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APPLICATIONS OF DEGREE FOR S^1-EQUIVARIANT GRADIENT MAPS TO VARIATIONAL NONLINEAR PROBLEMS WITH S^1-SYMMETRIES

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1. Introduction

There are developed many topological methods which are powerful tools in the theory of critical points of functionals; see for example [2], [3], [5], [6], [9], [13]–[20], [22]–[25], [27]–[29], [32], [33], [43]–[45], [47], [48], [51], [59]. It happens quite often that functionals whose critical points are important in the theory of differential equations are invariant under an action of a compact Lie group G. Symmetric variational nonlinear problems have been considered by many mathematicians; see for instance [7], [8], [10]–[12], [15], [21], [29]–[31], [34], [37], [41], [46], [55]–[57], [62], [63].

In [55] the author has constructed a degree theory for S^1-equivariant, orthogonal maps (the known class of S^1-equivariant gradient maps is included in the class of S^1-equivariant orthogonal maps). Moreover, we have applied this degree to research of bifurcations of solutions of S^1-equivariant nonlinear variational problems. For other definitions of degree theories for equivariant gradient maps we refer the reader to [21] (in case of S^1-symmetries) and to [37] (in case of symmetries of any compact Lie group). Degree theories for (not gradient) equivariant maps have been constructed in [26], [39], [40].

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In [55] we have only shown how to compute the degree for S^1-equivariant orthogonal isomorphisms (see Lemma 4.1, Theorem 4.2 and Corollary 4.3 of [55]). Till now nothing has been known about this degree in the degenerate case, i.e. if the linearization of an S^1-equivariant orthogonal map at the origin in not an isomorphism. Therefore the aim of this article is to develop methods of computing the degree for S^1-equivariant orthogonal maps and apply these methods to finite-dimensional nonlinear variational problems with S^1-symmetries.

Let us recall an interesting result concerning the Brouwer topological degree. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^1-map such that $0 \in \mathbb{R}^n$ is an isolated point in $f^{-1}(0)$ and $\dim(\ker Df(0)) = 1$. It is known that the Brouwer topological degree $\deg(f, D^n_\alpha, 0)$ is equal to ± 1 or 0, where $D^n_\alpha = \{x \in \mathbb{R}^n : \|x\| < \alpha\}$ and α is such that $f^{-1}(0) \cap D^n_\alpha = \{0\}$. If $\dim(\ker Df(0)) > 1$ then, generally, we know nothing about $\deg(f, D^n_\alpha, 0)$. Let us restrict the class of maps considered. Assume additionally that $f = \nabla g$ is the gradient of a C^2-function $g : \mathbb{R}^n \to \mathbb{R}$ and that $\dim(\ker(Df(0))) = 2$. Then it is known that $\deg(f, D^n_\alpha, 0) = (-1)^{m^-(Df(0))} p$, where $m^-(Df(0))$ denotes the negative Morse index of $Df(0) = \nabla^2 g(0)$ and $p \le 1$ is the degree of a two-dimensional gradient map which is obtained by the Lyapunov–Schmidt reduction. If $\dim(\ker Df(0)) > 2$ then, generally, we know nothing about $\deg(f, D^n_\alpha, 0)$.

Some computations of the Brouwer degree have been done in [4], [38], [54] in the nonequivariant case, and for G-equivariant maps in [36], [42], [48]–[50], [60], [61].

From now on assume that V is a finite-dimensional orthogonal representation of the group S^1 and that $\nabla f : (V, 0) \to (V, 0)$ is the gradient of an S^1-equivariant C^2-function $f : V \to \mathbb{R}$ such that the origin $0 \in V$ is its isolated critical point. Fix $\alpha > 0$ such that $(\nabla f)^{-1}(0) \cap D_\alpha(V) = \{0\}$ and denote by $\mathrm{DEG}(\nabla f, D_\alpha(V)) \in \mathbb{Z} \oplus (\bigoplus_{i=1}^\infty \mathbb{Z})$ the degree for S^1-equivariant orthogonal maps, where $D_\alpha(V) = \{x \in V : \|x\| < \alpha\}$.

The above remarks on the Brouwer degree show that it would be interesting to answer the following question:

How to compute $\mathrm{DEG}(\nabla f, D_\alpha(V))$ when $\dim(\ker(\nabla^2(0))) > 1$?

Unfortunately, generally, we do not know how to do it. Nevertheless, it turned out that, under some additional assumptions, we are still able to compute some coordinates of this degree. Sometimes this allows us to distinguish two S^1-equivariant gradient maps $\nabla f_1, \nabla f_2 : (V, 0) \to (V, 0)$ by their degrees as S^1-equivariant orthogonal maps.

This paper is organized as follows.

In Section 2 we prove some new results concerning the degree theory for S^1-equivariant, orthogonal maps. Lemma 2.4 is a known fact in the case of gradient C^1-maps and continuous homotopies (see [52]) and gradient homotopies (see

[22]). In this paper we formulate it for the class of S^1-equivariant gradient C^1-maps and S^1-equivariant gradient homotopies. We will use Lemma 2.4 in order to simplify a map whose degree as S^1-equivariant orthogonal map will be computed. Theorem 2.11 gives a useful formula for the degree in the case of S^1-equivariant gradient product maps. Combining Lemma 2.4 with Theorem 2.11 we distinguish two S^1-equivariant gradient maps by our degree (see Theorems 2.13, 2.15, 2.17, 2.19, 2.21).

In Section 3 we apply the results of the previous section to S^1-equivariant variational bifurcation problems. Namely, in Theorems 3.1–3.5 we formulate and prove sufficient conditions for the existence of bifurcation points of solutions of nonlinear problems. Under additional assumptions we are also able to control the isotropy group of bifurcating sequences (see Remark 3.6). The last bifurcation theorem in this section is the finite-dimensional version of the global Rabinowitz bifurcation theorem for S^1-equivariant gradient maps (see Theorem 3.7).

Section 4 is devoted to the study of S^1-equivariant nonlinear variational problems. In this section we apply the results of Section 2 to asymptotically linear problems. In Theorems 4.1–4.3 we formulate sufficient conditions for the existence of nontrivial solutions of asymptotically linear S^1-equivariant variational problems. Under additional assumptions we distinguish solutions of asymptotically linear problems by their isotropy groups (see Remark 4.4).

In Section 5 we give some final remarks and comments concerning this article and the further applications of results of this paper.

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2. Degree for S^1-equivariant gradient maps

In [55] we constructed a degree theory for S^1-equivariant orthogonal maps and using this degree we proved some global and local results in bifurcation theory. The aim of this section is to prove some new theorems in the degree theory of S^1-equivariant orthogonal maps. We formulate and prove them only for gradient maps. Any gradient S^1-equivariant map is orthogonal. Therefore from now on the degree for S^1-equivariant orthogonal maps will be called the degree for S^1-equivariant gradient maps.

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the group of complex numbers of modulus one and let

$$Z_j = \{g \in S^1 : g = e^{i\theta} \text{ for } \theta = 2\pi k/j \text{ and } k = 0, 1, \ldots, j-1\}$$

be a cyclic subgroup. We denote by V a finite-dimensional real representation of S^1. If $v \in V$ then $S^1_v = \{g \in S^1 : g \cdot v = v\}$ is the *isotropy group* of $v \in V$. Moreover, if V is orthogonal, let $\langle \cdot\,, \cdot \rangle$ be an S^1-invariant inner product in V. If $V_0 \subset V$ is a subrepresentation of V then $(V_0)^\perp = \{v \in V : \forall v_0 \in V_0, \langle v, v_0 \rangle = 0\}$.

For $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$ define a map $\varrho^j : S^1 \to \mathrm{GL}(2, \mathbb{R})$ as follows:

$$\varrho^{\,j}(e^{i\theta}) = \begin{bmatrix} \cos j\theta & -\sin j\theta \\ \sin j\theta & \cos j\theta \end{bmatrix}, \qquad 0 \le \theta < 2\pi.$$

For $k, j \in \mathbb{N}$ we denote by $R[k, j]$ the direct sum of k copies of $(\mathbb{R}^2, \varrho^j)$; we also denote by $R[k, 0]$ the trivial k-dimensional representation of S^1. We say that two representations V and W are *equivalent* if there exists an equivariant linear isomorphism $T : V \to W$.

The following classical result gives a complete classification (up to equivalence) of finite-dimensional representations of the group S^1 (see [1]).

THEOREM 2.1 (Classification theorem). *If V is a representation of S^1 then there exist finite sequences $\{k_i\}$, $\{j_i\}$ satisfying*

$$(*) \qquad j_i \in \{0\} \cup \mathbb{N}, \quad k_i \in \mathbb{N}, \quad 1 \le i \le r, \ j_1 < \ldots < j_r,$$

such that V is equivalent to $\bigoplus_{i=1}^r R[k_i, j_i]$. Moreover, the equivalence class of V is uniquely determined by $\{j_i\}$, $\{k_i\}$ satisfying $(*)$.

Let $f : V \times \mathbb{R} \to V$ be an S^1-equivariant gradient map such that $f(0, \lambda) = 0$ for any $\lambda \in \mathbb{R}$.

DEFINITION 2.2. A solution $(v, \lambda) \in V \times \mathbb{R}$ of the equation $f(v, \lambda) = 0$ is said to be *nontrivial* if $v \neq 0$. The set of nontrivial solutions will be denoted by $\mathcal{N}(f)$. A point $\lambda_0 \in \mathbb{R}$ is said to be a *bifurcation point* of solutions of the equation $f(v, \lambda) = 0$ if $(0, \lambda_0) \in \mathrm{cl}(\mathcal{N}(f))$.

The next theorem is Krasnosel'skiĭ's local bifurcation theorem for S^1-equivariant gradient maps. In this theorem instead of the Brouwer topological degree we use the degree for S^1-equivariant gradient maps. For the proof in the infinite-dimensional case see [55].

