, is denoted by SOC(n). For two given matrices A and B, we use ‘$A \succ B$’ (‘$A \succeq B$’) to indicate that $A - B$ is positive (semi)-definite, and $A \bullet B := \sum_{i,j} A_{ij} B_{ij} = \mathrm{Tr}\,(AB^T)$ to indicate the matrix inner product. The Frobenius norm of A is denoted by $\|A\|_F = \sqrt{\mathrm{Tr}\,(AA^T)}$.

2. Review of conic optimization and robust optimization

2.1. Conic optimization model and interior point method

Consider a primal-dual pair of linear conic optimization problems:

$$\begin{array}{l} \text{minimize } C \bullet X \\ \text{subject to } \mathcal{A}X = b, \ X \in \mathcal{C} \end{array} \tag{1}$$

and

$$\begin{array}{l} \text{maximize } b^T y \\ \text{subject to } \mathcal{A}^* y + S = C, \ S \in \mathcal{C}^* \end{array} \tag{2}$$

where $\mathcal{A}$ is a linear operator mapping an Euclidean space *onto* another Euclidean space, $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, $\mathcal{C}$ signifies a pointed, closed convex cone, and $\mathcal{C}^*$ is its dual cone. The problems (1)–(2) include many well known special cases such as

- Linear Programming (LP): $\mathcal{C} = \mathbb{R}^n_+$,
- Second Order Cone programming (SOCP): $\mathcal{C} = \prod_{i=1}^{n} \mathrm{SOC}(n_i)$,
- Semidefinite Programming (SDP): $\mathcal{C} = \mathcal{S}^n_+$ or $(\mathcal{H}^n_+)$.

For ease of exposition, we will focus on the SDP case with $\mathcal{C} = \mathcal{S}^n_+$. The other cases can be treated similarly (in fact they are special case of SDP). In practice, sometimes it is more convenient to work with the so-called *rotated* Second Order Cone: $\{(t, s, x) \in \mathbb{R}^n \mid ts \geq \|x\|^2,\ t \geq 0,\ s \geq 0\}$. This cone is equivalent to the standard SOC(n) via a simple linear transformation.

Assume that the feasible regions of the SDP pair (1)–(2) have nonempty interiors. Then we can define the central path of (1)–(2) as $\{(X(\mu), S(\mu))\}$ satisfying

$$\begin{aligned} \mathcal{A}^* y(\mu) + S(\mu) &= c \\ \mathcal{A}X(\mu) &= b \\ X(\mu)S(\mu) &= \mu I \end{aligned} \tag{3}$$

where μ is a positive parameter. By driving $\mu \to 0$ and under mild assumptions, the central path converges to an optimal primal-dual solution pair for (1)–(2). Notice that the central path condition (3) is exactly the necessary and sufficient optimality condition for the following convex problem:

$$\begin{aligned} &\text{minimize } C \bullet X - \mu \log\det(X) \\ &\text{subject to } \mathcal{A}X = b,\ X \in \mathcal{S}^n_+. \end{aligned} \tag{4}$$

In other words, the points on the central path corresponds to the optimal solution of (4) and the associated optimal dual solution. Here the function $-\log\det(X)$ is called the barrier function for the positive semidefinite matrix cone $\mathcal{S}^n_+$.

Many interior point algorithms follow (approximately) the central path to achieve optimality. As a result, the iterates are required to remain in a neighborhood of the central path which can be defined as:

$$\mathcal{N}(\gamma) = \Big\{(X, y, S) \mid \mathcal{A}X = b,\ \mathcal{A}^* y + S = c,\ X \succeq 0,\ S \succeq 0, \\ \|X^{1/2} S X^{1/2} - \frac{X \bullet S}{n} I\|_F \leq \gamma \frac{X \bullet S}{n}\Big\}$$

With this definition, a generic interior point path-following algorithm can be stated as follows.

GENERIC PATH-FOLLOWING ALGORITHM

Given a strictly feasible primal-dual pair $(X^0, y^0, S^0) \in \mathcal{N}(\gamma)$ with $0 < \gamma < 1$.
Let $k = 0$.
`REPEAT` (main iteration)
  Let $X = X^k,\ y = y^k,\ S = S^k$ and $\mu_k = X \bullet S/n$.
  Compute a search direction $(\Delta X^k, \Delta y^k, \Delta S^k)$.
  Compute the largest step t_k such that
    $(X + t^k \Delta X^k, y + t^k \Delta y^k, S + t^k \Delta S^k) \in \mathcal{N}(\gamma)$.
  Set $X^{k+1} = X + t^k \Delta X^k$, $y^{k+1} = y + t^k \Delta y^k$, $S^{k+1} = S + t^k \Delta S^k$.
  Set $k = k + 1$.
`UNTIL` convergence.

There are many choices for the search direction $(\Delta X, \Delta y, \Delta S)$. For example, we can take it as the solution of the following linear system of equations:

$$\begin{aligned} \mathcal{A}^* \Delta y + \Delta S &= C - S - \mathcal{A}^* A y \\ \mathcal{A} \Delta X &= b \\ \mathcal{H}_P(\Delta X S + X \Delta S) &= \mu I - \mathcal{H}_P(XS) \end{aligned} \tag{5}$$

where P is a nonsingular matrix and

$$\mathcal{H}_P(U) = \frac{1}{2}(PUP^{-1} + (PUP^{-1})^T).$$

Different choices of P lead to different search directions. For example, $P = I$ corresponds to the so-called AHO direction [85].

The standard analysis of path-following interior point methods shows that a total of $O(\sqrt{n} \log \mu_0/\epsilon)$ main iterations are required to reduce the duality gap $X \bullet S$ to less than ϵ. Each main iteration involves solving the linear system of equations (5) whose size depends on the underlying cone $\mathcal{C}$. If $\mathcal{C} = \mathbb{R}^n_+$ (linear programming), the linear system is of size $O(n)$, implying each main iteration has an arithmetic complexity of $O(n^3)$. In the case where $\mathcal{C} = \prod_{i=1}^n \mathrm{SOC}(n_i)$ (SOCP), the linear system (5) will have size $O(\sum_i n_i)$, so the complexity of solving (5) is $O((\sum_i n_i)^3)$. For the SDP case where $\mathcal{C} = \mathcal{S}^n_+$, the size of the linear system (5) is $O(n^2)$, so the amount of work required to solve (5) is $O(n^6)$. Combining the estimates of the number of main iterations with the complexity estimate per each iteration yields the overall complexity of interior point methods. In general, the computational effort required to solve SDP is more than that of SOCP, which in turn is more than that of LP. However, the expressive power of these optimization models rank in the reverse order.

2.2. *Robust optimization*

Robust optimization models in mathematical programming have received much attention recently; see, e.g. [7, 8, 29]. In this subsection we will briefly review some of these models and some extensions.

Consider a convex optimization for the form:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, 2, ..., m, \end{array} \tag{6}$$

where each f_i is convex. In many engineering design applications, the data defining the constraint and the objective functions may be inexact, corrupted by noise or may fluctuate with time. In such cases, the traditional approach is simply to use the nominal form of each f_i in the formulation of the design problem (6). However, an optimal solution for the nominal formulation (6) may yield poor performance or become infeasible when each f_i is perturbed. In other words, optimal solutions for (6) may be misleading or even useless in practice. A more appropriate thing to do is to seek a high quality solution which can remain feasible and deliver high quality performance in all possible realizations of unknown perturbation. This principle was formulated rigorously in [7, 8, 29]. Specifically, the data perturbation can be modelled using a parameter vector δ, with $\delta = 0$ representing the nominal unperturbed situation. In other words, we consider a family of perturbed functions parameterized by δ: $f_i(x; \delta)$, with δ taken from an uncertainty set Δ containing the origin. Then a robustly feasible solution x is the one that satisfies

$$f_i(x; \delta) \le 0, \quad \forall\, \delta \in \Delta \quad \text{or equivalently} \quad \max_{\delta \in \Delta} f_i(x; \delta) \le 0.$$

Thus, a robustly feasible solution x is, in a sense, strongly feasible, since it is required to satisfy all slightly perturbed version of the nominal constraint $f_i(x; 0) = f_i(x) \le 0$. The robust optimal solution can now be defined as a robust feasible solution which minimizes the worst case objective value $\max_{\delta \in \Delta} f_0(x; \delta)$. This gives rise to the following formulation:

$$\begin{array}{ll} \text{minimize} & \max_{\delta \in \Delta} f_0(x; \delta) \\ \text{subject to} & f_i(x; \delta) \le 0, \ \forall\, \delta \in \Delta, \ i = 1, 2, ..., m. \end{array} \tag{7}$$

Let us assume the perturbation vector δ enters the objective and the constraint functions f_i in such a way that preserves convexity, i.e., each $f_i(x; \delta)$ remains a convex function for each $\delta \in \Delta$. As a result, the robust counterpart (7) of the original (nominal case) convex problem (6) remains convex since its constraints are convex (for each i and δ) and the objective function $\max_{\delta \in \Delta} f_0(x; \delta)$ is also convex.

Much of the research in robust optimization is focussed on finding a finite representation of the feasible region of (7) which is defined in terms of infinitely many constraints (one for each $\delta \in \Delta$). Assume that the uncertainty parameter δ can be partitioned as $\delta = (\delta_0, \delta_1, \delta_2 ..., \delta_m)^T$ and that the uncertainty set has a Cartesian product structure $\Delta = \Delta_0 \times \Delta_1 \times \cdots \times \Delta_m$, with $\delta_i \in \Delta_i$. Moreover, assume that δ enters $f_i(x; \delta)$ in an affine manner. Under these assumptions, it is possible to characterize the robust feasible set of many well known classes of optimization problems in a finite way. In particular, consider the robust linear programming model proposed by Ben-Tal and Nemirovskii [8]:

$$\begin{array}{ll} \text{minimize} & \max_{\|\Delta c\| \le \epsilon_0} (c + \Delta c)^T x \\ \text{subject to} & (a_i + \Delta a_i)^T x \ge (b_i + \Delta b_i), \\ & \qquad \text{for all } \|(\Delta a_i, \Delta b_i)\| \le \epsilon_i, \ i = 1, 2, ..., m, \end{array} \tag{8}$$

where each $\epsilon_i > 0$ is a pre-specified scalar. In the above formulation, we have $\delta_i = (\Delta a_i, \Delta b_i)$ and $\Delta_i = \{(\Delta a_i, \Delta b_i) \mid \|(\Delta a_i, \Delta b_i)\| \le \epsilon_i\}$. It is known that the above robust LP can be reformulated as a SOCP [8]. The references [7, 8, 29] have shown that the robust counterpart of some other well known convex optimization problems can also be reformulated in a finite way as a conic optimization problem, often as an SOCP or SDP.

An important tool in converting a semi-infinite problem to one with finitely many constraints is the following well-known S-lemma of Yakubovich [90].

Proposition 1 (S-lemma, level set). *Suppose* $p(x) = \bar{c} + 2\bar{b}^T x + x^T \bar{A} x$ *is a quadratic function with* $p(\bar{x}) > 0$ *for some* $\bar{x} \in \mathbb{R}^n$*, and let* $\Delta := \{x \mid p(x) \ge 0\}$*. Then*

$$q(x) = c + 2b^T x + x^T A x \ge 0 \text{ for all } x \in \Delta$$

if and only if there exists $s \ge 0$ *such that*

$$q(x) - sp(x) \ge 0 \text{ for all } x \in \mathbb{R}^n.$$

Recently, a matrix version of the S-lemma has been established [55] for the following robust QMI:

$$C + X^T B + B^T X + X^T A X \succeq 0 \quad \text{for all } X \text{ with } I - X^T D X \succeq 0. \tag{9}$$

Here, the perturbation parameter X enters the LMI in a quadratic manner. The case where $A = 0$ (so the perturbation to the LMI is affine) has been previously considered in [7].