THEOREM 2.3 (Krasnosel'skiĭ's bifurcation theorem). *Let $f : V \times \mathbb{R} \to V$ be an S^1-equivariant gradient map such that $f(0, \lambda) = 0$ for any $\lambda \in \mathbb{R}$. Fix $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(0, \lambda_1)$ and $(0, \lambda_2)$ are not bifurcation points of solutions of the equation $f(v, \lambda) = 0$. If for any sufficiently small $\alpha > 0$,*

$$\mathrm{DEG}(f(\,\cdot\,, \lambda_1), D_\alpha(V)) \neq \mathrm{DEG}(f(\,\cdot\,, \lambda_2), D_\alpha(V)),$$

then in $[\lambda_1, \lambda_2]$ there is a bifurcation point of solutions of $f(v, \lambda) = 0$.

The following lemma will prove extremely useful in the second part of this section.

LEMMA 2.4 (Splitting lemma at the origin). *Let V be a finite-dimensional, real, orthogonal representation of S^1. Let $f : (V, 0) \to (V, 0)$ be an S^1-equivariant gradient C^1-map such that*

1. *$0 \in V$ is isolated in $f^{-1}(0)$,*
2. *$Df(0)$ is degenerate.*

Then there is $\alpha > 0$ and an S^1-equivariant gradient map

$$f_0 : (D_\alpha(V), \partial D_\alpha(V)) \to (V, V - \{0\})$$

such that

1. *there is an S^1-equivariant gradient homotopy H_t such that*
   (a) *$H_t : (D_\alpha(V), \partial D_\alpha(V)) \to (V, V - \{0\})$,*
   (b) *$H_0 = f$ and $H_1 = f_0$,*
   (c) *$H_t^{-1}(0) \cap D_\alpha(V) = \{0\}$ for any $t \in [0, 1]$,*
2. *there is an S^1-equivariant, gradient map $\varphi : (D_\alpha(\ker(Df(0))), 0) \to (\ker(Df(0)), 0)$ such that if $A = Df(0)_{|\mathrm{im}(Df(0))}$ and $v = (v_1, v_2) \in V = \ker(Df(0)) \oplus \mathrm{im}(Df(0))$ then*

$$f_0(v) = f_0(v_1, v_2) = (\varphi(v_1), A(v_2)).$$

An example of an appropriate homotopy can be found in [22].

Let V and W be finite-dimensional, real, orthogonal representations of S^1. Let Ω (resp. $\mathcal{W}$) be an open, bounded and S^1-invariant subset of V (resp. W).

LEMMA 2.5. *Let $f = \nabla\psi : (\Omega, \partial\Omega) \to (V, V - \{0\})$ be an S^1-equivariant gradient map. Then there is an S^1-equivariant gradient homotopy $H_t : (\Omega, \partial\Omega) \to (V, V - \{0\})$ such that*

1. *$H_0(v) = f(v)$,*
2. *$H_t^{S^1} = f^{S^1}$ for all $t \in [0, 1]$,*
3. *$H_1(v_1, v_2) = (f^{S^1}(v_1), v_2)$ for $v = (v_1, v_2) \in V = V^{S^1} \oplus (V^{S^1})^\perp$ in a sufficiently small neighbourhood of $(f^{S^1})^{-1}(0) \cap \Omega$.*

The proof is left to the reader.

We will need the notion of S^1-regular value.

DEFINITION 2.6. Let $f : (\Omega, \partial\Omega \cup \Omega^{S^1}) \to (V, V - \{0\})$ be an S^1-equivariant gradient map. The point $0 \in V$ is said to be an S^1-*regular value* of f if

1. $f^{-1}(0)$ consists of a finite number of orbits in Ω,
2. if $v_0 \in f^{-1}(0)$ then $\dim(\ker(Df(v_0))) = 1$.

LEMMA 2.7. *Let $f:(\Omega,\partial\Omega\cup\Omega^{S^1})\to(V,V-\{0\})$ be an S^1-equivariant gradient map. Then f can be approximated in C^0-norm by S^1-equivariant gradient maps such that $0\in V$ is their S^1-regular value.*

REMARK 2.8. The above lemma was formulated in [21]. See also Theorem 7.8, p. 80 of [15].

The following two technical lemmas will be needed in the proof of Theorem 2.11. Denote by $D_\beta(\mathbb{R}^n,a)$ the open disc of radius β centered at $a\in\mathbb{R}^n$.

LEMMA 2.9. *Let $f:(\Omega,\partial\Omega\cup\Omega^{S^1})\to(V,V-\{0\})$ be an S^1-equivariant gradient map such that $0\in V$ is its S^1-regular value and that $f^{-1}(0)$ consists of one orbit of the action of S^1. Let $g:(\mathbb{R}^n,a)\to(\mathbb{R}^n,0)$ be such that $Dg(a)$ is an isomorphism. Choose $\beta>0$ such that $g^{-1}(0)\cap D_\beta(\mathbb{R}^n,a)=\{a\}$. Then*

$$\mathrm{DEG}((f,g),\Omega\times D_\beta(\mathbb{R}^n,a))=\mathrm{sign}(\det(Dg(a)))\cdot\mathrm{DEG}(f,\Omega).$$

LEMMA 2.10. *Let $\Omega\subset V$ and $\mathcal{W}\subset W$ be open, bounded, S^1-invariant subsets such that $\Omega^{S^1}=\emptyset$ and $\mathcal{W}^{S^1}=\emptyset$. Assume that $f:(\Omega,\partial\Omega)\to(V,V-\{0\})$ and $g:(\mathcal{W},\partial\mathcal{W})\to(W,W-\{0\})$ are S^1-equivariant gradient maps. Then*

$$\mathrm{DEG}((f,g),\Omega\times\mathcal{W})=\Theta\in\mathbb{Z}\oplus\Big(\bigoplus_{i=1}^{\infty}\mathbb{Z}\Big).$$

The Brouwer topological degree $\deg((f,g),\Omega_1\times\Omega_2,(0,0))$ of a continuous product map $(f,g):(\Omega_1\times\Omega_2,\partial(\Omega_1\times\Omega_2))\to(\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}^n\times\mathbb{R}^m-\{(0,0)\})$ is equal to $\deg(f,\Omega_1,0)\cdot\deg(g,\Omega_2,0)$. The following theorem is an analogue of this property for S^1-equivariant gradient maps. Let us equip $\mathbb{Z}\oplus\bigoplus_{i=1}^{\infty}\mathbb{Z}$ with a ring structure by defining multiplication in the following way:

$$\alpha\star\beta=(\alpha_0\cdot\beta_0,\alpha_0\cdot\beta_1+\beta_0\cdot\alpha_1,\alpha_0\cdot\beta_2+\beta_0\cdot\alpha_2,\alpha_0\cdot\beta_3+\beta_0\cdot\alpha_3,\ldots)$$

for $\alpha=(\alpha_0,\alpha_1,\alpha_2,\ldots)$, $\beta=(\beta_0,\beta_1,\beta_2,\ldots)\in\mathbb{Z}\oplus\bigoplus_{i=1}^{\infty}\mathbb{Z}$.

THEOREM 2.11 (Cartesian product formula). *Let $\Omega_1\subset V$ and $\Omega_2\subset W$ be open, bounded, S^1-invariant subsets of representations V and W, respectively. Let $f:(\Omega_1,\partial\Omega_1)\to(V,V-\{0\})$ and $g:(\Omega_2,\partial\Omega_2)\to(W,W-\{0\})$ be S^1-equivariant gradient maps. Then*

$$\mathrm{DEG}((f,g),\Omega_1\times\Omega_2)=\mathrm{DEG}(f,\Omega_1)\star\mathrm{DEG}(g,\Omega_2).$$

PROOF. Using Lemma 2.5 without loss of generality one can assume that there exist open S^1-invariant subsets $\mathcal{U}_1\subset\Omega_1$ and $\mathcal{U}_2\subset\Omega_2$ such that

1. $f^{-1}(0)\cap\mathcal{U}_1^{S^1}=f^{-1}(0)\cap\Omega_1^{S^1}$,
2. $f(v_1,v_2)=(f^{S^1}(v_1),v_2)$ for $(v_1,v_2)\in\mathcal{U}_1\subset\Omega_1\subset V=V^{S^1}\oplus(V^{S^1})^{\perp}$,
3. $g^{-1}(0)\cap\mathcal{U}_2^{S^1}=g^{-1}(0)\cap\Omega_2^{S^1}$,

4. $g(w_1, w_2) = (g^{S^1}(w_1), w_2)$ for $(w_1, w_2) \in \mathcal{U}_2 \subset \Omega_2 \subset W = W^{S^1} \oplus (W^{S^1})^\perp$.

Set $\mathcal{W}_1 = \Omega_1 - (\mathrm{cl}(\mathcal{U}_1) \cup V^{S^1})$ and $\mathcal{W}_2 = \Omega_2 - (\mathrm{cl}(\mathcal{U}_2) \cup W^{S^1})$ and notice that $f^{-1}(0) \subset \mathcal{U}_1 \cup \mathcal{W}_1$, $\mathcal{U}_1 \cap \mathcal{W}_1 = \emptyset$, $g^{-1}(0) \subset \mathcal{U}_2 \cup \mathcal{W}_2$, $\mathcal{U}_2 \cap \mathcal{W}_2 = \emptyset$. By the properties of degree (see Theorem 3.9 of [55]),

$$\begin{aligned}\mathrm{DEG}((f,g),\Omega_1 \times \Omega_2) = {} & \mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{U}_2) + \mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{W}_2) \\ & + \mathrm{DEG}((f,g),\mathcal{W}_1 \times \mathcal{U}_2) + \mathrm{DEG}((f,g),\mathcal{W}_1 \times \mathcal{W}_2).\end{aligned}$$

To compute $\mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{U}_2)$, notice that if

$$(v_1, v_2, w_1, w_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \subset V^{S^1} \oplus (V^{S^1})^\perp \oplus W^{S^1} \oplus (W^{S^1})^\perp,$$

then

$$(f,g)(v_1, v_2, w_1, w_2) = (f(v_1), v_2, g(w_1), w_2) = (f^{S^1}(v_1), v_2, g^{S^1}(w_1), w_2).$$

Thus by Theorem 3.9 of [55] we obtain

$$\begin{aligned}\mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{U}_2) &= \mathrm{DEG}((f,g)^{S^1}, (\mathcal{U}_1 \times \mathcal{U}_2)^{S^1}) \\ &= \mathrm{DEG}((f,g)^{S^1}, (\Omega_1 \times \Omega_2)^{S^1}).\end{aligned}$$

To compute $\mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{W}_2)$, first notice that if

$$(v_1, v_2, w_1, w_2) \in \mathcal{U}_1 \times \mathcal{W}_2 \subset V^{S^1} \oplus (V^{S^1})^\perp \oplus W^{S^1} \oplus (W^{S^1})^\perp,$$

then

$$(f,g)(v_1, v_2, w_1, w_2) = (f(v_1), v_2, g(w_1, w_2)).$$

In view of Lemma 2.7, without loss of generality, one can assume that

1. $0 \in V^{S^1}$ is a regular value of $f^{S^1} : (\Omega_1^{S^1}, \partial\Omega_1^{S^1}) \to (V^{S^1}, V^{S^1} - \{0\})$,
2. $0 \in W$ is an S^1-regular value of $g : (\mathcal{W}_2, \partial\mathcal{W}_2) \to (W, W - \{0\})$.