Proposition 2. *The robust QMI* (9) *is equivalent to*

$$\begin{bmatrix} C & B^T \\ B & A \end{bmatrix} \in \left\{ Z \;\middle|\; Z - s \begin{bmatrix} I & 0 \\ 0 & -D \end{bmatrix} \succeq 0,\ s \ge 0 \right\}. \tag{10}$$

This shows that the robust QMI (9) holds if and only if the data matrices (A, B, C, D) satisfy a certain LMI relation.

Another popular technique to convert certain semi-infinite constraints into a single convex constraint is the Positive Real Lemma [6] (and the closely related Kalman-Yakubovich-Popov [KYP] Lemma). For discrete time Finite Impulse Response (FIR) systems, this lemma can be expressed as follows [5, 24, 27].

Proposition 3. *Let* $\{r_m \in \mathbb{C},\ -M+1 \le m \le M-1\}$ *be a sequence satisfying* $r_{-m} = \bar{r}_m$*, and define its Fourier transform as* $R(e^{j\theta}) = \sum_{m=-M+1}^{M-1} r_m e^{jm\theta}$*. Then* $R(e^{j\theta}) \ge 0$ *for all* $\theta \in [0, 2\pi)$ *if and only if there exists an* $X \in \mathcal{H}_+^{M\times M}$ *such that* $\mathrm{Tr}\,(X) = r_0$ *and* $\sum_{\ell=0}^{M-1-m} X_{\ell+m,\ell} = r_m$*, for* $1 \le m \le M-1$.

The importance of Proposition 3 lies in the fact that in signal processing applications one often encouters design constraints of the spectral mask type which are specified in terms of the lower and upper bounds on the Fourier transform of the FIR system to be designed. Such semi-infinite constraints can be appropriately transformed into an LMI using Proposition 3. Recently there have been some useful extensions of Proposition 3 which show that the condition $R(e^{j\theta}) \ge 0$ for all $\theta \in [\alpha, \beta] \subset [0, 2\pi)$ can also be represented as an LMI (see Proposition 5 and [5, 23]).

3. Efficient optimization methods for digital signal processing

In this section, we describe some successful applications of convex optimization methods to digital signal processing. As it turns out, the theory of interior point methods and robust optimization needs to be appropriately modified or extended to account for the effect of noise and non-convexity often encountered in signal processing applications. This is especially true for real-time signal processing applications where data are collected dynamically and are corrupted by noise. Previously the robust optimization techniques and interior point algorithms have only been studied in the noise-free deterministic and convex setting. The major technical issues to be studied are

(a) Convergence (in the stochastic sense) and convergence speed;
(b) Robustness to modelling error and noise statistics;
(c) Numerical stability.

3.1. Stochastic convergence and rate of convergence of IPLS (Interior Point Least Square) or its variants in a time-varying environment

Consider a discrete-time linear system as follows:

$$y_i = x_i^T w_* + v_i, \qquad i = 1, 2, \ldots$$

where $y_i \in \mathbb{R}$ is the sequence of observations, $x_i \in \mathbb{R}^M$ denotes the sequence of input vectors of size M, $w_* \in \mathbb{R}^M$ is the unknown deterministic parameter vector or filter that we wish to estimate, and v_i is the additive measurement noise. The parameter estimation problem is to identify the parameter vector w_* from input/output pairs $\{x_i, y_i\}$. When the estimate is to be updated sequentially with the arrival of new data, the problem is often referred to as adaptive filtering [40]. Define the mean-squared error of a particular estimate w (at time n) as

$$\mathcal{F}_n(w) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - x_i^T w \right)^2 = \frac{1}{n} y_n^T y_n - 2w^T p_{xy}(n) + w^T R_{xx}(n) w \tag{11}$$

where

$$y_n = [y_1, y_2, \ldots, y_n]^T, \quad p_{xy}(n) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i, \quad R_{xx}(n) = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T.$$

The least squares estimate of the parameter vector w_* is the estimate that minimizes $\mathcal{F}_n$. By solving $\nabla \mathcal{F}_n(w) = 0$ we find the well-known optimum linear filter $w_n^{ls} = R_{xx}(n)^{-1} p_{xy}(n)$.

The above problem is dynamic in that the data are collected one at a time. The traditional approach for identifying w_* is the Recursive Leaset Square (RLS) method which works well asymptotically but requires proper initialization in the transient phase. It turns out that we can use the concept of analytic center (cf. (3)–(4)) to devise an efficient Interior Point Least Square (IPLS) algorithm which not only enjoys an optimal asymptotic convergence rate but also fast transient convergence. To develop the IPLS algorithm, let us define the following convex feasibility region:

$$\Omega_n = \left\{ w \in \mathbb{R}^M \mid \mathcal{F}_n(w) \leq \tau_n,\ \|w\|^2 \leq R^2 \right\}, \tag{12}$$

where $\tau_n > 0$ is an appropriately chosen scalar and $R > 0$ is a fixed number. Any vector $w \in \Omega_n$ is said to be feasible at time n. Obviously, we would like τ_n to approach zero, because then any feasible vector w will approximately minimize $\mathcal{F}_n(w)$. The additional inequality $\|w\|^2 \leq R^2$ is important since it ensures that the region Ω_n is bounded and provides the "correct" regularization.

For the above convex feasible region Ω_n, we define its *analytic center* as the vector w_n^a such that the following *logarithmic barrier function*

$$\phi_n(w) = -\log(\tau_n - \mathcal{F}_n(w)) - \log(R^2 - \|w\|^2), \qquad w \in \Omega_n$$

is minimized. It remains to specify how τ_n is updated in each iteration. We choose

$$\tau_n \triangleq \mathcal{F}_n(w_{n-1}^a) + \beta \frac{R}{\sqrt{2}} \|\nabla \mathcal{F}_n(w_{n-1}^a)\|_2, \tag{13}$$

where $\beta > 0$ is a constant chosen by the user. An important consequence of (13) is that w_{n-1}^a remains feasible with respect to the updated region Ω_n so that w_{n-1}^a can be used as a starting point for a Newton procedure to find the analytic center w_n^a of Ω_n. The condition $\mathcal{F}_n(w_{n-1}^a) \leq \tau_n$ is easily verified with (13). The $(R/\sqrt{2})$ term in τ_n provides a normalization that is required to show the convergence of the algorithm. The parameter β is typically used in interior point methods to control the "aggressiveness" with which constraints are generated. For example, a small β corresponds to a small allowed slack, i.e., w_{n-1}^a is left close to the boundary of Ω_n. By increasing β we allow more slack and hence define Ω_n more conservatively.

A major advantage of the analytic center based adaptive filtering algorithm is its fast transient convergence. Generally speaking, the convergence behaviour of an estimation algorithm is influenced by two factors: the speed of the statistical averaging process and the decay rate of the initialization parameters. The former is usually dictated by the law of large numbers and therefore exhibits $O(1/n)$ type of convergence rate. The latter, which is called the transient behavior, is dependent on how fast the estimation algorithm can phase out the effect of initialization and is therefore more algorithm specific. Fast transient convergence is important for accurate estimation of time-varying parameters. It has been shown [2] that our new adaptive filtering algorithm enjoys a geometric rate of convergence in the transient phase while the traditional Recursive Least Squares (RLS) algorithm converges only at the rate of $O(1/n)$.

The details of the Interior Point Least Squares (IPLS) algorithm for adaptive filtering are given below.

IPLS: An Analytic Center Based Adaptive Filtering Procedure

Step 1: Initialization. Select $R > 0$, $\beta > 0$. Let $\mathcal{F}_0(w) = 0$ and $\Omega_0 = \{\|w\|^2 \leq R^2\}$. Let $w_0^a = 0$.

Step 2: Updating. For $n \geq 1$, update the feasible region Ω_n using the new data sample (x_n, y_n) and the new value τ_n according to (13).

Step 3: Recentering. Compute the analytic center w_n^a of Ω_n by taking Newton steps starting from the previous center w_{n-1}^a. Set $n := n + 1$ and return to Step 2.

It has been shown [2, 3] the updating of τ_n and the computation of an approximate analytic center w_n of Ω_n can be efficiently performed with per sample complexity of $O(M^{2.2})$. In fact, a constant number of Newton updating iterations are needed at each recentering step. It was also shown that $w_n \to w_*$ as $n \to \infty$. The effectiveness of IPLS algorithm was demonstrated in a channel tracking application in a Code Division Multiple Access (CDMA) uplink application [3].

3.2. Robust beamforming

In recent decades, adaptive beamforming has been widely used in wireless communications, microphone array speech processing, radar, sonar, medical imaging, radio astronomy, and other areas. Such techniques employ multiple sensors (or antennas) for signal transmission/reception to improve the system performance. Since the output from multiple sensors possess distinct spatial correlations depending on the angle of signal arrival, we can appropriately process the sensor outputs to separate signals arriving from different directions. This signal processing step is called beamforming.

Mathematically, the output of a narrowband beamformer is given by

$$y(k) = \mathbf{w}^H \mathbf{x}(k)$$

where k is the time index, $\mathbf{x}(k) = [x_1(k), \ldots, x_M(k)]^T \in \mathbb{C}^{M\times 1}$ is the complex vector of array observations, $\mathbf{w} = [w_1, \ldots, w_M]^T \in \mathbb{C}^{M\times 1}$ is the complex vector of beamformer weights, M is the number of array sensors. The observation (training snapshot) vector is given by

$$\mathbf{x}(k) = \mathbf{s}(k) + \mathbf{i}(k) + \mathbf{n}(k) = s(k)\mathbf{a} + \mathbf{i}(k) + \mathbf{n}(k) \tag{14}$$

where $\mathbf{s}(k)$, $\mathbf{i}(k)$, and $\mathbf{n}(k)$ are the desired signal, interference, and noise components, respectively. Here, $s(k)$ is the signal waveform, and $\mathbf{a}$ is the signal steering vector. The

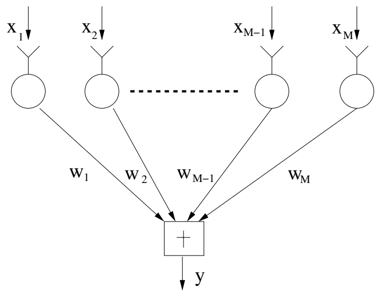


Fig. 1. Beamforming using a linear array of antennas

optimal weight vector can be found by the maximization of the Signal-to-Interference-plus-Noise Ratio (SINR) [59]

$$\mathrm{SINR} = \frac{\sigma_s^2\, |\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \mathbf{R}_{\mathrm{i+n}} \mathbf{w}} \tag{15}$$

where

$$\mathbf{R}_{\mathrm{i+n}} = \mathrm{E}\left\{ (\mathbf{i}(t) + \mathbf{n}(t))\, (\mathbf{i}(t) + \mathbf{n}(t))^H \right\} \tag{16}$$

is the $M \times M$ interference-plus-noise covariance matrix and σ_s^2 is the signal power. It is easy to find the solution for the weight vector by maintaining a distortionless response towards the desired signal and minimizing the output interference-plus-noise power [59]. Hence, the maximization of (15) is equivalent to [59]

$$\min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R}_{\mathrm{i+n}} \mathbf{w} \qquad \text{subject to} \qquad \mathbf{w}^H \mathbf{a} = 1\,. \tag{17}$$

In practice, $\mathbf{R}_{\mathrm{i+n}}$ is not available and is usually approximated by the data covariance matrix $\mathbf{R}$. Moreover, since the steering vector $\mathbf{a}$ is typically estimated from data and therefore known only approximately, we are led to consider the following robust formulation of (17):

$$\min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R} \mathbf{w} \qquad \text{subject to} \qquad |\mathbf{w}^H (\mathbf{a} + \Delta a)| \geq 1 \qquad \text{for all} \quad \|\Delta a\| \leq \epsilon, \tag{18}$$

where $|\cdot|$ denotes the magnitude of a complex number. Note that (18) represents a modified version of (17): we impose distortionless response constraints for all steering vectors within a distance ϵ from a, instead of requiring distortionless response for just a single steering vector $\mathbf{a}$. Unfortunately, the optimization problem (18) involves *infinitely many nonconvex quadratic constraints*, which suggests that it might be an intractable problem. Therefore, the robust convex optimization framework (6)–(7) of Ben-Tal and Nemirovskii does not apply. Surprisingly, it turns out we can reformulate this problem exactly as a convex SOCP [80].