Then

1. $(f^{S^1})^{-1}(0) = \{a_1, \ldots, a_p\}$ and $Df^{S^1}(a_i)$ is a nonsingular matrix for any $i = 1, \ldots, p$,
2. $g^{-1}(0) = M_1 \cup \ldots \cup M_q$ and $\dim(\ker(Dg(a))) = 1$ for any $a \in M_j$, $j = 1, \ldots, q$.

Fix $\beta > 0$ such that $D_\beta(\mathbb{R}^n, a_i) \cap D_\beta(\mathbb{R}^n, a_j) = \emptyset$, for all $i, j \in \{1, \ldots, p\}$, $i \neq j$. Moreover, let $\mathcal{Q}_1, \ldots, \mathcal{Q}_q \subset \mathcal{W}_2$ be open, S^1-invariant, disjoint subsets such that $M_j \subset \mathcal{Q}_j$ for all $j \in \{1, \ldots, q\}$. Thus we obtain

$$\mathrm{DEG}((f,g),\mathcal{U}_1 \times \mathcal{W}_2) = \sum_{j=1}^{q} \sum_{i=1}^{p} \mathrm{DEG}((f^{S^1}, g), D_\beta(V^{S^1}, a_i) \times \mathcal{Q}_j).$$

By Lemma 2.9 and Theorem 3.9 of [55] we obtain

$$\begin{aligned}
&\sum_{j=1}^{q}\sum_{i=1}^{p}\mathrm{DEG}((f^{S^1},g),D_\beta(V^{S^1},a_i)\times\mathcal{Q}_j)\\
&\qquad=\sum_{j=1}^{q}\sum_{i=1}^{p}\mathrm{sign}(\det(Df^{S^1}(a_i)))\cdot\mathrm{DEG}(g,\mathcal{Q}_j)\\
&\qquad=\sum_{j=1}^{q}\mathrm{DEG}(g,\mathcal{Q}_j)\cdot\sum_{i=1}^{p}\mathrm{sign}(\det(Df^{S^1}(a_i)))\\
&\qquad=\sum_{j=1}^{q}\mathrm{DEG}(g,\mathcal{Q}_j)\cdot\deg(f^{S^1},\Omega_1^{S^1},0)\\
&\qquad=\deg(f^{S^1},\Omega_1^{S^1},0)\cdot\sum_{j=1}^{q}\mathrm{DEG}(g,\mathcal{Q}_j)\\
&\qquad=\deg(f^{S^1},\Omega_1^{S^1},0)\cdot\mathrm{DEG}(g,\mathcal{W}_2)=\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot\mathrm{DEG}(g,\mathcal{W}_2),
\end{aligned}$$

where $\deg(f^{S^1},\Omega_1^{S^1},0)$ denotes the Brouwer topological degree. Repeating the above reasoning we show that

$$\begin{aligned}
\mathrm{DEG}((f,g),\mathcal{W}_1\times\mathcal{U}_2)&=\deg(g^{S^1},\Omega_2^{S^1})\cdot\mathrm{DEG}(f,\mathcal{W}_1)\\
&=\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot\mathrm{DEG}(f,\mathcal{W}_1).
\end{aligned}$$

Finally, from Lemma 2.10 it follows that $\mathrm{DEG}((f,g),\mathcal{W}_1\times\mathcal{W}_2)=0$.

The above computations and the properties of degree now yield

$$\begin{aligned}
\mathrm{DEG}((f,g),\Omega_1\times\Omega_2)&=\mathrm{DEG}((f,g)^{S^1},(\Omega_1\times\Omega_2)^{S^1})\\
&\quad+\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot\mathrm{DEG}(g,\mathcal{W}_2)+\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot\mathrm{DEG}(f,\mathcal{W}_1)\\
&=\mathrm{DEG}((f,g)^{S^1},(\Omega_1\times\Omega_2)^{S^1})\\
&\quad+\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot(\mathrm{DEG}(g,\Omega_2)-\mathrm{DEG}(g^{S^1},\Omega_2^{S^1}))\\
&\quad+\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot(\mathrm{DEG}(f,\Omega_1)-\mathrm{DEG}(f^{S^1},\Omega_1^{S^1}))\\
&=(\mathrm{DEG}((f,g)^{S^1},(\Omega_1\times\Omega_2)^{S^1})-\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot\mathrm{DEG}(g^{S^1},\Omega_2^{S^1}))\\
&\quad+\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot\mathrm{DEG}(g,\Omega_2)+\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot\mathrm{DEG}(f,\Omega_1)\\
&\quad-\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot\mathrm{DEG}(f^{S^1},\Omega_1^{S^1})\\
&=\mathrm{DEG}_{S^1}(f,\Omega_1)\cdot\mathrm{DEG}(g,\Omega_2)+\mathrm{DEG}_{S^1}(g,\Omega_2)\cdot\mathrm{DEG}(f,\Omega_1)\\
&\quad-\mathrm{DEG}((f,g)^{S^1},(\Omega_1\times\Omega_2)^{S^1}).
\end{aligned}$$

□

The Brouwer topological degree $\deg(f,(a,b),0)$ of a continuous map $f:(a,b)\to\mathbb{R}$ is equal to ±1 or 0. The following lemma gives an analogous property for the degree of S^1-equivariant gradient maps.

Lemma 2.12. *Let V be a two-dimensional nontrivial representation of S^1 such that $V \approx R[1,j_0]$. Assume that $f:(D_\alpha(V),\partial D_\alpha(V)) \to (V, V-\{0\})$ is an S^1-equivariant gradient map. Then*

$$\mathrm{DEG}_Q(f,D_\alpha(V)) = \begin{cases} 1, & Q = S^1,\\ 1 \text{ or } 0, & Q = Z_{j_0},\\ 0, & Q \neq Z_{j_0},\, S^1.\end{cases}$$

We omit an easy proof.

The point of the following theorem is that it allows one to distinguish two S^1-equivariant gradient maps with isolated zeros at the origin by their degree. The distinct coordinates of degrees correspond to the isotropy groups of points which do not occur in the kernels of the linearizations of the maps considered.

If A is a symmetric matrix then we denote by $\sigma_-(A)$ the set of negative eigenvalues of A and by $\mu(\lambda)$ the multiplicity of $\lambda \in \sigma_-(A)$.

Theorem 2.13. *Let $f_1, f_2 : (V,0) \to (V,0)$ be S^1-equivariant gradient maps such that*

1. *$0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$,*
2. *$\ker(Df_1(0)) \cap V^{S^1} = \{0\}$ and $\ker(Df_2(0)) \cap V^{S^1} = \{0\}$,*
3. *if $V \approx \bigoplus_{i=1}^n R[k_i,j_i]$ and $\ker(Df_1(0)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1]$, $\ker(Df_2(0)) \approx \bigoplus_{i=1}^{n_2} R[k_i^2, j_i^2]$ then there is $j_{i_0} \in \{j_1,\dots,j_n\} - \{0\}$ such that*
   (a) *$j_{i_0} \neq \gcd\{a_1,\dots,a_k\}$ for any $a_1,\dots,a_k \in \{j_1^1,\dots,j_{n_1}^1\}$,*
   (b) *$j_{i_0} \neq \gcd\{a_1,\dots,a_k\}$ for any $a_1,\dots,a_k \in \{j_1^2,\dots,j_{n_2}^2\}$,*
   (c)

$$\sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \frac{\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))}))}{\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda).$$

Then there is $\alpha > 0$ such that $\mathrm{DEG}(f_1,D_\alpha(V)) \neq \mathrm{DEG}(f_2,D_\alpha(V))$. More precisely,

$$\mathrm{DEG}_{Z_{j_{i_0}}}(f_1,D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2,D_\alpha(V)).$$

Proof. Fix $i \in \{1,2\}$. Set $A_i = Df_i(0)_{|\mathrm{im}(Df_i(0))}$ and take an S^1-equivariant C^1-map $\varphi_i : (D_\alpha(\ker(Df_i(0))),0) \to (\ker(Df_i(0))),0)$ as in Lemma 2.4. Then for any sufficiently small $\alpha > 0$ we have

$$\mathrm{DEG}(f_i,D_\alpha(V)) = \mathrm{DEG}(f_0^i,D_\alpha(V)),$$

where $f_0^i : D_\alpha(\ker(Df_i(0)) \oplus \mathrm{im}(Df_i(0))) \to \ker(Df_i(0)) \oplus \mathrm{im}(Df_i(0))$ is given by the formula

$$f_0^i(v_1,v_2) = (\varphi_i(v_1), A_i(v_2)).$$

Notice that by assumption 2 and from the definition of degree (see [55]) it follows that

$$\mathrm{DEG}_{S^1}(\varphi_i, D_\alpha(\ker(Df_i(0)))) = 1.$$

Moreover, from assumptions 3(a) and 3(b) it follows that the isotropy group of any point in $\ker(Df_1(0))$ and $\ker(Df_2(0))$ is different from $Z_{j_{i_0}}$. Therefore from the definition of degree it follows that $\mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_i, D_\alpha(\ker(Df_i(0)))) = 0$, $i = 1, 2$.

Applying now Theorem 2.11 we obtain

$$\begin{aligned}
&\mathrm{DEG}_{Z_{j_{i_0}}}(f_0^i, D_\alpha(V)) \\
&\quad= \mathrm{DEG}_{S^1}(\varphi_i, D_\alpha(\ker(Df_i(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_i, D_\alpha(\mathrm{im}(Df_i(0)))) \\
&\qquad+ \mathrm{DEG}_{S^1}(A_i, D_\alpha(\mathrm{im}(Df_i(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_i, D_\alpha(\ker(Df_i(0)))) \\
&\quad= \mathrm{DEG}_{Z_{j_{i_0}}}(A_i, D_\alpha(\mathrm{im}(Df_i(0)))) \\
&\qquad+ \mathrm{sign}(\det(A_i)) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_i, D_\alpha(\ker(Df_i(0)))) \\
&\quad= \mathrm{DEG}_{Z_{j_{i_0}}}(A_i, D_\alpha(\mathrm{im}(Df_i(0)))) \\
&\quad= \frac{1}{2}\,\mathrm{sign}(\det(Df_i(0)_{|\mathrm{im}(Df_i(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_i(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda).
\end{aligned}$$

The last equality is a consequence of Corollary 4.3 of [55]. In other words, we have just computed the coordinate of $\mathrm{DEG}(f_i, D_\alpha(V))$ which corresponds to the isotropy group $Z_{j_{i_0}}$. Invoking assumption 3(c) completes our proof. □

In order to illustrate the above theorem we consider the following example.

EXAMPLE 2.14. Let $f_1, f_2 : V = R[4,0] \oplus R[4,1] \oplus R[4,2] \to V$ be S^1-equivariant gradient maps such that $0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$, and

$$Df_1(0) = \begin{bmatrix} -\mathrm{Id}_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & +\mathrm{Id}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\mathrm{Id}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \cdot \mathrm{Id}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_4 \end{bmatrix},$$

$$Df_2(0) = \begin{bmatrix} -\mathrm{Id}_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\mathrm{Id}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & +\mathrm{Id}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_4 \end{bmatrix}.$$

It is evident that $\ker(Df_1(0)) = R[1,2]$, $\ker(Df_2(0)) = \{0\}$, and that for $j_{i_0} = 1$, we have $j_{i_0} \neq \gcd\{2\} = 2$ and

$$6 = \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))}))}{\operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) = \frac{1}{1} \cdot 8.$$

All the assumptions of Theorem 2.13 are fulfilled, and therefore

$$3 = \mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) = 4. \qquad \square$$

The following theorem is analogous to Theorem 2.13. However, the set $\ker(Df_2(0))^{S^1}$ is a nonzero linear subspace of V^{S^1}.

THEOREM 2.15. *Let $f_1, f_2 : (V,0) \to (V,0)$ be S^1-equivariant gradient maps such that*

1. $0 \in V$ *is isolated in* $f_1^{-1}(0)$ *and in* $f_2^{-1}(0)$,
2. $\ker(Df_1(0)) \cap V^{S^1} = \{0\}$ *and* $\dim(\ker(Df_2(0)) \cap V^{S^1}) = k$,
3. *if* $V \approx \bigoplus_{i=1}^n R[k_i, j_i]$ *and* $\ker(Df_1(0)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1]$, $\ker(Df_2(0)) \approx R[k,0] \oplus \bigoplus_{i=1}^{n_2} R[k_i^2, j_i^2]$ *then there is* $j_{i_0} \in \{j_1, \ldots, j_n\} - \{0\}$ *such that*
   (a) $j_{i_0} \neq \gcd\{a_1, \ldots, a_k\}$ *for any* $a_1, \ldots, a_k \in \{j_1^1, \ldots, j_{n_1}^1\}$,
   (b) $j_{i_0} \neq \gcd\{a_1, \ldots, a_k\}$ *for any* $a_1, \ldots, a_k \in \{j_1^2, \ldots, j_{n_2}^2\}$,
   (c) *if* $k = 1$, *then*
   $$\sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),$$
   for $\gamma = 0, 1$,
   (d) *if* $k = 2$ *then*
   $$\sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \frac{\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))}))}{\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda)$$
   for $\gamma \leq 1$,
   (e) *if* $k > 2$, *then*
   $$\sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda)$$
   for $\gamma = 0, 1, 2, \ldots$

Then there is $\alpha > 0$ such that

$$\mathrm{DEG}(f_1, D_\alpha(V)) \neq \mathrm{DEG}(f_2, D_\alpha(V)).$$

More precisely, $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$.

PROOF. Repeating the same reasoning as in the proof of Theorem 2.13 we obtain

$$\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) = \frac{1}{2}\,\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))})) \cdot \sum_{\lambda\in\sigma_-(Df_1(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda)$$

and

$$\begin{aligned}
&\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))\\
&\quad= \mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0))))\\
&\qquad\cdot \mathrm{DEG}_{Z_{j_{i_0}}}(Df_2(0)_{|R[k_{i_0},j_{i_0}}, D_\alpha(\mathrm{im}(Df_2(0))))\\
&\quad= \mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0))))\\
&\qquad\cdot \frac{1}{2}\,\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \cdot \sum_{\lambda\in\sigma_-(Df_2(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda).
\end{aligned}$$

From assumption 2 it follows that

- if $k = 1$, then $\mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0)))) = \pm 1$ or 0,
- if $k = 2$, then $\mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \leq 1$,
- if $k > 2$, then $\mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \in \mathbb{Z}$.

The rest of this proof is a consequence of assumption 3(c)–(e). $\square$

In order to illustrate the above theorem we consider the following example.

EXAMPLE 2.16. Let $f_1, f_2 : V = R[k,0] \oplus R[3,2] \oplus R[1,4] \to V$ be S^1-equivariant gradient maps such that $0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$, and

$$Df_1(0) = \begin{bmatrix} -\mathrm{Id}_k & 0 & 0 & 0 & 0\\ 0 & -\mathrm{Id}_2 & 0 & 0 & 0\\ 0 & 0 & -\mathrm{Id}_2 & 0 & 0\\ 0 & 0 & 0 & -\mathrm{Id}_2 & 0\\ 0 & 0 & 0 & 0 & 0\cdot\mathrm{Id}_2 \end{bmatrix},$$

$$Df_2(0) = \begin{bmatrix} 0\cdot\mathrm{Id}_k & 0 & 0 & 0 & 0\\ 0 & -\mathrm{Id}_2 & 0 & 0 & 0\\ 0 & 0 & -\mathrm{Id}_2 & 0 & 0\\ 0 & 0 & 0 & +\mathrm{Id}_2 & 0\\ 0 & 0 & 0 & 0 & 0\cdot\mathrm{Id}_2 \end{bmatrix}.$$

It is evident that $\ker(Df_1(0)) = R[1,4]$, $\ker(Df_2(0)) = R[k,0] \oplus R[1,4]$ and that for $j_{i_0} = 2$, we have $j_{i_0} \neq \gcd\{4\} = 4$ and

$$6 = \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) = \gamma \cdot 4.$$

All the assumptions of Theorem 2.15 are fulfilled, and therefore

$$\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)). \qquad \square$$

In the following theorem we distinguish coordinates of degrees which correspond to the isotropy group appearing in the kernel of the linearization of one of the maps considered.

Theorem 2.17. *Let $f_1, f_2 : (V, 0) \to (V, 0)$ be S^1-equivariant gradient maps such that*

1. $0 \in V$ *is isolated in* $f_1^{-1}(0)$ *and in* $f_2^{-1}(0)$,
2. $\ker(Df_1(0)) \cap V^{S^1} = \{0\}$ *and* $\ker(Df_2(0)) \cap V^{S^1} = \{0\}$,
3. $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$ *and* $\ker(Df_1(0)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1]$, $\ker(Df_2(0)) \approx R[1, j_{i_0}]$,
4. $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^1, \dots, j_{n_1}^1\}$,
5.

$$\sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))}))}{\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma$$

for $\gamma = 2, 0$.

Then there is $\alpha > 0$ such that

$$\mathrm{DEG}(f_1, D_\alpha(V)) \neq \mathrm{DEG}(f_2, D_\alpha(V)).$$

More precisely, $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$.

Proof. Repeating the proof of Theorem 2.13, for a sufficiently small positive α, we obtain

$$
\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}((\varphi_1, A_1), D_\alpha(V)) \\
&= \mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_1, D_\alpha(\mathrm{im}(Df_1(0)))) \\
&\quad + \mathrm{DEG}_{S^1}(A_1, D_\alpha(\mathrm{im}(Df_1(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \\
&= \mathrm{DEG}_{Z_{j_{i_0}}}(A_1, D_\alpha(\mathrm{im}(Df_1(0)))) \\
&\quad + \mathrm{sign}(\det(A_1)) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \\
&= \mathrm{DEG}_{Z_{j_{i_0}}}(A_1, D_\alpha(\mathrm{im}(Df_1(0)))) \\
&= \frac{1}{2}\,\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),
\end{aligned}
$$

where $A_1 = Df_1(0)_{|\mathrm{im}(Df_1(0))}$ and φ_1 is as in Lemma 2.4. Moreover,

$$
\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}((\varphi_2, A_2), D_\alpha(V)) \\
&= \mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\mathrm{im}(Df_2(0)))) \\
&\quad + \mathrm{DEG}_{S^1}(A_2, D_\alpha(\mathrm{im}(Df_2(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \\
&= \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\mathrm{im}(Df_2(0)))) \\
&\quad + \mathrm{sign}(\det(A_2)) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0)))),
\end{aligned}
$$

where $A_2 = Df_2(0)_{|\mathrm{im}(Df_2(0))}$ and φ_2 is as in Lemma 2.4. From assumption 3 it follows that $\dim(\ker(Df_2(0))) = 2$, therefore by Lemma 2.12 we know that $\mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0))))$ is equal to 1 or 0. Consequently, we obtain

$$
\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\mathrm{im}(Df_2(0)))) + \delta \cdot \mathrm{sign}(\det(A_2)) \\
&= \frac{1}{2}\Bigg(\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda)\Bigg) \\
&\quad + \delta \cdot \mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \\
&= \frac{1}{2}\Bigg(\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \\
&\quad + 2 \cdot \delta \cdot \mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))}))\Bigg) \\
&= \mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \cdot \frac{1}{2}\Bigg(\sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + 2 \cdot \delta\Bigg)
\end{aligned}
$$

for $\delta = 1$ or 0.