Proposition 4. *The robust beamforming problem* (18) *can be transformed into the following SOCP:*

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w} \qquad \text{subject to} \qquad \mathbf{w}^H \mathbf{a} \geq \epsilon \|\mathbf{w}\| + 1\,, \quad \mathrm{Im}\left\{ \mathbf{w}^H \mathbf{a} \right\} = 0\,. \tag{19}$$

which can be solved via interior point methods with a complexity of $O(M^{3.5})$.

The argument for Proposition 4 is simple. First, we use triangle inequality to show that the nonconvex semi-infinite problem (18) is equivalent to:

$$\min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R} \mathbf{w} \qquad \text{subject to} \qquad |\mathbf{w}^H \mathbf{a}| - \epsilon \|\mathbf{w}\| \geq 1\,. \tag{20}$$

The nonlinear constraint in (20) is still nonconvex due to the absolute value operation on the left hand side. An important observation is that the cost function in (20) is *unchanged* when w undergoes an arbitrary phase rotation. Therefore, if w_0 is an optimal solution to

(20), we can always rotate, without affecting the objective function value, the phase of $\mathbf{w}_0$ so that $\mathbf{w}^H\mathbf{a}$ is real. Thus, we can, without any loss of generality, choose w such that

$$\mathrm{Re}\left\{\mathbf{w}^H\mathbf{a}\right\} \geq 0, \quad \mathrm{Im}\left\{\mathbf{w}^H\mathbf{a}\right\} = 0.$$

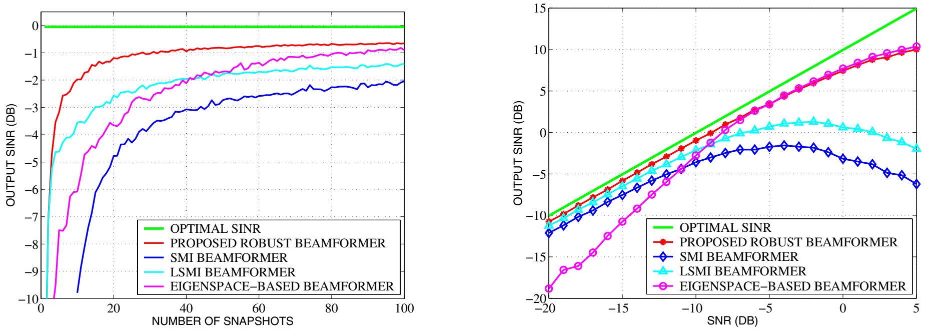
It is not yet known if one can extend Proposition 4 by incorporating other robust constraints in (18) to account for the placement of nulls and sidelobe control. The answer to this is quite relevant to practical digital beamforming applications. However, mathematically, introducing new robust nonconvex quadratic constraints in (18) may destroy the hidden structure of SOCP. It is not clear how one can reformulate the problem while retaining the SOCP structure.

Prior to the discovery of the SOCP formulation given in Proposition 4, the same robust optimization technique was used to design a powerful SOCP based blind multiuser detector [20]. A closely related (but independent) work [51] used the standard robust LP model (8) to formulate the robust beamforming problem by replacing the constraint set of (17) with Re $(\mathbf{w}^H(\mathbf{a} + \Delta a)) \geq 1$, for all $\|\Delta a\| \leq \epsilon$. While this formulation does not appear to correspond to the maximization of the worst case SINR (15), it can be shown, surprisingly, to be also equivalent to the same SOCP (19). Similarly, the robust approach [15] offered an interesting solution to beamforming using quadratic programming, although no explicit attempt was made to ensure the worst case SINR is maximized. Several recent work have further extended the robust beamforming method (18) to nonstationary and general rank models [46, 48, 64, 67, 81].

Example: Signal look direction mismatch

We now present a working example illustrating the effectiveness of the robust beamforming approach. Let us assume a uniform linear array with $M = 10$ omnidirectional sensors spaced half a wavelength apart. For each scenario, 100 simulation runs are used to obtain each simulated point. We assume two interfering sources with plane wavefronts and the Directions Of Arrival (DOA's) 30° and 50°, respectively. Both the presumed and actual signal spatial signatures are plane waves impinging from the DOA's 3° and 5°, respectively. This corresponds to a 2° mismatch in the signal look direction. The Interference-to-Noise Ratio (INR) in a single sensor is equal to 15 dB and the signal is always present in the training data cell. Four methods are compared in terms of the mean output SINR: the proposed robust beamformer (19), the SMI beamformer [65], the LSMI beamformer [19] and the eigenspace-based beamformer [16]. The optimal theoretic SINR is also shown in the figures. The `SeDuMi` convex optimization MATLAB toolbox [70] has been used to compute the weight vector of our robust beamformer, where the constant $\epsilon = 3$ has been chosen assuming that the nominal steering vector is normalized so that $\mathbf{a}^H\mathbf{a} = M$ $(= 10)$. The diagonal loading factor $\xi = 0.5$ is taken in the LSMI beamformer. Furthermore, diagonal loading with the same parameter is applied to our robust technique as well, but only in the case when the data covariance matrix $\mathbf{R}$ is rank deficient (i.e., in the case when the number of snapshots N is less than M).

Fig. 2(a) shows the performance of the methods tested versus the number of training snapshots N for the fixed SNR $= -10$ dB. The performance of these algorithms versus



(a) Output SINR versus training sample size N.

(b) Output SINR versus SNR.

Fig. 2. Performance comparison of different robust beamformers

the SNR for the fixed training data size $N = 30$ is shown in Fig. 2(b). These figures show a substantial performance gain of the new robust methods as compared to the existing approaches.

3.3. Magnitude filter design

In the design of finite impulse response (FIR) filters, one often encounters a spectral mask constraint on the magnitude of the frequency response of the filter (e.g., [63, 66, 68]). That is, for given $L(e^{j\omega})$ and $U(e^{j\omega})$, constrain the (possibly complex) filter coefficients g_k so that

$$L(e^{j\omega}) \leq |G(e^{j\omega})| \leq U(e^{j\omega}) \qquad \text{for all } 0 \leq \omega < 2\pi, \tag{21}$$

or determine that the constraint cannot be satisfied. Here, $j = \sqrt{-1}$ and $G(e^{j\omega}) = \sum_k g_k e^{-j\omega k}$ is the frequency response of the filter. The mask functions $L(e^{j\omega})$ and $U(e^{j\omega})$ are typically given by industrial standards or by the overall system design specification of a particular application. Fig. 3 shows a graphical example of a spectral mask constraint (the dashed lines) for a pulse shaping filter (length $L = 48$) suggested by a wireless communication standard (IS95 [42]). A spectral mask constraint can be rather awkward to accommodate into general optimization-based filter design techniques for two reasons. First, it is semi-infinite in the sense that there are two inequality constraints for every $\omega \in [0, 2\pi)$. Second, the set of feasible filter coefficients is in general non-convex due to the lower bound on $|G(e^{j\omega})|$. In order to efficiently solve filter design problems employing such constraints, we must find a way in which (21) can be represented in a finite and convex manner.

There are two established approaches [63] to deal with the problem of non-convexity of (21). The first is to enforce additional constraints on the parameters g_k so that $G(e^{j\omega})$ has 'linear phase' (i.e., g_k possesses a symmetry property). In that case

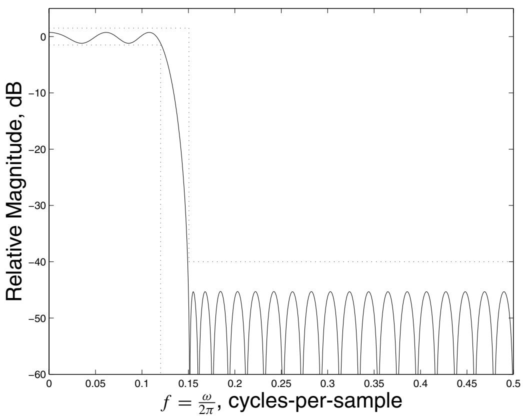


Fig. 3. An example of spectral mask constraint: the frequency magnitude response of the pulse shaping filter suggested by the IS95 standard

$|G(e^{j\omega})|$ becomes a linear function of approximately half the g_k's, and hence (21) can be reduced to two semi-infinite linear (and hence convex) constraints. The second approach to deal with non-convexity is to reformulate (21) in terms of the autocorrelation of the filter [17, 24, 41, 87, 88]. In particular, if

$$r_m = \sum_k g_k \bar{g}_{k-m}$$

represents the autocorrelation of the filter, then taking Fourier transform on both sides of above equation yields $R(e^{j\omega}) = \sum_m r_m e^{-j\omega m} = |G(e^{j\omega})|^2$, and hence (21) becomes

$$L(e^{j\omega})^2 \le R(e^{j\omega}) \le U(e^{j\omega})^2 \qquad \text{for all } 0 \le \omega < 2\pi, \tag{22}$$

which amounts to two semi-infinite linear constraints on r_m. (Observe that $r_{-m} = \bar{r}_m$ and hence $R(e^{j\omega})$ is real.) Hence, by reformulating the mask constraint in terms of r_m, $m \ge 0$, we obtain convex (in fact linear) constraints. Note that the constraint that $R(e^{j\omega}) \ge L(e^{j\omega})^2 \ge 0$ is sufficient to ensure that a filter g_k can be extracted (though not uniquely) from a designed autocorrelation r_m via spectral factorization (also known as Riesz-Féjer Theorem) [88]. Therefore the nonconvex feasibility problem (21) is completely equivalent to the convex (linear) feasibility problem (22).

The problem of representing (21) or (22) in a finite manner is more challenging. One standard, but ad-hoc, approach is to approximate the constraints by discretizing them uniformly in frequency and enforcing the $2N$ linear constraints

$$L(e^{j\omega_i})^2 + \epsilon \le R(e^{j\omega_i}) \le U(e^{j\omega_i})^2 - \epsilon \qquad \text{for } \omega_i = 2\pi i/N,\ i = 0, 1, \ldots, N-1, \tag{23}$$

where N and ϵ are chosen heuristically. For a fixed N, one must choose ϵ to be small enough so that the over-constraining of the problem at frequencies ω_i does not result in significant performance loss, yet one must choose ϵ to be large enough for satisfaction of (23) to guarantees satisfaction of (22) for all $0 \le \omega < 2\pi$. Recently, a dual parameterization method [21] is introduced to treat the semi-infinite mask constraint. However, this approach may still result in non-convex design problems and thus is subject to risks of local optima.

If the lower and upper spectral bounds $L(e^{j\omega})$ and $U(e^{j\omega})$ are constant over $[0, 2\pi]$, then we can invoke the Positive Real Lemma (Proposition 3) to transform the spectral mask constraints into some equivalent finite LMI constraints. However, the bounds $L(e^{j\omega})$ and $U(e^{j\omega})$ are typically piecewise constant over $[0, 2\pi]$, so the Positive Real Lemma does not apply. Fortunately, it turns out that one can derive LMI formulations of general constraints of the form Re $(R(e^{j\omega})) \ge$ Re $(A(e^{j\omega}))$ for all $\omega \in [\alpha, \beta]$, where $A(e^{j\omega}) = \sum_{k=0}^{M_A-1} a_k e^{-j\omega k}$ is a fixed trigonometric polynomial and Re $(\cdot)$ denotes the real part. Since these LMI formulations apply to segments of the unit circle they naturally incorporate piecewise constant lower/upper bounds in (22) and can be considered as generalizations of the Positive Real Lemma (cf. Proposition 3). Thus, there exists a precise finite LMI representation of a large class of spectral mask constraints that results in convex design problems [5, 23].