Combining formulas for $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V))$ and $\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$ with assumption 5 completes the proof. $\square$

In order to illustrate the above theorem we consider the following example.

EXAMPLE 2.18. Let $f_1, f_2 : V = R[4,1] \oplus R[4,2] \to V$ be S^1-equivariant gradient maps such that $0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$, and

$$Df_1(0) = \begin{bmatrix} -\mathrm{Id}_2 & 0 & 0 & 0 \\ 0 & +\mathrm{Id}_6 & 0 & 0 \\ 0 & 0 & 0\cdot\mathrm{Id}_6 & 0 \\ 0 & 0 & 0 & +\mathrm{Id}_2 \end{bmatrix},$$

$$Df_2(0) = \begin{bmatrix} 0\cdot\mathrm{Id}_2 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_6 & 0 & 0 \\ 0 & 0 & +\mathrm{Id}_6 & 0 \\ 0 & 0 & 0 & +\mathrm{Id}_2 \end{bmatrix}.$$

Notice that $\ker(Df_1(0)) = R[3,2]$, $\ker(Df_2(0)) = R[1,1]$, and for $j_{i_0} = 1$ we have $j_{i_0} \neq \gcd\{2\} = 2$, and

$$2 = \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))}))}{\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma = \frac{1}{1} \cdot 6 + \gamma,$$

where $\gamma = 2, 0$. All the assumptions of Theorem 2.17 are fulfilled, and therefore

$$\mathrm{DEG}_{\{e\}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{\{e\}}(f_2, D_\alpha(V)).$$

The following theorem is similar to Theorem 2.17. The only difference is that we assume that $(\ker(Df_1(0)))^{S^1}$ is a nonzero linear subspace of V^{S^1}.

THEOREM 2.19. *Let $f_1, f_2 : (V,0) \to (V,0)$ be S^1-equivariant gradient maps such that*

1. *$0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$,*
2. *$\dim(\ker(Df_1(0)) \cap V^{S^1}) = k$ and $\ker(Df_2(0)) \cap V^{S^1} = \{0\}$,*
3. *$\ker(Df_2(0)) \approx R[1, j_{i_0}]$ and $\ker(Df_1(0)) \approx R[k,0] \oplus \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1]$, $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$,*
4. *$j_{i_0} \neq \gcd\{a_1, \ldots, a_k\}$ for any $a_1, \ldots, a_k \in \{j_1^1, \ldots, j_{n_1}^1\}$,*
5. *either*

$$\gamma_1 \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma_2,$$

where

if $k = 1$, then $\gamma_1 = \pm 1, 0$ and $\gamma_2 = 2, 0$,

if $k > 2$, then $\gamma_1 \in \mathbb{Z}$ and $\gamma_2 = 2, 0$, *or*

$$\gamma_1 \cdot \frac{\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))}))}{\operatorname{sign}(\det Df_2(0)_{|\operatorname{im}(Df_2(0))}))} \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma_2,$$

where $k = 2$, $\gamma_1 \leq 1$ *and* $\gamma_2 = 2, 0$.

Then there is $\alpha > 0$ *such that*

$$\mathrm{DEG}(f_1, D_\alpha(V)) \neq \mathrm{DEG}(f_2, D_\alpha(V)).$$

More precisely, $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$.

Proof. Repeating the proof of Theorem 2.17, for a sufficiently small positive α, we obtain

$$\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}((\varphi_1, A_1), D_\alpha(V)) \\
&= \mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_1, D_\alpha(\operatorname{im}(Df_1(0)))) \\
&\quad + \mathrm{DEG}_{S^1}(A_1, D_\alpha(\operatorname{im}(Df_1(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \\
&= \mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_1, D_\alpha(\operatorname{im}(Df_1(0)))) \\
&= \mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \\
&\quad \cdot \frac{1}{2} \operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),
\end{aligned}$$

where $A_1 = Df_1(0)_{|\operatorname{im}(Df_1(0))}$ and φ_1 is as in Lemma 2.4. Moreover,

$$\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}((\varphi_2, A_2), D_\alpha(V)) \\
&= \mathrm{DEG}_{S^1}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\operatorname{im}(Df_2(0)))) \\
&\quad + \mathrm{DEG}_{S^1}(A_2, D_\alpha(\operatorname{im}(Df_2(0)))) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0)))) \\
&= \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\operatorname{im}(Df_2(0)))) \\
&\quad + \operatorname{sign}(\det(A_2)) \cdot \mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0)))),
\end{aligned}$$

where $A_2 = Df_2(0)_{|\operatorname{im}(Df_2(0))}$ and φ_2 is as in Lemma 2.4. From assumption 3 it follows that $\dim(\ker(Df_2(0))) = 2$, therefore by Lemma 2.12 we know that $\mathrm{DEG}_{Z_{j_{i_0}}}(\varphi_2, D_\alpha(\ker(Df_2(0))))$ is equal to 1 or 0. Additionally, from assumption 2 it follows that

- if $k = 1$, then $\mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) = \pm 1$ or 0,
- if $k = 2$, then $\mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \leq 1$,
- if $k > 2$, then $\mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \in \mathbb{Z}$.

Consequently, we obtain

$$\begin{aligned}
&\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \\
&\quad = \mathrm{DEG}_{S^1}(\varphi_1, D_\alpha(\ker(Df_1(0)))) \\
&\qquad \frac{1}{2}\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \\
&\quad = \delta_1 \cdot \frac{1}{2}\operatorname{sign}(\det(Df_1(0)_{|\operatorname{im}(Df_1(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) &= \mathrm{DEG}_{Z_{j_{i_0}}}(A_2, D_\alpha(\operatorname{im}(Df_2(0)))) + \delta_2 \cdot \operatorname{sign}(\det(A_2)) \\
&= \frac{1}{2}\operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \\
&\quad + \delta_2 \cdot \operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))})) \\
&= \frac{1}{2}\Bigg(\operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))})) \cdot \sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \\
&\quad + 2 \cdot \delta_2 \cdot \operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))}))\Bigg) \\
&= \operatorname{sign}(\det(Df_2(0)_{|\operatorname{im}(Df_2(0))})) \\
&\quad \cdot \frac{1}{2}\Bigg(\sum_{\lambda \in \sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + 2 \cdot \delta_2\Bigg),
\end{aligned}$$

where

- if $k = 1$, then $\delta_1 = \pm 1$ or 0 and $\delta_2 = 1$ or 0,
- if $k = 2$, then then $\delta_1 \leq 1$ and $\delta_2 = 1$ or 0,
- if $k > 2$, then $\delta_1 \in Z$ and $\delta_2 = 1$ or 0.

Combining formulas for $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V))$ and $\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$ with assumption 5 completes the proof. □

Example 2.20. Let $f_1, f_2 : V = R[1,0] \oplus R[4,1] \oplus R[4,2] \to V$ be S^1-equivariant gradient maps such that $0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$, and

$$Df_1(0) = \begin{bmatrix} 0\cdot \mathrm{Id}_1 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_2 & 0 & 0 & 0 \\ 0 & 0 & +\mathrm{Id}_6 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot\mathrm{Id}_6 & 0 \\ 0 & 0 & 0 & 0 & +\mathrm{Id}_2 \end{bmatrix},$$

$$Df_2(0) = \begin{bmatrix} +\mathrm{Id}_1 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot\mathrm{Id}_2 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_6 & 0 & 0 \\ 0 & 0 & 0 & +\mathrm{Id}_6 & 0 \\ 0 & 0 & 0 & 0 & +\mathrm{Id}_2 \end{bmatrix}.$$

Notice that $\ker(Df_1(0)) = R[1,0]\oplus R[3,2]$, $\ker(Df_2(0)) = R[1,1]$, and for $j_{i_0} = 1$ we have $j_{i_0} \neq \gcd\{2\} = 2$, and

$$\begin{aligned} \gamma_1 \cdot 2 &= \gamma_1 \cdot \sum_{\lambda\in\sigma_-(Df_1(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \\ &\neq \sum_{\lambda\in\sigma_-(Df_2(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma_2 = 6+\gamma_2, \end{aligned}$$

where $\gamma_1 = \pm 1, 0$, $\gamma_2 = 2, 0$. All the assumptions of Theorem 2.19 are fulfilled, and therefore

$$\mathrm{DEG}_{\{e\}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{\{e\}}(f_2, D_\alpha(V)).$$

In the following theorem points with isotropy group $Z_{j_{i_0}}$ occur in the linear spaces $\ker(Df_1(0))$ and $\ker(Df_2(0))$. Nevertheless, we can distinguish coordinates of degree which correspond to this isotropy group.

THEOREM 2.21. *Let $f_1, f_2 : (V,0) \to (V,0)$ be S^1-equivariant gradient maps such that*

1. *$0\in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$,*
2. $\ker(Df_1(0))\cap V^{S^1} = \{0\}$ *and* $\ker(Df_2(0))\cap V^{S^1} = \{0\}$,
3. $V \approx \bigoplus_{i=1}^n R[k_i,j_i]$ *and* $\ker(Df_1(0)) \approx R[1,j_{i_0}]$, $\ker(Df_2(0)) \approx R[1,j_{i_0}]$,
4. $$\sum_{\lambda\in\sigma_-(Df_1(0)_{|R[k_{i_0},j_{i_0}})} \mu(\lambda) \neq \frac{\mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))}))}{\mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))}))} \cdot \sum_{\lambda\in\sigma_-(Df_2(0)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma,$$

   where $\gamma = \pm 2, 0$.