Proposition 5. *Let* $0 \le \alpha < \beta < 2\pi$. *Let* $r = (r_{-M+1}, \cdots, r_{M-1})^T \in \mathbb{C}^{2M-1}$ *and its Fourier transform as* $R(e^{j\omega}) = \sum_{m=-M+1}^{M-1} r_m e^{jm\omega}$. *Then* Re $(R(e^{j\omega})) \ge 0$ *for all* $\omega \in [\alpha, \beta]$ *if and only if there exist* $X \in \mathcal{H}_+^{M\times M}$, $Z \in \mathcal{H}_+^{(M-1)\times(M-1)}$, $\xi \in \mathbb{R}$ *such that*

$$r + \xi j e_0 = \mathbf{L}^*(X) + \mathbf{\Lambda}^*(Z; \alpha, \beta), \tag{24}$$

where $\mathbf{L}^*$ *and* $\mathbf{\Lambda}^*$ *are some linear operators, and* e_0 *is the* M*-th column of the* $(2M-1) \times (2M-1)$ *identity matrix.*

Proposition 5 provides a theoretically satisfying characterization of the mask constraint which avoids the heuristic approximation of discretization techniques, yet generates practically competitive design algorithms [23].

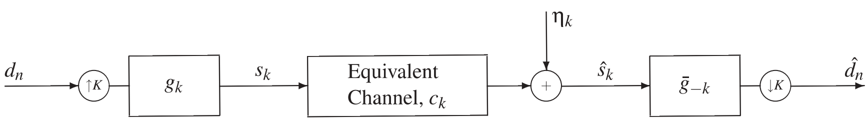


Fig. 4. Discrete-time model of baseband PAM communication channel

We can apply this LMI formulation to design several families of FIR filters, including those which generate robust 'chip' waveforms for digital wireless telephony systems based on code division multiple access [22, 23]. Specifically, consider a match filtered baseband PAM (Pulse Amplitude Modulation) digital communication system described by Figure 4, where the integer $K > 0$ denotes the oversampling factor and $\{g_k\}$ denotes the pulse shaping filter (or chip waveform) to be designed. The data estimates after match filtering $\hat{d}_n = \sum_k \bar{g}_{k-Kn}\hat{s}_k$, can be written as

$$\hat{d}_n = \sum_q c_{n-q}^{\mathrm{ISI}} d_q + \eta_n^d, \tag{25}$$

where $c_q^{\mathrm{ISI}} = \sum_i c_i r_{i-Kq}$ is the equivalent channel from an inter-symbol interference (ISI) perspective, $r_m \triangleq \sum_k g_k \bar{g}_{k+m}$ is the autocorrelation sequence of the filter g_k, and $\eta_n^d = \sum_k \bar{g}_{k-Kn} \eta_k$ is the effect of the noise on $\hat{d}_n$. A common design goal is to find a pulse shaping filter $\{g_k\}$ which minimizes the spectral occupation of the communication scheme subject to the constraint that the filters are self-orthogonal at translations of integer multiples of K. The orthogonality constraint ensures that there is no inter-symbol interference in a distortionless channel, and that the receiver filter neither amplifies nor correlates the white noise component of the external interference. Given positive integers K, L and spectral mask functions $L(e^{\mathrm{j}\omega})$, $U(e^{\mathrm{j}\omega})$, and assuming the channel is distortionless, we wish to find a filter vector (possibly complex) $\{g_k\}$ of length L satisfying

$$\sum_{k=\ell K}^{L-1} g_k \bar{g}_{k-K\ell} = \delta(\ell), \quad \ell = 0, 1, \ldots, \lfloor (L-1)/K \rfloor, \tag{26a}$$

$$L(e^{\mathrm{j}\omega}) \le |G(e^{\mathrm{j}\omega})| \le U(e^{\mathrm{j}\omega}), \quad \text{for all } 0 \le \omega < 2\pi, \tag{26b}$$

where $\delta(\cdot)$ denotes the Kronecker Delta function, $\lfloor x \rfloor$ denotes the greatest integer $\le x$. Unfortunately, both these constraints are non-convex in the parameters g_k. However, we can parameterize the problem in terms of the autocorrelation r_m, resulting in the linear constraints

$$r_{\ell K} = \delta_\ell, \qquad \text{for } \ell = 0, 1, \ldots, \lfloor (L-1)/K \rfloor, \tag{27a}$$

$$L^2(e^{\mathrm{j}\omega}) \le R(e^{\mathrm{j}\omega}) \le U^2(e^{\mathrm{j}\omega}), \qquad \text{for all } 0 \le \omega < 2\pi. \tag{27b}$$

Notice that by Proposition 5 the constraint (27b) can be reformulated as a finite LMI, so the overall design problem (27a)–(27b) can be cast as a LMI feasibility problem.

Example. We design a filter to compete with the filter specified for the synthesis of the chip waveform in the IS95 standard [42]. The standard requires a filter with a ± 1.5dB ripple in the pass-band $f \in [0, f_p]$, and 40dB attenuation in the stop-band $f \in [f_s, 1/2]$, where $f_p = 590/(1228.8K)$ and $f_s = 740/(1228.8K)$. The filter chosen in the standard has linear phase, $K = 4$ and $L = 48$, and hence $f_p \approx 0.12$ and $f_s \approx 0.15$. Whilst that filter satisfies the spectral mask, it does not satisfy the orthogonality constraints (26a). Hence, the IS95 filter can induce substantial 'inter-chip' interference even when the physical channel is benign. Therefore, we seek a minimal length filter such that *both* the frequency response mask is satisfied *and* the filter is orthogonal. A globally optimal solution to this problem was found using an SDP formulation coupled with a binary search technique [24]. Each SDP was solved using `SeDuMi` [70] in about just under 7 minutes on a 400 MHz PENTIUM II PC. The above procedure resulted in a length $L = 51$, so orthogonality is achieved for the price of a mild increase in filter length (from $L = 48$). The frequency response of the designed filter is shown in Fig. 5 (compare with Fig. 3 of the nonorthogonal filter suggested by the IS95 standard).

The formulation in the above example (see (27)) was based on a nominally ideal channel model. In practice, robustness to channel perturbations will be required. Fortunately, robustness to quite diverse class of perturbations can be enforced using a semidefinite

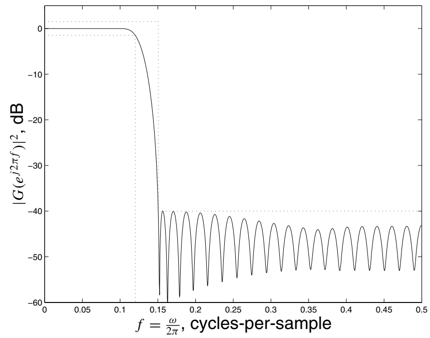


Fig. 5. Relative power spectra (in decibels) of the designed orthogonal filter ($L = 51$) with the spectral mask from the IS95 standard

programming framework [22]. Moreover, for computational reason, it is more efficient to deal with the dual formulation of the cone of autocorrelation sequences [4]. This is due to the fact that the dual cone can be easily parameterized by $O(n)$ parameters, while the primal cone (cf. Proposition 3) requires $O(n^2)$ parameters. This results in a substantial reduction in the arithmetic operation count (from $O(n^6)$ to $O(n^3)$) per each interior point iteration.

3.4. Robust magnitude filter design

In this subsection, we study the problem of designing magnitude filters which are robust to quantization errors. To motivate our robust model, let us consider the design of a linear phase FIR (Finite Impulse Response) filter for digital signal processing. Here, for a linear phase FIR filter $h = (h_1 \ldots, h_n) \in \mathbb{R}^n$ (symmetric part ignored), the frequency response is (essentially)

$$H(e^{\mathrm{j}\omega}) = e^{\mathrm{j}n\omega}(h_1 + h_2 \mathbf{cos}\ \omega + \cdots + h_n \mathbf{cos}\ (n\omega)) = e^{\mathrm{j}n\omega}(\mathbf{cos}\ \omega)^T h,$$

where $\mathbf{cos}\ \omega = (1, \mathbf{cos}\ \omega, \cdots, \mathbf{cos}\ (n\omega))^T$. The FIR filter usually must satisfy a given spectral envelope constraint (typically specified by design requirements or industry standards, see Figure 3).

$$L(e^{\mathrm{j}\omega}) \le (\mathbf{cos}\ \omega)^T h \le U(e^{\mathrm{j}\omega}), \quad \forall\ \omega \in [0, \pi] \tag{28}$$

Finding a discrete h (say, 4-bit integer vector) satisfying (28) is NP-hard. Ignoring discrete structure of h, we can find a h satisfying (28) in polynomial time [23] by exactly reformulating (28) as a LMI system using Proposition 5. However, rounding such a solution to the nearest discrete h may degrade performance significantly. Our design strategy is then to first discretize the frequency $[0, \pi]$, then find a solution robust to discretization and rounding errors. This leads to the following notion of robustly feasible solution:

$$L(e^{\mathrm{j}\omega_i}) \le (\mathbf{cos}\ \omega_i + \Delta_i)^T (h + \Delta h) \le U(e^{\mathrm{j}\omega_i}), \quad \text{for all } \|\Delta_i\| \le \epsilon,\ \|\Delta h\| \le \delta, \tag{29}$$

where Δ_i accounts for discretization error, while Δh models the rounding errors.

The above example motivates us to consider the following formulation of a robust linear program:

$$\begin{aligned}
&\text{minimize} \quad \max_{\|\Delta x\|\le\delta,\|\Delta c\|\le\epsilon_0} (c+\Delta c)^T(x+\Delta x)\\
&\text{subject to } (a_i+\Delta a_i)^T(x+\Delta x) \ge (b_i+\Delta b_i), \\
&\qquad\qquad \text{for all } \|(\Delta a_i, \Delta b_i)\| \le \epsilon_i,\ \|\Delta x\| \le \delta,\ i=1,2,...,m.
\end{aligned} \tag{30}$$

Here two types of perturbation are considered. First, the problem data $(\{a_i\}, \{b_i\}, c)$ might be affected by unpredictable perturbation (e.g., measurement error). Second, the optimal solution x^{opt} is subject to implementation errors caused by the finite precision arithmetic of digital hardware. That is, we have $x^{actual} := x^{opt} + \Delta x$, where x^{acutal} is the actually implemented solution. To ensure x^{actual} remains feasible and delivers a performance comparable to that of x^{opt}, we need to make sure that x^{opt} is robust against both types of perturbations. This is essentially the motivation of the above robust linear programming model. Notice that the above model (30) is more general than the robust LP formulation (8) proposed by Ben-Tal and Nemirovskii [8] in that the latter only considers perturbation error in the data $(\{a_i\}, \{b_i\}, c)$.

We now reformulate the robust linear program (30) as a semidefinite program. We say the solution x is *robustly feasible* if, for all $i = 1, 2, ..., m$,

$$\begin{aligned}
&(a_i+\Delta a_i)^T(x+\Delta x) \ge (b_i+\Delta b_i),\\
&\text{for all } \|(\Delta a_i, \Delta b_i)\| \le \epsilon_i,\ \|\Delta x\| \le \delta,\ i=1,2,...,m.
\end{aligned}$$

It can be shown [8] that x is robustly feasible if and only if

$$a_i^T(x+\Delta x) - b_i - \epsilon_i\sqrt{\|x+\Delta x\|^2+1} \ge 0, \quad \forall\ \|\Delta x\| \le \delta,\ i=1,2,...,m. \tag{31}$$

Constraint (31) can be formulated as

$$\begin{bmatrix} (a_i^T(x+\Delta x)-b_i)I & \epsilon_i\begin{bmatrix} x+\Delta x\\ 1\end{bmatrix}\\ \epsilon_i\begin{bmatrix}(x+\Delta x)^T & 1\end{bmatrix} & a_i^T(x+\Delta x)-b_i\end{bmatrix} \succeq 0, \quad \forall\ \|\Delta x\| \le \delta,\ i=1,2,...,m. \tag{32}$$

Now the objective function can also be modelled by introducing an additional variable t to be minimized, and at the same time set as a constraint $t-(c+\Delta c)^T(x+\Delta x) \ge 0$, for all $\|\Delta c\| \le \epsilon_0$ and $\|\Delta x\| \le \delta$. Then the objective can be modelled by $t - c^T(x+\Delta x) \ge \epsilon_0\|x+\Delta x\|$, for all $\|\Delta x\| \le \delta$, which is equivalent to

$$\begin{bmatrix} (t-c^T(x+\Delta x))I & \epsilon_0(x+\Delta x)\\ \epsilon_0(x+\Delta x)^T & t-c^T(x+\Delta x)\end{bmatrix} \succeq 0, \quad \forall\ \|\Delta x\| \le \delta. \tag{33}$$

Using Proposition 2, we can show that (32) is equivalent to the following: there exists a $\mu_i \ge 0$ such that

$$\begin{bmatrix} (a_i^Tx-b_i)I & \epsilon_i\begin{bmatrix} x\\ 1\end{bmatrix} & \epsilon_i\begin{bmatrix} I\\ 0\end{bmatrix}\\ \epsilon_i\begin{bmatrix} x^T & 1\end{bmatrix} & a_i^Tx-b_i & \frac{1}{2}a_i^T\\ \epsilon_i\begin{bmatrix} I & 0\end{bmatrix} & \frac{1}{2}a_i & 0\end{bmatrix} - \mu_i\begin{bmatrix} 0 & 0 & 0\\ 0 & \delta^2 & 0\\ 0 & 0 & -I\end{bmatrix} \succeq 0. \tag{34}$$

Similarly, (33) holds for all $\|\Delta x\| \le \delta$ if and only if there is a $\mu_0 \ge 0$ such that

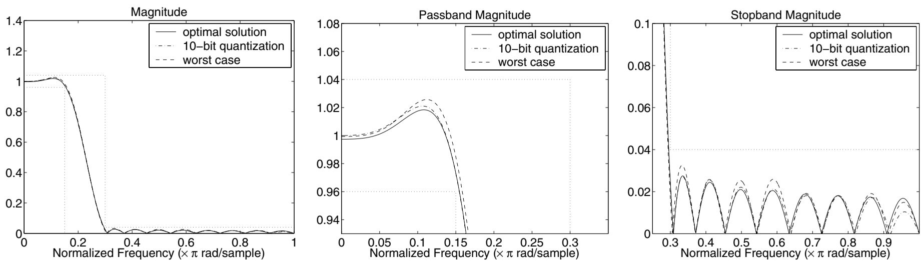
$$\begin{bmatrix} (t - c^T x)I & \epsilon_0 x & \epsilon_0 I \\ \epsilon_0 x^T & t - c^T x & -\frac{1}{2}c^T \\ \epsilon_0 I & -\frac{1}{2}c & 0 \end{bmatrix} - \mu_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & -I \end{bmatrix} \succeq 0. \qquad (35)$$

Therefore, the robust linear programming model becomes a semidefinite program.

Proposition 6. *The robust linear program* (30) *can be reformulated as the following SDP:*

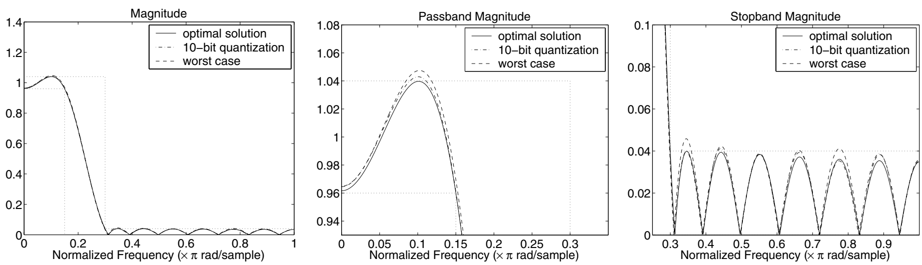
$$\text{minimize } t, \ \text{ subject to (34) and (35).}$$

Figures 6 and 7 show a design example obtained using the robust formulation (29) and the nonrobust LMI formulation of spectral mask constraint (28), respectively. In both cases, the dotted lines denote the spectral mask constraints which the designed filter must satisfy, the solid curves correspond to the magnitude responses of optimal robust and non-robust filters designed from the formulation (29) with $\delta = 2^{-10}$ and



(a) Magnitude responses (b) Detailed passband responses (c) Detailed stopband responses

Fig. 6. Magnitude responses of robust FIR filter ($n = 22$) subject to different quantization errors



(a) Magnitude responses (b) Detailed passband responses (c) Detailed stopband responses

Fig. 7. Magnitude responses of nonrobust FIR filter ($n = 19$) subject to quantization errors

0 respectively, the dash-dot and dashed curves correspond to the 10-bit truncation of the optimal filters. To satisfy the spectral mask specification, the minimum filter length is 22 for the robust filter and 19 for the nonrobust filter. We consider both the nearest neighbor quantization and the farthest neighbor quantization (or worst case rounding). As shown in Figs. 6 and 7, when the optimal coefficients are truncated to 10 bits, the quantized versions of the robust filter still satisfy the spectral mask constraints, while the resulting nonrobust filters violate both the passband and the stopband spectral mask specifications.

4. Efficient optimization methods for digital communication

Conic and robust optimization techniques have also been successfully applied to some of the fundamental system design and real time processing problems in the area of digital communication. In this section, we describe two such applications arising from multi-user communication and wireless Multi-Input Multi-Output (MIMO) communication.

4.1. The application of SDP to transceiver design for a multi-user communication system

Consider a quasi-synchronous vector multiple access scheme with two users whose data vectors s_1 and s_2, are uncorrelated (see Figure 8). The channel matrices H_1 and H_2, which are of size $p \times n$, are assumed to be known, and n is a zero mean additive Gaussian noise vector which is uncorrelated with s_1 and s_2 and has known correlation matrix $E(nn^H) = R$. With square transmitter precoding matrices F_1 and F_2, the received signal takes the form

$$x = H_1 F_1 s_1 + H_2 F_2 s_2 + n. \tag{36}$$

In our development, each data block s_i will be treated as white with identity correlation matrix. From the received signal x, we wish to extract the transmitted signals $s_i, i = 1, 2$. This can be accomplished in various ways. A popular approach is to use a linear receiver G_i whereby the equalized signal $G_i x$ is quantized according to the finite alphabet of s_i, with $G_i, i = 1, 2$, being the block (matrix) equalizers.

The above vector multiple access channel model arises naturally in the so-called generalized multi-carrier block transmission scheme [84]. When the data blocks are appropriately padded with cyclic prefix, the channel matrices H_i become circulant and can be diagonalized by Fourier transforms, resulting the well known OFDM (Orthogonal Frequency Division Multiplexing) scheme. We will develop an SDP formulation of the MMSE transceiver design problem for general H_j and R as well as a SOCP formulation for the diagonalized OFDM channels. We point out that similar models to that in Fig. 8 have been considered in [94] where the capacity region for the above multi-access communication channel is evaluated using the tool of semidefinite programming.

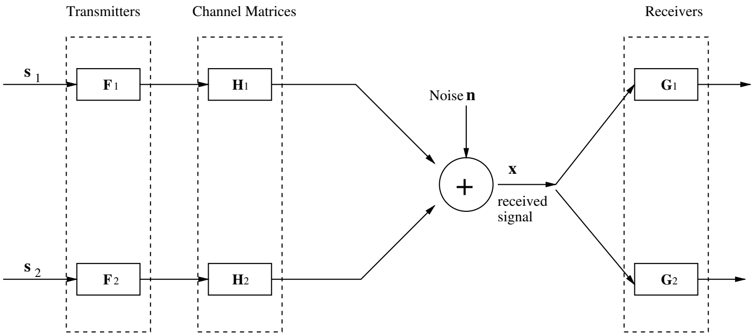


Fig. 8. Two user multiple access scheme (uplink)

SDP formulation of MMSE Transceiver Design

For the system in Figure 8, let e_i denote the error vector (before making the hard decision) for user i, $i = 1, 2$. Then

$$\begin{aligned} e_1 &= G_1 x - s_1 = G_1(H_1 F_1 s_1 + H_2 F_2 s_2 + n) - s_1 \\ &= (G_1 H_1 F_1 - I)\, s_1 + G_1 H_2 F_2 s_2 + G_1 n. \end{aligned}$$

Assuming statistical independence of signals and noises, this further implies that

$$E(e_1 e_1^H) = (G_1 H_1 F_1 - I)\,(G_1 H_1 F_1 - I)^H + (G_1 H_2 F_2)\,(G_1 H_2 F_2)^H + G_1 R G_1^H \tag{37}$$

where $E(\cdot)$ represents statistical expectation and we have used the fact that $E(nn^H) = R$ and $E(s_i s_i^H) = I$, $i = 1, 2$. Similarly, we have

$$E(e_2 e_2^H) = (G_2 H_2 F_2 - I)\,(G_2 H_2 F_2 - I)^H + (G_2 H_1 F_1)\,(G_2 H_1 F_1)^H + G_2 R G_2^H. \tag{38}$$

As is always the case in practice, there are power constraints on the transmitting matrix filters:

$$\mathrm{Tr}\left(F_1 F_1^H\right) \le p_1, \quad \mathrm{Tr}\left(F_2 F_2^H\right) \le p_2 \tag{39}$$

where $p_1 > 0$ and $p_2 > 0$ are user-specified bounds on the transmitting power for each user. Our goal is to design a set of transmitting matrix filters F_i satisfying the power constraints (39) and a set of matrix equalizers G_i such that the total mean squared error MSE $=\ \mathrm{Tr}(E(e_1 e_1^H)) +\ \mathrm{Tr}(E(e_2 e_2^H))$ is minimized. In other words, we aim to solve

$$\begin{array}{ll} \text{minimize}_{F_1,F_2,G_1,G_2} & \mathrm{Tr}(E(e_1 e_1^H)) +\ \mathrm{Tr}(E(e_2 e_2^H)) \\ \text{subject to} & \mathrm{Tr}(F_1 F_1^H) \le p_1, \quad \mathrm{Tr}(F_2 F_2^H) \le p_2, \end{array} \tag{40}$$

where $\mathrm{Tr}(E(e_1 e_1^H))$ and $\mathrm{Tr}(E(e_2 e_2^H))$ are given by (37) and (38) respectively. The receiver filters G_1 and G_2 in (40) are unconstrained. The objective function of (40) is a fourth order polynomial in $G_i, F_i, i = 1, 2$. It can be easily checked (even for the case where the block length n is one; i.e., each G_i, F_i is a scalar) that the Hessian matrix of this fourth-order polynomial is not positive semidefinite. Therefore, the objective function of (40) is nonconvex, and hence it can be difficult to minimize due to the usual difficulties with local solutions and the selection of a stepsize and starting point. In what follows, we will reformulate (40) as a convex semidefinite program.