Then there is $\alpha > 0$ such that

$$\mathrm{DEG}(f_1, D_\alpha(V)) \neq \mathrm{DEG}(f_2, D_\alpha(V)).$$

More precisely, $\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V))$.

PROOF. As in the proof of Theorem 2.17 we obtain the formulas

$$\begin{aligned}\mathrm{DEG}_{Z_{j_{i_0}}}(f_1, D_\alpha(V)) = {} & \mathrm{sign}(\det(Df_1(0)_{|\mathrm{im}(Df_1(0))})) \\ & \cdot \frac{1}{2}\bigg(\sum_{\lambda\in\sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + 2\cdot\delta_1\bigg)\end{aligned}$$

and

$$\begin{aligned}\mathrm{DEG}_{Z_{j_{i_0}}}(f_2, D_\alpha(V)) = {} & \mathrm{sign}(\det(Df_2(0)_{|\mathrm{im}(Df_2(0))})) \\ & \cdot \frac{1}{2}\bigg(\sum_{\lambda\in\sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + 2\cdot\delta_2\bigg),\end{aligned}$$

where $\delta_1, \delta_2 \in \{0,1\}$. Combining these with assumption 4, we complete the proof. $\square$

EXAMPLE 2.22. Let $f_1, f_2 : V = R[6,1] \to V$ be S^1-equivariant gradient maps such that $0 \in V$ is isolated in $f_1^{-1}(0)$ and in $f_2^{-1}(0)$, and

$$Df_1(0) = \begin{bmatrix} 0\cdot\mathrm{Id}_2 & 0 \\ 0 & +\mathrm{Id}_{10}\end{bmatrix}, \qquad Df_2(0) = \begin{bmatrix} 0\cdot\mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_{10}\end{bmatrix}.$$

Notice that $\ker(Df_1(0)) = R[1,1]$, $\ker(Df_2(0)) = R[1,1]$, and for $j_{i_0} = 1$ we have

$$0 = \sum_{\lambda\in\sigma_-(Df_1(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda\in\sigma_-(Df_2(0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma = 10 + \gamma,$$

where $\gamma = \pm 2, 0$. All the assumptions of Theorem 2.21 are fulfilled, therefore

$$\mathrm{DEG}_{\{e\}}(f_1, D_\alpha(V)) \neq \mathrm{DEG}_{\{e\}}(f_2, D_\alpha(V)).$$

3. Applications to bifurcation theory

In this section we formulate sufficient conditions for the existence of bifurcation points of S^1-equivariant variational bifurcation problems. In the proofs of these theorems, as the main tool, we use the degree for S^1-equivariant gradient maps. More precisely, we apply the Krasnosel'skiĭ theorem for S^1-equivariant gradient maps (see Theorem 2.3). Using Theorems 2.13, 2.15, 2.17, 2.19, 2.21 we distinguish degrees for S^1-equivariant gradient maps on two different levels of a parameter space. In other words, we verify the assumptions of the Krasnosel'skiĭ theorem.

THEOREM 3.1. *Let $f : V \times \mathbb{R} \to V$ be an S^1-equivariant gradient map such that $f(0, \lambda) = 0$ for any $\lambda \in \mathbb{R}$. Fix $\lambda_1, \lambda_2 \in \mathbb{R}$ and assume that*

1. $\ker(Df(0,\lambda_1)) \cap V^{S^1} = \{0\}$ *and* $\ker(Df(0,\lambda_2)) \cap V^{S^1} = \{0\}$,

2. *if* $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$ *and*

$$\ker(Df(0,\lambda_1)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1], \quad \ker(Df(0,\lambda_2)) \approx \bigoplus_{i=1}^{n_2} R[k_i^2, j_i^2],$$

then there is $j_{i_0} \in \{j_1, \dots, j_n\} - \{0\}$ *such that*

(a) $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^1, \dots, j_{n_1}^1\}$,

(b) $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^2, \dots, j_{n_2}^2\}$,

3.

$$\sum_{\lambda \in \sigma_-(Df(0,\lambda_1)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det(Df(0,\lambda_1)_{|\operatorname{im}(Df(0,\lambda_1))}))}{\operatorname{sign}(\det(Df(0,\lambda_2)_{|\operatorname{im}(Df(0,\lambda_2))}))} \cdot \sum_{\lambda \in \sigma_-(Df(0,\lambda_2)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda).$$

Then in $[\lambda_1, \lambda_2]$ *there is a bifurcation point of solutions of the equation* $f(v, \lambda) = 0$.

Proof. Without loss of generality one can assume that λ_1 and λ_2 are not bifurcation points. In Theorem 2.13, put $f_i(v) = f(v, \lambda_i)$ for $i = 1, 2$. We obtain

$$\operatorname{DEG}(f_1, D_\alpha(V)) \neq \operatorname{DEG}(f_2, D_\alpha(V)),$$

because the $Z_{j_{i_0}}$ coordinates of the above degrees are different. Applying Theorem 2.3 we complete the proof. □

Theorem 3.2. *Let* $f : V \times \mathbb{R} \to V$ *be an* S^1*-equivariant gradient map such that* $f(0, \lambda) = 0$ *for any* $\lambda \in \mathbb{R}$. *Fix* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and assume that*

1. $\ker(Df(0,\lambda_1)) \cap V^{S^1} = \{0\}$ *and* $\dim(\ker(Df(0,\lambda_2)) \cap V^{S^1}) = k$,
2. *if* $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$ *and*

$$\ker(Df(0,\lambda_1)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1], \quad \ker(Df(0,\lambda_2)) \approx R[k, 0] \oplus \bigoplus_{i=1}^{n_2} R[k_i^2, j_i^2],$$

then there is $j_{i_0} \in \{j_1, \dots, j_n\} - \{0\}$ *such that*

(a) $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^1, \dots, j_{n_1}^1\}$,

(b) $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^2, \dots, j_{n_2}^2\}$,

3. *if* $k = 1$, *then*

$$\sum_{\lambda \in \sigma_-(Df(0,\lambda_1)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-(Df(0,\lambda_2)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda)$$

where $\gamma = 0, 1$,

4. *if* $k = 2$, *then*

$$\sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \frac{\operatorname{sign}(\det(Df(0,\lambda_1)_{|\operatorname{im}(Df(0,\lambda_1))}))}{\operatorname{sign}(\det(Df(0,\lambda_2)_{|\operatorname{im}(Df(0,\lambda_2))}))} \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda)$$

for $\gamma \le 1$,

5. *if* $k > 2$, *then*

$$\sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda),$$

where $\gamma = 0, 1, 2, \dots$

Then in $[\lambda_1, \lambda_2]$ *there is a bifurcation point of solutions of the equation* $f(v, \lambda) = 0$.

PROOF. The proof is the same as that of Theorem 3.1, except that instead of Theorem 2.13 we apply Theorem 2.15. □

THEOREM 3.3. *Let* $f : V \times \mathbb{R} \to V$ *be an* S^1*-equivariant gradient map such that* $f(0, \lambda) = 0$ *for any* $\lambda \in \mathbb{R}$. *Fix* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and assume that*

1. $\ker(Df(0,\lambda_1)) \cap V^{S^1} = \{0\}$ *and* $\ker(Df(0,\lambda_2)) \cap V^{S^1} = \{0\}$,
2. $\ker(Df(0,\lambda_2)) \approx R[1, j_{i_0}]$ *and* $\ker(Df(0,\lambda_1)) \approx \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1]$, $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$,
3. $j_{i_0} \neq \gcd\{a_1, \dots, a_k\}$ *for any* $a_1, \dots, a_k \in \{j_1^1, \dots, j_{n_1}^1\}$,
4. $$\sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det(Df(0,\lambda_1)_{|\operatorname{im}(Df(0,\lambda_1))}))}{\operatorname{sign}(\det(Df(0,\lambda_2)_{|\operatorname{im}(Df(0,\lambda_2))}))} \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma,$$

where $\gamma = 2, 0$.

Then in $[\lambda_1, \lambda_2]$ *there is a bifurcation point of solutions of the equation* $f(v, \lambda) = 0$.

PROOF. The proof is the same as that of Theorem 3.1, with Theorem 2.17 used instead of Theorem 2.13. □

THEOREM 3.4. *Let* $f : V \times \mathbb{R} \to V$ *be an* S^1*-equivariant gradient map such that* $f(0, \lambda) = 0$ *for any* $\lambda \in \mathbb{R}$. *Fix* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and assume that*

1. $\dim(\ker(Df(0,\lambda_1)) \cap V^{S^1}) = k$ *and* $\ker(Df(0,\lambda_2)) \cap V^{S^1} = \{0\}$,

2. $\ker(Df(0,\lambda_2)) \approx R[1,j_{i_0}]$ *and*

$$\ker(Df(0,\lambda_1)) \approx R[k,0] \oplus \bigoplus_{i=1}^{n_1} R[k_i^1, j_i^1], \quad V \approx \bigoplus_{i=1}^{n} R[k_i, j_i],$$

3. $j_{i_0} \neq \gcd\{a_1,\ldots,a_k\}$ *for any* $a_1,\ldots,a_k \in \{j_1^1,\ldots,j_{n_1}^1\}$,
4. *if* $k=1$, *then*

$$\gamma_1 \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma_2,$$

where $\gamma_1 = \pm 1, 0$ *and* $\gamma_2 = 2, 0$,

5. *if* $k=2$, *then*

$$\gamma_1 \cdot \frac{\operatorname{sign}(\det(Df(0,\lambda_1)_{|\operatorname{im}(Df(0,\lambda_1))}))}{\operatorname{sign}(\det(Df(0,\lambda_2)_{|\operatorname{im}(Df(0,\lambda_2))}))} \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma_2,$$

where $\gamma_1 \leq 1$ *and* $\gamma_2 = 2, 0$,

6. *if* $k>2$, *then*

$$\gamma_1 \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma_2,$$

where $\gamma_1 \in \mathbb{Z}$ *and* $\gamma_2 = 2, 0$.