As the first step, we can eliminate G_1 and G_2 in (40) by first minimizing the total MSE with respect to G_1 and G_2 assuming F_1 and F_2 are fixed. The resulting receivers are the so called *linear Minimum Mean Squared Error* (*MMSE*) *receivers*:

$$G_1 = F_1^H H_1^H W, \quad G_2 = F_2^H H_2^H W \tag{41}$$

with

$$W = \left(H_1 F_1 F_1^H H_1^H + H_2 F_2 F_2^H H_2^H + R\right)^{-1}, \tag{42}$$

Substituting the MMSE equalizer (41) into (37) results in the following minimized (with respect to G_i) mean square error:

$$E(e_1 e_1^H) = -F_1^H H_1^H W H_1 F_1 + I, \quad E(e_2 e_2^H) = -F_2^H H_2^H W H_2 F_2 + I. \tag{43}$$

Substituting (43) into the total MSE and using the definition of W (42) gives rise to

$$\text{MSE} = \text{Tr}\,(WR) + n, \tag{44}$$

Now let us further define two new matrix variables $U_1 = F_1 F_1^H$ and $U_2 = F_2 F_2^H$. Then the MMSE (44) can be expressed as

$$\text{MSE} = \text{Tr}\left((H_1 U_1 H_1^H + H_2 U_2 H_2^H + R)^{-1} R\right) + n$$

and the power constraints (39) can be expressed as $\text{Tr}(U_1) \le p_1$ and $\text{Tr}(U_2) \le p_2$. Consequently, the optimal joint MMSE transmitter-receiver design problem can be stated as

$$\begin{aligned} &\text{minimize}_{U_1,U_2}\ g(U_1, U_2) := \text{Tr}\left((H_1 U_1 H_1^H + H_2 U_2 H_2^H + R)^{-1} R\right) \\ &\text{subject to} \quad \text{Tr}(U_1) \le p_1, \quad \text{Tr}(U_2) \le p_2, \\ &\qquad\qquad\quad U_1 \succeq 0, \quad U_2 \succeq 0. \end{aligned} \tag{45}$$

Using the auxiliary matrix variable W (42) and the fact that $R \succeq 0$, we can rewrite (45) in the following alternative (but equivalent) form:

$$\begin{aligned} &\text{minimize}_{W,U_1,U_2} \quad \text{Tr}\,(WR) \\ &\text{subject to} \quad \text{Tr}(U_1) \le p_1, \quad \text{Tr}(U_2) \le p_2, \\ &\qquad\qquad\quad W \succeq (H_1 U_1 H_1^H + H_2 U_2 H_2^H + R)^{-1} \\ &\qquad\qquad\quad U_1 \succeq 0, \quad U_2 \succeq 0. \end{aligned} \tag{46}$$

Notice that the constraint

$$W \succeq (H_1 U_1 H_1^H + H_2 U_2 H_2^H + R)^{-1}$$

can be replaced, via Schur's complement, by the following equivalent linear matrix inequality

$$\begin{bmatrix} W & I \\ I & H_1 U_1 H_1^H + H_2 U_2 H_2^H + R \end{bmatrix} \succeq 0. \tag{47}$$

Therefore, we obtain an equivalent Semidefinite Programming (SDP) formulation. This SDP formulation makes it possible to efficiently solve the optimal transmitter design problem using interior point methods [61, 76]. The advantage of the SDP formulation over the formulation (40) is that the former is convex while the latter is not. The arithmetic complexity of the interior point methods for solving the SDP is $O(n^{6.5}\log(1/\epsilon))$, where $\epsilon > 0$ is the solution accuracy [61, 76]. Once the optimal U_1 and U_2 have been determined, they can be factorized (e.g., Cholesky factorization) as $U_1 = F_1 F_1^H$ and $U_2 = F_2 F_2^H$ to obtain optimal MMSE transmitter matrices F_1 and F_2.

Diagonal designs

When the channel matrices H_1 and H_2 are diagonal (as in OFDM systems) and the noise covariance matrix R is also diagonal, we can show (see Proposition 7 below or [52]) that the optimal transmitters are also diagonal and can be computed more efficiently (faster than solving the SDP described earlier).

Proposition 7. *If the channel matrices H_1 and H_2 are diagonal and the noise covariance matrix R is diagonal, then the optimal transmitters U_1 and U_2 are also diagonal. Consequently, the MMSE transceivers for a multi-user OFDM system can be implemented by optimally allocating power to each subcarrier for all the users.*

Proposition 7 should be good news to practitioners since it says that in 'diagonal' scenarios, there is no need to implement full precoder matrices because diagonal precoders are optimal. Notice that diagonal precoders simply represent power loading/subcarrier allocation at the transmitters. Therefore, Proposition 7 implies that the MMSE transceivers for a multi-user OFDM system can be implemented by optimally assigning subcarriers and allocating power to them.

Another important implication of Proposition 7 is the significant simplification in the computation of the optimal MMSE transceivers. In particular, Proposition 7 suggests that we only need to search among all the diagonal transmitters in order to achieve the minimum MSE. Therefore, if H_1, H_2 and R are diagonal, it is only necessary to solve (48) below rather than the SDP described in the previous section. Before we state this formally, we point out that when the channel matrices H_j have been diagonalized using the FFT (Fast Fourier Transform) and IFFT (Inverse FFT), the ith diagonal element is $H_j(i)$, where $H_j(i)$ is the frequency response of user j's channel at the ith point on the FFT grid, $\omega_i = 2\pi i/n$. Define the diagonal entries of U_1, U_2 by $u_1 = \text{diag}\,(U_1)$,

$u_2 =$ diag (U_2). Then using u_1, u_2 as the new variables to be optimized, and letting $R = \text{diag}\{\rho_i^2\}$, the reduced optimization problem becomes

$$\begin{aligned} \text{minimize}_{u_1,u_2} \quad & \sum_{i=1}^{n} \rho_i^2 \left(|h_1(i)|^2 u_1(i) + |h_2(i)|^2 u_2(i) + \rho_i^2\right)^{-1} \\ \text{subject to} \quad & \sum_{i=1}^{n} u_1(i) \le p_1, \quad \sum_{i=1}^{n} u_2(i) \le p_2, \\ & u_1(i) \ge 0, \quad u_2(i) \ge 0, \quad i = 1, 2, ..., n. \end{aligned} \tag{48}$$

Introducing an auxiliary vector w, we can transform (48) into the following (rotated) second order cone program:

$$\begin{aligned} \text{minimize}_{w,u_1,u_2} \quad & \sum_{i=1}^{n} \rho_i^2 w(i) \\ \text{subject to} \quad & \sum_{i=1}^{n} u_1(i) \le p_1, \quad \sum_{i=1}^{n} u_2(i) \le p_2, \\ & w(i)\left(|h_1(i)|^2 u_1(i) + |h_2(i)|^2 u_2(i) + \rho_i^2\right) \ge 1, \\ & u_1(i) \ge 0, \quad u_2(i) \ge 0, \quad i = 1, 2, ..., n. \end{aligned} \tag{49}$$

There exist highly efficient (general purpose) interior point methods [61] to solve the above second order cone program with total computational complexity of $O(n^{3.5} \log(1/\epsilon))$, where $\epsilon > 0$ is the solution accuracy. This is a significant improvement from the complexity of $O(n^{6.5} \log(1/\epsilon))$ if we solve the MMSE transceiver design problem as an SDP.

Extensions to the sum-capacity formulation and to the MIMO channel as well as broadcast channel models have been obtained in a number of recent papers [34,52,62]. Another related work is given in [94].

4.2. Efficient optimal/sub-optimal detection methods for space-time processing in wireless communication

Recently the use of multiple antennas at both the transmitter and the receiver has been shown to provide several-fold increase in capacity. In a rich scattering environment such as indoor wireless scenario, the channel gains between any pair of transmitter antenna and receiver antenna can be assumed independent. In this case, the capacity increase is shown [71] to be proportional to the number of transmit antennas. To reap the benefits brought by the MIMO (Multi-Input-Multi-Output) channel, one must develop an efficient decoding algorithm used in such a system, especially for cases where channel matrix is ill conditioned. The existing V-Blast type of receiver algorithms [32] developed at Bell Labs suffers from significant performance loss when the channel matrix is ill conditioned.

Central to the MIMO maximum likelihood channel detection and decoding is the following constrained optimization problem

$$\min_{x \in \mathcal{V}^m} \|y - Hx\|^2 \tag{50}$$

where $H \in \mathcal{C}^{n \times m}$ denotes the (known) channel matrix, $y \in \mathcal{C}^n$ is the received channel output, x is the unknown (to be determined) transmitted information symbol vector from the signal constellation set $\mathcal{V}^m$. Usually the constellation set $\mathcal{V}$ is either a finite lattice set (QAM signalling) or a set of points uniformly distributed on the unit circle over the complex plane (PSK signalling):

$$e^{j\phi_i}, \text{ with } \phi_i = 2\pi i/M, \ \forall\, i = 1, \dots, M.$$

The current popular method to approximately solve (50) is with the Sphere Decoder [30, 79]. This method enjoys a cubic complexity on average [39], but is known to have poor performance in low SNR (signal to noise ratio) region.

Consider the case of PSK signalling (which is a popular choice in practice due to its constant modulus and superior performance). The discrete nature of the set $\{\phi_i\}$ makes problem (50) intractable. We can use a continuous relaxation of the set $\{\phi_i\}$ to include all possible angles in $[0, 2\pi]$. In other words, we relax the constraint $x \in \mathcal{V}^m$ to $x \in \mathcal{U}^m = \{\, x \mid |x_i| = 1, \ \forall\, i = 1, \dots, m\}$, so problem (50) becomes:

$$\min_{x \in \mathcal{U}^m} \|y - Hx\|^2 \tag{51}$$

In the combinatorial optimization context (which roughly corresponds to the case of Binary PSK modulation of $M = 2$), the above relaxation was first considered by Burer-Monteiro-Zhang [14] as a low-rank semidefinite programming relaxation of a Boolean quadratic maximization problem. In addition to applying this relaxation to the graph partitioning problem, Burer-Monteiro-Zhang also analyzed the structure of the local

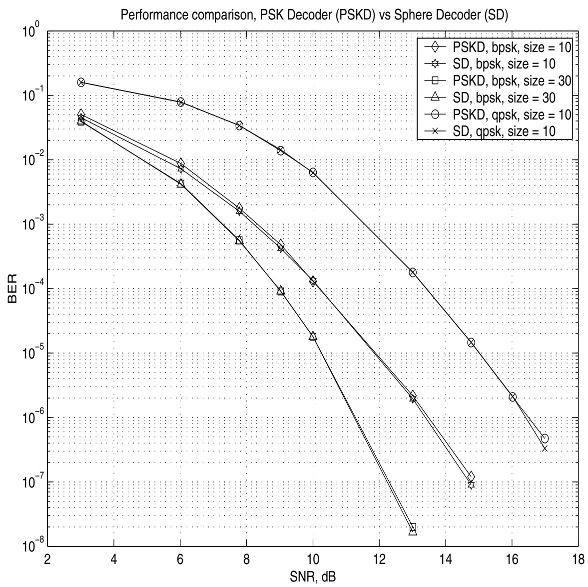


Fig. 9. Bit error performance comparison of PSK Decoder and the Sphere Decoder for uncoded channel with $m = n = 10$ and $m = n = 30$ for binary PSK modulation and $m = n = 10$ for 4-PSK modulation

maxima of (51), and showed that for $M = 2$ each local minimizer of (51) is also a global minimizer. For general M, we have the following result [53].

Proposition 8. *Every local minimizer $\hat{x}$ of* (51) *is a $\frac{1}{2}$-minimizer of* (51) *in the sense that*

$$(f(\hat{x}) - f_{\min})/(f_{\max} - f_{\min}) \leq \frac{1}{2},$$

where $f(x) := \|y - Hx\|^2$, $f_{\max}$ and $f_{\min}$ represent the global maximum and minimum value of f over $\mathcal{U}^m$.

Proposition 8 suggests that finding a local solution of (51) will provide a good approximate solution for the original maximum likelihood detection problem (50). This is the approach adopted first by [1, 56, 72, 73, 83] in the context of CDMA multiuser detection of binary symbols and further extended to the current MIMO channel model for general PSK modulation by [53]. In particular, given the channel matrix H and the vector of received signals y, we need to minimize the relaxed objective function $f(x)$ in (51). While various options exist, we have used the coordinate descent method (similar to [1]). The choice of a starting point x_0 can be random or obtained as an approximate solution to $Hx = y$ by running a fixed number of iterations of the conjugate gradient method, Gauss-Jacobi or Gauss-Seidel iterative method. Since each iteration of these methods have a complexity of $O(k^2)$, where $k = \max\{m, n\}$, the overall complexity of the initialization process does not exceed $O(k^2)$.