Then in $[\lambda_1,\lambda_2]$ *there is a bifurcation point for the equation* $f(v,\lambda)=0$.

PROOF. Repeat the proof of Theorem 3.3, using Theorem 2.19 instead of Theorem 2.17. □

THEOREM 3.5. *Let* $f : V\times\mathbb{R} \to V$ *be an* S^1*-equivariant gradient map such that* $f(0,\lambda)=0$ *for any* $\lambda\in\mathbb{R}$. *Fix* $\lambda_1,\lambda_2\in\mathbb{R}$ *and assume that*

1. $\ker(Df(0,\lambda_1)) \cap V^{S^1} = \{0\}$ *and* $\ker(Df(0,\lambda_2)) \cap V^{S^1} = \{0\}$,
2. $V \approx \bigoplus_{i=1}^n R[k_i,j_i]$ *and* $\ker(Df(0,\lambda_1)) \approx R[1,j_{i_0}]$, $\ker(Df(0,\lambda_2)) \approx R[1,j_{i_0}]$,
3.

$$\sum_{\lambda\in\sigma_-(Df(0,\lambda_1)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det(Df(0,\lambda_1)_{|\operatorname{im}(Df(0,\lambda_1))}))}{\operatorname{sign}(\det(Df(0,\lambda_2)_{|\operatorname{im}(Df(0,\lambda_2))}))} \cdot \sum_{\lambda\in\sigma_-(Df(0,\lambda_2)_{|R[k_{i_0},j_{i_0}]})} \mu(\lambda) + \gamma,$$

where $\gamma = \pm 2, 0$.

Then in $[\lambda_1,\lambda_2]$ *there is a bifurcation point for* $f(v,\lambda)=0$.

PROOF. Repeat the proof of Theorem 3.1, using Theorem 2.21 instead of Theorem 2.13. □

REMARK 3.6. Notice that in Theorems 3.1–3.5 we do not control the isotropy group of the bifurcating sequence of orbits of solutions. Under some additional assumptions one can compute these isotropy groups. In other words, we can prove symmetry breaking bifurcation theorems. Namely, assume additionally in Theorems 3.1–3.5 that

1. for fixed $\lambda_1, \lambda_2 \in \mathbb{R}$ there are no bifurcation points for $f^{S^1}(v, \lambda) = 0$ for $\lambda \in [\lambda_1, \lambda_2]$, where $f^{S^1} : V^{S^1} \times \mathbb{R} \to V^{S^1}$ is the restriction of f to the set of fixed points of the S^1 action,
2. $\{j \in \{j_1, \ldots, j_n\} : j/j_{i_0} \in \mathbb{N}\} = \{j_{i_0}\}$.

Then we can show that in $[\lambda_1, \lambda_2]$ there is a bifurcation point for $f(v, \lambda) = 0$ with the bifurcating sequence of orbits of zeros of f having isotropy group $Z_{j_{i_0}}$.

Additionally, notice that in order to exclude the existence of bifurcation points of $f^{S^1}(v, \lambda) = 0$ it is enough to replace assumption 1 of this remark by the assumption that $Df^{S^1}(0, \lambda)$ is an isomorphism for any $\lambda \in [\lambda_1, \lambda_2]$. This assumption is more restrictive but it seems to be easier to verify. □

The last bifurcation theorem we formulate in this paper is the finite-dimensional version of the global Rabinowitz bifurcation theorem for S^1-equivariant gradient maps (an infinite-dimensional version of this theorem can be found in [55]). Let V be a finite-dimensional representation of S^1 and let $\Omega \subset V \times \mathbb{R}$ be open and S^1-invariant. Additionally, assume that $f : \Omega \to \mathbb{R}$ is an S^1-equivariant C^2-function such that for any $\alpha < \beta$,

$$\#(\{(0, \lambda) \in \Omega \cap (\{0\} \times \mathbb{R}) : \det(\nabla^2 f(0, \lambda)) = 0\} \cap (\{0\} \times [\alpha, \beta])) < \infty$$

and that $\nabla f : (V \times \mathbb{R}, \{0\} \times \mathbb{R}) \to (V, \{0\})$, where ∇f denotes the gradient of f and $\#(A)$ denotes the number of elements of the set A.

Fix $\lambda_0 \in \mathcal{A} = \{\lambda \in \mathbb{R} : (0, \lambda) \in \Omega \cap (\{0\} \times \mathbb{R}) : \det(\nabla^2 f(0, \lambda)) = 0\}$ and $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \cap \mathcal{A} = \{\lambda_0\}$. Define the *bifurcation index* $\eta(\lambda_0)$ by

$$\eta(\lambda_0) = \mathrm{DEG}(\nabla f(\,\cdot\,, \lambda_0 + \varepsilon), D_\alpha(V)) - \mathrm{DEG}(\nabla f(\,\cdot\,, \lambda_0 - \varepsilon), D_\alpha(V)),$$

where $D_\alpha(V)$ denotes the disc of a sufficiently small radius α centered at the origin. Denote by $C(\lambda_0)$ the connected component of the set

$$\mathrm{cl}(\{(v, \lambda) \in \Omega : \nabla f(v, \lambda) = 0 \text{ and } v \neq 0\})$$

such that $(0, \lambda_0) \in C(\lambda_0)$.

Without loss of generality one can assume that $V = R[k, 0] \oplus R[k_1, j_1] \oplus \ldots \oplus R[k_r, j_r]$, where $0 \leq k$, $0 < k_i$, $0 < j_1 < \ldots < j_r$.

It is known that

$$\nabla^2 f(0,\lambda_0 \pm \varepsilon) = \begin{bmatrix} A_0^{\pm} & 0 & 0 & 0 \\ 0 & A_1^{\pm} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_r^{\pm} \end{bmatrix}.$$

By Corollary 4.3 of [55] the coordinate $\eta(\lambda_0)_Q$ of the bifurcation index $\eta(\lambda_0)$ which corresponds to the isotropy group Q is equal to

$$\eta(\lambda_0)_Q = \begin{cases} \operatorname{sign}(\det(A_0^+)) - \operatorname{sign}(\det(A_0^-)), & Q = S^1, \\ 0, & Q = Z_{j_i} \text{ for } j_i \notin K_r^j, \\ \frac{1}{2}\operatorname{sign}(\det(A_0^+)) \cdot \sum_{\lambda \in \sigma_-(A_i^+)} \mu(\lambda) & \\ \qquad - \operatorname{sign}(\det(A_0^-)) \cdot \sum_{\lambda \in \sigma_-(A_i^-)} \mu(\lambda), & Q = Z_{j_i} \text{ for } j_i \in K_r^j, \end{cases}$$

where $K_r^j = \{j_1, \ldots, j_r\}$. It is understood that if $k = 0$ then $\operatorname{sign}(\det A_0^{\pm}) = 1$.

Theorem 3.7 (Global Rabinowitz bifurcation theorem). *If $\eta(\lambda_0) \neq \Theta \in Z \oplus (\bigoplus_{i=1}^{\infty} Z)$ then either*

(a) *the component $C(\lambda_0)$ is not compact in Ω (if $\Omega = V \times \mathbb{R}$ it means that $C(\lambda_0)$ is unbounded), or*
(b) $C(\lambda_0) \cap (\{0\} \times \mathbb{R}) \cap \Omega = \{0\} \times \{\lambda_{i_1}, \ldots, \lambda_{i_s}\} \subset \{0\} \times \mathcal{A}$, $\sum_{j=1}^{s} \eta(\lambda_{i_j}) = \Theta \in \mathbb{Z} \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z})$.

In order to prove this theorem it is enough to repeat the reasoning presented in the case of the infinite-dimensional version of the Rabinowitz global bifurcation theorem for S^1-equivariant orthogonal operators (see [55]).

Remark 3.8. All the theorems of this section give sufficient conditions for the existence of bifurcation points of solutions of S^1-equivariant gradient nonlinear problems. As a tool we have used the degree theory for S^1-equivariant gradient maps. It is natural to pose the question of global bifurcations, i.e. whether connected sets of nontrivial solutions bifurcate from the set of trivial solutions and if they are global in the sense of Rabinowitz. The linearizations of our maps at $(0,\lambda) \in V \times \mathbb{R}$ are degenerate, therefore all the trivial solutions of $f(v,\lambda) = 0$ are suspected of being bifurcation points. That is why in this situation we cannot apply the global bifurcation theorem presented in this section, Theorem 3.7. Nevertheless, following [35], we can still work with the notion of global bifurcation from an interval.

We say that there is a *global bifurcation* of solutions of $f(v,\lambda) = 0$ *from* $[\lambda_1,\lambda_2]$ if there is a connected set C of nontrivial solutions of $f(v,\lambda) = 0$ whose closure intersects $\{0\} \times [\lambda_1,\lambda_2]$ and such that either C is unbounded or $\operatorname{cl}(C)$ contains a trivial solution outside $\{0\} \times [\lambda_1,\lambda_2]$. In fact, using the degree for S^1-equivariant gradient maps in place of the Brouwer degree we can adapt the proof of [53] to obtain a global bifurcation from a fixed interval in the parameter space.

Degree theory is the only topological tool which is used in the proofs of global bifurcation theorems. For instance, the change of the Conley index does not imply that the set of nontrivial solutions bifurcating from the set of trivial solutions is connected. Therefore, the theorems proved in this section have an advantage over approaches via classical topological invariants other than the degree. □

The following question seems to be important in view of Remark 3.8.