We have compared the performance of the new PSK decoder and the Sphere Decoder [79] in terms of BER and computational complexity for different dimensions

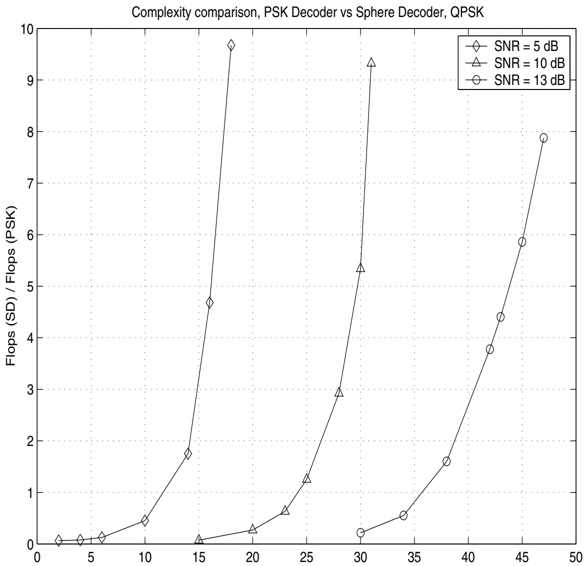


Fig. 10. Complexity comparison, ratio of the Sphere Decoder flops over the PSK Decoder flops for SNR = 5, 10 and 13 dB, 4-PSK modulation

and SNR levels. Notice that PSK decoder works only for PSK constellations, while the Sphere Decoder only for QAM signals. Therefore, the comparison of the two methods is possible only for binary PSK and 4-PSK constellations since these are the cases that can be handled by both decoders. Figure 9 shows the BER comparison of the Sphere Decoder and the PSK Decoder for the case of equal number of transmitters and receivers $m = n = 10$ and $m = n = 30$ with binary PSK signalling, and for the case of $m = n = 10$ for 4-PSK signalling. Both decoders exhibit similar performance in terms of BER. However, the two decoders are significantly different in their complexity. The running time of both decoders depends on the problem size and the type of constellation. For example, Figure 10 shows the ratio of floating-point operations for the Sphere Decoder over the new PSK Decoder as a function of m (we set $m = n$). Three different SNRs (= 5, 10 and 13 dB) are considered and the modulation is 4-PSK. For SNR=10 dB, Sphere Decoder performs faster for problem sizes up to $m = n = 24$, after which the the new PSK decoder becomes substantially faster. For SNR = 5 dB and with 4-PSK modulation, the new PSK Decoder becomes faster for dimensions $m = n = 12$ and above. The efficiency of our PSK decoder can also be seen from CPU time: decoding 100 4-PSK symbols takes 0.29 sec on a 1.5 GHz PENTIUM IV PC.

5. Concluding remarks

The recent development and application of modern optimization techniques has made it possible to solve many of the core problems in signal processing and digital communication efficiently. This is remarkable because not long ago many of these problems were still considered difficult or intractable by signal processing and communication engineers. Encouraged by these successes, some of the mainstream signal processing and communications conferences have started to sponsor special sessions and tutorials to feature the role of optimization and its application potential. This is good news for the mathematical programming community because it not only brings recognition to our field but also will provide new impetus for the continued innovation in algorithm design and numerical software development in mathematical programming. Moreover, the interplay between these fields will ultimately benefit both sides: the modern optimization techniques and related software can help solve some well known difficult engineering problems efficiently, and at the same time, the process of formulating and solving practical engineering problems will also inspire optimizers to refine and extend the current theory and algorithms. The latter point is best represented by the robust beamforming example and the robust magnitude filter design example in Section 3, where the existing robust convex optimization techniques were found to be inadequate and had be generalized.

This paper presented a few examples where the use of conic and robust optimization techniques played an essential role in their formulation and numerical solution. Needless to say, the selected examples only reflect author's limited experience in this burgeoning research area, and they by no means represent all the currently known engineering applications of optimization. Indeed, the list of important applications of convex optimization (e.g., linear/quadratic programming, SOCP/SDP and robust optimization) in electrical engineering is long and fast growing. They include blind channel equalization [26, 54, 57], statistical estimation and moment problems [4, 12, 27, 31, 69], VLSI layout and circuit design [14, 18, 77], designing optimal communication systems (e.g., minimum bit error rate) [25, 44, 45, 82, 89, 94], and robust Kalman filtering [33, 47]. Another fascinating application of convex optimization is in multi-user information theory where SDP duality theory and interior point algorithms have been used to establish the dual correspondence of the capacity region of a multi-access channel and the so called "dirty paper" region of a vector broadcast channel [43,78,93]. Undoubtedly, much more remains to be done to harness the power of convex optimization in the field of signal processing and communication.

Acknowledgements. The author wishes to thank his collaborators who have helped make many of the results presented in this paper possible. These include Kaywan Afkhamie, Robert Cui, Tim Davidson, Zhi Ding, Minyue Fu, Alex Gershman, Georgios Giannakis, Nick Kisialiou, Lingjie Li, Jing Liu, Boris Maricic, Mei Meng, Sharahm Shahbazpanahi, Jos Sturm, Sergiy Vorobyov, Max Wong and Shuzhong Zhang.

References

[1] Abdi, M., El Nahas, H., Jard, A., Moulines, E.: Semidefinite Positive Relaxation of the Maximum-Likelihood Criterion Applied to Multiuser Detection in a CDMA Context. IEEE Signal Process. Lett. **9**, 165–167 (2002)

[2] Afkhamie, K., Luo, Z.-Q., Wong, K.M.: Adaptive Linear Filtering Using Interior Point Optimization Techniques. IEEE Transactions on Signal Process. **48**(6), 1637–1648 (2000)

[3] Afkhamie, K.H., Luo, Z.-Q., Wong, K.M.: MMSE Decision-Feedback Equalization with Short Training Sequences: An Application of Interior Point Least Squares. IEEE Transactions on Signal Process. **49**, 1543–1555 (2001)

[4] Alkire, B., Vandenberghe, L.: Handling Nonnegative Constraints In Spectral Estimation. Proceedings of the Thirty-Fourth Asilomar Conference on Signals, Systems and Computers **1**, 202–206 (2000)

[5] Alkire, B., Vandenberghe, L.: Interior-Point Methods For Magnitude Filter Design. Proceedings of 2001 IEEE International Conference on Acoustics, Speech, and Signal Processing **6**, 3821–3824 (2001)

[6] Anderson, B.D.O., Hitz, K.L., Diem, N.D.: Recursive Algorithm for Spectral Factorization. IEEE Transactions Circuits Syst. **CAS–21**, 742–750 (1974)

[7] Ben-Tal, A., El Ghaoui, L., Nemirovskii, A.: Robust Semidefinite Programming. In: Handbook of Semidefinite Programming, edited by Wolkowicz, H., Saigal, R. and Vandenberghe, L., Kluwer Academic Publishers, March 2000

[8] Ben-Tal, A., Nemirovskii, A.: Robust Convex Optimization. Math. Operations Res. 23, 769–805 (1998)

[9] Ben-Tal, A., Nemirovski, A.: Lectures on Modern Convex Optimization: Analysis, Algorithms and Engineering Applications. SIAM-MPS Series on Optimization, SIAM publications, Philadelphia, 2001

[10] Ben-Tal, A., Nemirovski, A.: Robust Optimization Methodology and Applications. Math. Programming, Ser. B **92**, 453–480 (2002)

[11] Ben-Tal, A., Nemirovskii, A.: Structural Design. In: Handbook of Semindefinite Programming, Theory, Algorithms, and Applications, edited by Wolkowicz, H., Saigal, R. and Vandenberghe, L., Kluwer Academic Press, 2000

[12] Bertsimas, D., Sethuraman, J.: Moment Problems and Semidefinite Optimization. In: Handbook of Semindefinite Programming, Theory, Algorithms, and Applications, edited by Wolkowicz, H., Saigal, R. and Vandenberghe, L., Kluwer Academic Press, 2000

[13] Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory. SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1994

[14] Burer, S., Monteiro, R.D.C., Zhang, Y.: Rank-Two Relaxation Heuristics For Max-Cut And Other Binary Quadratic Programs. SIAM J. Optim. **12**, 503–521 (2001)

[15] Cantoni, A., Lin, X.G., Teo, K.L.: A New Approach to the Optimization of Robust Antenna Array Processors. IEEE Transactions on Antennas and Propagation **41**(4), 403–411 (1993)

[16] Chang, L., Yeh, C.C.: Performance of DMI and Eigenspace-based Beamformers. IEEE Transactions Antennas Propagation **40**, 1336–1347 (1992)

[17] Chen, X., Parks, T.W.: Design of Optimal Minimum Phase FIR Filters by Direct Factorization. Signal Process. **10**, 369–383 (1986)

[18] Choi, C., Ye, Y.: Application of Semidefinite Programming to Circuit Partitioning. In: Pardalos, P. eds., Approximation and Complexity in Numerical Optimization, pp. 130–136, Kluwer Academics Publishers, 2000
[19] Cox, H., Zeskind, R.M., Owen, M.H.: Robust Adaptive Beamforming. IEEE Transactions on Acoustic Speech Signal Processing **35**, 1365–1376 (1987)
[20] Cui, S., Maricic, B., Luo, Z.-Q., Ding, Z.: Robust Blind Multiuser Detection Against Signature Waveform Mismatch. Submitted to IEEE Transactions on Wireless Communication
[21] Dam, H.H., Teo, K.L., Nordebo, S., Cantoni, A.: The Dual Parameterization Approach to Optimal Least Square FIR Filter Design Subject to Maximum Error Constraints. IEEE Transactions Signal Process. **48**(8), 2314–2320 (2000)
[22] Davidson, T.N.: Efficient Design of Waveforms for Robust Pulse Amplitude Modulation. IEEE Transactions on Signal Process. **4**(12), 3098–3111 (2001)
[23] Davidson, T.N., Luo, Z.-Q., Sturm, J.: Linear Matrix Inequality Formulation of Spectral Mask Constraints. IEEE Transactions on Signal Process. **50**, 2702–2715 (2002)
[24] Davidson, T.N., Luo, Z.-Q., Wong, K.M.: Spectrally-Efficient Orthogonal Pulse Shape Design via Semidefinite Programming. IEEE Transactions on Signal Process. **48**(5), 1433–1445 (2000)
[25] Ding, Y.-W., Davidson, T.N., Luo, Z.-Q., Wong, K.M.: Minimum BER Block Precoders for Zero-forcing Equalization. Accepted for publication in IEEE Transactions on Signal Process.
[26] Ding, Z., Luo, Z.-Q.: A Fast Linear Programming Algorithm for Blind Equalization. IEEE Transactions on Commun. **48**, 1432–1436 (2000)
[27] Dumitrescu, B., Tabus, I., Stoica, P.: On the Parameterization of Positive Real Sequences and MA Parameter Estimation. IEEE Transactions on Signal Process. **49**(11), 2630–2639 (2001)
[28] El Ghaoui, L., Lebret, H.: Robust Solutions To Least-Squares Problems With Uncertain Data. SIAM J. Matrix Anal. Appl. 1997
[29] El Ghaoui, L., Oustry, F., Lebret, H.: Robust Solutions to Uncertain Semidefinite Programs. SIAM J. Optim. **9**, 1 (1998)
[30] Fincke, U., Pohst, M.: Improved Methods for Calculating Vectors of Short Length in a Lattice. Including a Complexity Analysis. Math. Comput. **44**, 463–471 (1985)
[31] Fedorov. V., Lee, J.: Design of Experiments in Statistics. In: Handbook of Semindefinite Programming, Theory, Algorithms, and Applications, edited by Wolkowicz, H., Saigal, R. and Vandenberghe, L., Kluwer Academic Press, 2000
[32] Foschini, G.J.: Layered Space-Time Architecture for Wireless Communications in a Fading Environment When Using Multi-Element Antennas. Bell Labs. Tech. J. 41–59 (1996)
[33] Fu, M., Souza, C., Luo, Z.-Q.: Finite Horizon Robust Kalman Filter Design. IEEE Transactions on Signal Process. **49**, 2103–2112 (2001)
[34] Gohary, R.H., Davidson, T.N., Luo, Z.-Q.: On Efficient Design of Vector Broadcast Communications Systems, Manuscript, Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario L8S 4K1, 2002
[35] Goemans, M.X., Rendl, R.: Combinatorial Optimization. In: Handbook of Semindefinite Programming, Theory, Algorithms, and Applications, edited by Wolkowicz, H., Saigal, R. and Vandenberghe, L., Kluwer Academic Press, 2000
[36] Goemans, M.X., Williamson, D.P.: Improved Approximation Algorithms for Maximum Cut and Satisfiability Problem Using Semi-definite Programming. J. Association Comput. Machinery **42**, 1115–1145 (1995)
[37] Goldfarb, D., Iyengar, G.: Robust Convex Quadratic Constrained Programs. Manuscript, Department of Industrial Engineering and Operations Research, Columbia University, July, 2002
[38] Hassibi, B., Hochwald, B.M.: Cayley Differential Unitary Space-Time Codes. IEEE Transactions on Inf. Theor. **48**(6), 1485–1503 (2002)
[39] Hassibi, B., Vikalo, H.: On the Expected Complexity of Sphere Decoding. Proceedings of the Thirty-Fifth Asilomar Conference on Signals, Systems and Computers **2**, 1051–1055 (2001)
[40] Haykin, S.: Adaptive Filter Theory. Third Edition, Prentice Hall, New Jersey, 1998
[41] Herrmann, O., Schuessler, W.: Design of Nonrecursive Digital Filters with Minimum Phase. Electronics Lett. **6**(11), 329–330 (1970)
[42] Proposed EIA/TIA Interim Standard. Wideband Spread Spectrum Digital Cellular System Dual-Mode Mobile Station-Base Station Compatibility Standard. Technical Report TR45.5, QUALCOMM, Inc., San Diego, CA, USA, Apr. 1992
[43] Jindal, N., Vishwanath, S., Goldsmith, A.: On the Duality of Gaussian Multiple-Access and Broadcast Channels. Submitted to IEEE Transactions on Inf. Theor. August 2002
[44] Jindal, N., Goldsmith, A.: Capacity and Optimal Power Allocation for Fading Broadcast Channels with Minimum Rates. Submitted to IEEE Transactions on Inf. Theor. Sept. 2001

[45] Kosut, R., Chung, W., Johnson, C., Boyd, S.: On Achieving Reduced Error Propagation Sensitivity In DFE Design Via Convex Optimization. Proceedings of the 39th IEEE Conference on Decision and Control **5**, 4320–4323 (2000)
[46] Lebret, H., Boyd, S.: Antenna Array Pattern Synthesis via Convex Optimization. IEEE Transactions on Signal Process. **45**(3), 526–532 (1997)
[47] Li, L., Luo, Z.-Q., Davidson, T.N., Wong, K.M., Bossé, E.: Robust Filtering via Semidefinite Programming with Applications to Target Tracking. SIAM J. Optim. **12**, 740–755 (2002)
[48] Liu, J., Gershman, A., Luo, Z.-Q.: Adaptive Beamforming with Sidelobe Control: A Second Order Cone Programming Approach. Accepted for publication in IEEE Signal Processing Letters
[49] Liu, Y., Luo, Z.-Q.: Design of Robust IIR Magnitude Filters via Semidefinite Programming, to appear in Proceedings of ICASSP 2003, Hong Kong
[50] Lobo, M., Vandenberghe, L., Boyd, S., Lebret, H.: Applications of Second-Order Cone Programming. Linear Algebra and its Appl. **284**, 193–228 (1998)
[51] Lorenz, R., Boyd, S.: Robust Minimum Variance Beamforming. Submitted to IEEE Transactions on Signal Process.
[52] Luo, Z.-Q., Davidson, T.N., Giannakis, G.B., Wong, K.M.: Transceiver Optimization for Multiple Access through ISI Channels. Accepted for publication in IEEE Transactions on Signal Process.
[53] Luo, Z.-Q., Luo, X.-D., Kisialiou, M.: An Efficient Quasi-Maximum Likelihood Decoder for PSK Signals, to appear in Proceedings of ICASSP'2003, Hong Kong
[54] Luo, Z.-Q., Meng, M., Wong, K.M., Zhang, J.-K.: A Fractionally Spaced Blind equalizer Based on Linear Programming. IEEE Transactions on Signal Process. **50**, 1650–1660 (2002)
[55] Luo, Z.-Q., Sturm, J., Zhang, S.: Multivariate Nonnegative Quadratic Mappings, Manuscript, Department of Electrical and Computer Engineering, McMaster University, 2002
[56] Ma, W.-K., Davidson, T.N., Wong, K.M., Luo, Z.-Q., Ching, P.-C.: Quasi-maximum-likelihood Multiuser Detection Using Semi-definite Relaxation. IEEE Transactions on Signal Process. **50**, 912–922 (2002)
[57] Maricic, B., Luo, Z.-Q., Davidson, T.N.: Blind Constant Modulus Equalization via Convex Optimization, IEEE Transactions on Signal Process. **51**(3), 805–818 (2003)
[58] Mittelmann, H.D.: An Independent Benchmarking of SDP and SOCP Solvers, preprint, Department of Mathematics, Arizona State University, 2001
[59] Monzingo, R.A., Miller, T.W.: Introduction to Adaptive Arrays, Wiley, NY, 1980
[60] Nesterov, Y.: Squared Functional Systems And Optimization Problems. In: High Performance Optimization (H. Frenk et al., eds.), Kluwer Academic Publishers 405–440 (2000)
[61] Nesterov, Y., Nemirovskii, A.: Interior Point Polynomial Methods in Convex Programming SIAM Studies in Applied Mathematics 13, Philadelphia, PA, 1994
[62] Ohno, S., Giannakis, G., Luo, Z.-Q.: Multi-Carrier Multiple Access is Sum-Rate Optimal for Block Transmissions over Circulant ISI Channels. Submitted to IEEE Transactions on Information Theory
[63] Parks, T.W., Burrus, C.S.: Digital Filter Design. J. Wiley & Sons, New York, 1987
[64] Pesavento, M., Gershman, A., Luo, Z.-Q.: Robust Array Interpolation Using Second-Order Cone Programming. IEEE Signal Process. Lett. **9**, 8–11 (2002)
[65] Reed, I.S., Mallett, J.D., Brennan, L.E.: Rapid Convergence Rate in Adaptive Arrays. IEEE Transactions Aerospace and Electron. Syst. **10**, 853–863 (1974)
[66] Selesnick, I.W., Lang, M., Burrus, C.S.: Constrained Least Square Design for FIR Filters Without Specified Transition Bands. IEEE Transactions Signal Process. **44**(8), 1879–1892 (1996)
[67] Shahbazpanahi, S., Gershman, A., Luo, Z.-Q., Wong, K.M.: Robust Adaptive Beamforming for General-rank Signal Models, Submitted to IEEE Transactions on Signal Process.
[68] Steiglitz, K., Parks, T.W., Kaiser, J.F.: METEOR: A Constraint-based FIR Filter Design Program. IEEE Transactions Signal Process. **40**(8), 1901–1909 (1992)
[69] Stoica, P., McKelvey, T., Mari, J.: MA Estimation In Polynomial Time. IEEE Transactions on Signal Process. **48**(7), 1999–2012 (2000)
[70] Sturm, J.F.: Using SeDuMi 1.02, A Matlab Toolbox For Optimization Over Symmetric Cones. Optim. Meth. Software **11–12**, 625–653 (1999). See `http://fewcal.kub.nl/sturm/software/sedumi.html` for updates
[71] Telatar, I.E.: Capacity of Multi-Antenna Gaussian Channels. European Transactions on Telecommunication **10**, 585–595 (1999)
[72] Tan, P., Rasmussen, L.: The Application of Semidefinite Programming for Detection in CDMA. IEEE J. Selected Areas Commun. **19**, 1142–1149 (2001)
[73] Tan, P., Rasmussen, L., Lim, T.: Constrained Maximum Likelihood Detection in CDMA. IEEE Transactions Commun. **49**, 142–153 (2001)
[74] Toh, K.C., Todd, M.J., Tutuncu, R.H.: SDPT3 – a Matlab Software Packgage for Semidefinite Programming, version 3.0, Technical Report, National University of Singapore, June 2001

[75] Tuqan, J., Vaidyanathan, P.P.: Globally Optimal Two Channel FIR Orthonormal Filter Banks Adapted to the Input Signal Statistics. IEEE Transactions Signal Process. **48**, 2822–2833 (2000)
[76] Vandenberghe, L., Boyd, S.: Semidefinite Programming. SIAM Rev. **31**, 49–95 (1996)
[77] Vandenberghe, Boyd, S., El Gamal, A.: Optimizing Dominant Time Constant in RC Circuits IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems **17**, 110–125 (1998)
[78] Vishwanath, S., Jindal, N., Goldsmith, A.: Duality, Achievable Rates, and Sum-Rate Capacity of Gaussian MIMO Broadcast Channels. Submitted to IEEE Transactions on Information Theory, August, 2002
[79] Viterbo, E., Bouros, J.: A Universal Lattice Code Decoder For Fading Channels. IEEE Transactions Inf. Theor. **45**(5), 1639–1642 (1999)
[80] Vorobyov, S., Gershman, A., Luo, Z.-Q.: Robust Adaptive Beamforming Using Worst-Case Performance Optimization: A Solution to the Signal Mismatch Problem. IEEE Transactions on Signal Process. **51**(2), 33–323 (2003)
[81] Vorobyov, S., Gershman, A., Luo, Z.-Q., Ma, N.: Adaptive Beamforming with Joint Robustness Against Mismatched Steering Vector and Inteference Nonstationarity. Accepted for publication in IEEE Signal Processing Letters
[82] Wang, X., Lu, W.S., Antoniou, A.: Constrained Minimum-BER Multiuser Detection. IEEE Transactions on Signal Process. **48**(10), 2903–2909 (2000)
[83] Wang, X., Lu, W.S., Antoniou, A.: A Near-optimal Multiuser Detector for CDMA Channels Using Semidefinte Programming Relaxation. Proceedings of 2001 IEEE Intenational Sympsium on Circuits and Systems **4**, 298–301 (2001)
[84] Wang, Z., Giannakis, G.: Wireless Multicarrier Communications – Where Fourier Meets Shannon. IEEE Signal Process. Mag. 29–48 (2000)
[85] Wolkowicz, H., Saigal, R., Vandenberghe, L.: Handbook of Semidefinite Programming: Theory, Algorithms and Applications, Kluwer Academic Press, pp. 163–188, 1999
[86] Wu, S.-P., Boyd, S.: SDPSOL: A Parser/Solver For Semidefinite Programs with Matrix Structure. in: Recent Advances in LMI Methods for Control, L. El Ghaoui and S.-I. Niculescu, editors, SIAM, chapter 4, pp. 79–91, 2000
[87] Wu, S.-P., Boyd, S., Vandenberghe, L.: FIR Filter Design via Semidefinite Programming and Spectral Factorization. Proceedings of the 35th IEEE Conference on Decision and Control **1**, 271–276 (1996)
[88] Wu, S.-P., Boyd, S., Vandenberghe, L.: FIR Filter Design via Spectral Factorization and Convex Optimization. Applied and Computational Control, Signals, and Circuits, B. Datta, Ed., Vol. 1. Birkhauser, Boston, May 1999
[89] Xiao, L., Johansson, M., Hindi, H., Boyd, S., Goldsmith, A.: Joint Optimization of Communication Rates and Linear Systems, Proceedings 2001 Conference on Decision and Control, December 2001
[90] Yakubovich, V.A.: S-Procedure In Nonlinear Control Theory. Vestnik Leningrad University **4**, 73–93 (1977) English translation; original Russian publication in Vestnik Leningradskogo Universiteta, Seriya Matematika 62–77 (1971)
[91] Ye, Y.: A .699 Approximation Algorithm for Max-Bisection. Math. Programming **90**(1), 101–111 (2001)
[92] Ye, Y.: Interior-Point Algorithm: Theory and Analysis, John Wiley and Sons, New York, 1997
[93] Yu, W., Cioffi, J.: Sum Capacity of Gaussian Vector Broadcast Channel. Submitted IEEE Transactions Inf. Theor., Nov. 2001
[94] Yu, W., Rhee, W., Cioffi, J.: Multiuser Transmitter Optimization for Vector Multiple Access Channels, manuscript, STAR laboratory, Department of Electrical Engineering, Stanford University, CA 94305–9515, 2002