> *Is it possible to get the results of Section* 3 *working with subspaces fixed by various isotropy groups and using standard topological invariants like the Brouwer degree, the Morse theory and the Conley index?*

In order to answer this question we will consider some examples of families of S^1-equivariant gradient maps. Let $m^0(B), m^+(B), m^-(B)$ denote the nullity, positive Morse index and negative Morse index of a symmetric matrix B, respectively.

EXAMPLE 3.9. Consider a 1-parameter family of S^1-equivariant C^2-functions $f : (V = R[4,0] \oplus R[4,1] \oplus R[4,2]) \times \mathbb{R} \to R[1,0]$. Assume additionally that $\nabla f(0,\lambda) = 0$ for any $\lambda \in \mathbb{R}$, and there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

(a) $0 \in V$ is isolated in $\nabla f(\,\cdot\,,\lambda_i)^{-1}(0)$ for $i = 1, 2$,
(b) $\nabla^2 f(0,\lambda_i) = Df_i(0)$ for $i = 1, 2$, where $Df_i(0)$ are given in Example 2.14.

In Example 2.14 it has been shown that

$$3 = \mathrm{DEG}_{\{e\}}(\nabla f(\,\cdot\,,\lambda_1), D_\alpha(V)) \neq \mathrm{DEG}_{\{e\}}(\nabla f(\,\cdot\,,\lambda_2), D_\alpha(V)) = 4.$$

It is easy to check that all the assumptions of Theorem 3.1 are fulfilled. Hence in $[\lambda_1, \lambda_2]$ there is a bifurcation point of $\nabla f(v,\lambda) = 0$. Referring to Remark 3.8 we can say that we have proved the existence of a global bifurcation from $[\lambda_1, \lambda_2]$.

Coming back to our question, there are three possible isotropy groups of points in V, namely, S^1, $\{e\}$ and Z_2.

Isotropy group S^1. After restriction we obtain a map $(\nabla f)^{S^1} : V^{S^1} \times \mathbb{R} = R[4,0] \times \mathbb{R} \to V^{S^1}$ such that $(\nabla^2 f)^{S^1}(0,\lambda_i) = -\operatorname{Id}_4$ for $i = 1, 2$. Since on two levels of the parameter space, λ_1 and λ_2, we have the same map $-\operatorname{Id}_4$, we cannot distinguish them by any topological invariant.

Isotropy group $\{e\}$. After restriction we obtain a map $(\nabla f)^{\{e\}} : V \times \mathbb{R} \to V$. Since $m^0(\nabla^2 f)(0,\lambda_1) = 2$ and $m^-(\nabla^2 f)(0,\lambda_1) = 14$ is even, we have

$\deg(\nabla f(\,\cdot\,,\lambda_1), D_\alpha(V)) \leq 1$. Since $m^-(\nabla^2 f)(0,\lambda_2) = 16$, $m^+(\nabla^2 f)(0,\lambda_2) = 4$ and $m^0(\nabla^2 f)(0,\lambda_2) = 0$, we have $\deg(\nabla f(\,\cdot\,,\lambda_2), D_\alpha(V)) = 1$. It can happen that both degrees are equal to 1. This means that we cannot distinguish the maps $\nabla f(\,\cdot\,,\lambda_1)$ and $\nabla f(\,\cdot\,,\lambda_2)$ using the Brouwer degree.

Since the intersection of the intervals

$$[m^+(\nabla^2 f)(0,\lambda_1), m^+(\nabla^2 f)(0,\lambda_1) + m^0(\nabla^2 f)(0,\lambda_1)] = [4,6]$$

and

$$[m^+(\nabla^2 f)(0,\lambda_2), m^+(\nabla^2 f)(0,\lambda_2) + m^0(\nabla^2 f)(0,\lambda_2)] = [4,4]$$

is not empty we cannot distinguish $\nabla f(\,\cdot\,,\lambda_1)$ and $\nabla f(\,\cdot\,,\lambda_2)$ using the Conley index and Morse theory.

Isotropy group Z_2. After restriction we obtain a map $(\nabla f)^{Z_2} : V^{Z_2} \times \mathbb{R} = (R[4,0] \oplus R[4,2]) \times \mathbb{R} \to V^{Z_2}$ such that

$$(\nabla^2 f)^{Z_2}(0,\lambda_1) = \begin{bmatrix} -\mathrm{Id}_4 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_2 & 0 & 0 \\ 0 & 0 & 0\cdot\mathrm{Id}_2 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_4 \end{bmatrix},$$
$$(\nabla^2 f)^{Z_2}(0,\lambda_2) = \begin{bmatrix} -\mathrm{Id}_4 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_2 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_2 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_4 \end{bmatrix}.$$

Since $m^0((\nabla^2 f)^{Z_2}(0,\lambda_1)) = 2$ and $m^-((\nabla^2 f)^{Z_2}(0,\lambda_1)) = 8$ is even, we obtain $\deg((\nabla f)^{Z_2}(\,\cdot\,,\lambda_1), D_\alpha(V^{Z_2})) \leq 1$.

Since $m^-((\nabla^2 f)^{Z_2}(0,\lambda_2)) = 8$, $m^+((\nabla^2 f)^{Z_2}(0,\lambda_2)) = 4$, $m^0((\nabla^2 f)^{Z_2}(0,\lambda_2)) = 0$, we have $\deg((\nabla f)^{Z_2}(\,\cdot\,,\lambda_2), D_\alpha(V^{Z_2})) = 1$. It can happen that both degrees are 1 so we cannot distinguish $(\nabla f)^{Z_2}(\,\cdot\,,\lambda_1)$ and $(\nabla f)^{Z_2}(\,\cdot\,,\lambda_2)$ using the Brouwer degree.

Since the intersection of the intervals

$$[m^+((\nabla^2 f)^{Z_2}(0,\lambda_1)), m^+((\nabla^2 f)^{Z_2}(0,\lambda_1)) + m^0((\nabla^2 f)^{Z_2}(0,\lambda_1))] = [2,4]$$

and

$$[m^+((\nabla^2 f)^{Z_2}(0,\lambda_2)), m^+((\nabla^2 f)^{Z_2}(0,\lambda_2)) + m^0((\nabla^2 f)^{Z_2}(0,\lambda_2))] = [4,4]$$

is not empty we cannot distinguish the two maps using the Conley index and Morse theory. $\square$

Similar examples can be obtained starting from Examples 2.16, 2.18, 2.20 and 2.22.

4. Applications to asymptotically linear problems

In this section we formulate sufficient conditions for the existence of nontrivial orbits of zeros (different from the origin) of S^1-equivariant gradient maps. We apply the results proved in Section 2.

Theorem 4.1. *Let $f : (V,0) \to (V,0)$ be an S^1-equivariant gradient map such that*

1. $f(x) = A_0(x) + o|x|$ *as* $|x| \to 0$,
2. *there is* $\beta > 0$ *such that there is an S^1-equivariant gradient homotopy*
$$H : (D_\beta(V) \times [0,1], \partial D_\beta(V) \times [0,1]) \to (V, V - \{0\})$$
such that $H(\,\cdot\,,0) = f$ *and* $H(\,\cdot\,,1) = A_\infty$,
3. A_∞ *is a nonsingular symmetric matrix, and* A_0 *is a symmetric matrix,*
4. $\ker(Df(0)) \cap V^{S^1} = \ker(A_0) \cap V^{S^1} = \{0\}$,
5. *if* $V \approx \bigoplus_{i=1}^{n} R[k_i, j_i]$ *and* $\ker(A_0) \approx \bigoplus_{i=1}^{n_0} R[k_i^0, j_i^0]$, *then there is* $j_{i_0} \in \{j_1, \ldots, j_n\} - \{0\}$ *such that*
   (a) $j_{i_0} \neq \gcd\{a_1, \ldots, a_k\}$ *for any* $a_1, \ldots, a_k \in \{j_1^0, \ldots, j_{n_0}^0\}$,
   (b)
$$\sum_{\lambda \in \sigma_-((A_0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det((A_0)_{|\operatorname{im}(A_0)}))}{\operatorname{sign}(\det(A_\infty))} \cdot \sum_{\lambda \in \sigma_-((A_\infty)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda).$$

Then there is a nontrivial (*different from the origin*) *zero of f.*

Proof. Suppose that the origin is the only zero of f. From the assumptions for $\beta > 0$ and sufficiently small $\alpha > 0$ we have

$$\mathrm{DEG}(f, D_\beta(V)) = \mathrm{DEG}(A_\infty, D_\beta(V)) = \mathrm{DEG}(A_\infty, D_\alpha(V)).$$

Putting in Theorem 2.13, $f_1 = A_\infty$ and $f_2 = f$, we show that

$$\mathrm{DEG}(f_1, D_\beta(V)) = \mathrm{DEG}(f_1, D_\alpha(V)) \neq \mathrm{DEG}(f_2, D_\alpha(V)),$$

and consequently, by Theorem 3.9 of [55],

$$\begin{aligned}\mathrm{DEG}(f, D_\beta(V) - \operatorname{cl}(D_\alpha(V))) &= \mathrm{DEG}(f, D_\beta(V)) - \mathrm{DEG}(f, D_\alpha(V)) \\ &= \mathrm{DEG}(A_\infty, D_\beta(V)) - \mathrm{DEG}(f, D_\alpha(V)) \\ &= \mathrm{DEG}(f_1, D_\beta(V)) - \mathrm{DEG}(f_2, D_\alpha(V))\end{aligned}$$

is a nontrivial element in $\mathbb{Z} \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z})$. Applying Theorem 3.9 of [55] we complete the proof. □

THEOREM 4.2. *Let $f : (V,0) \to (V,0)$ be an S^1-equivariant gradient map such that*

1. $f(x) = A_0(x) + o|x|$, *when* $|x| \to 0$,
2. *there is $\beta > 0$ such that there is an S^1-equivariant gradient homotopy*

$$H : (D_\beta(V) \times [0,1], \partial D_\beta(V) \times [0,1]) \to (V, V - \{0\})$$

such that $H(\,\cdot\,,0) = f$ and $H(\,\cdot\,,1) = A_\infty$,

3. *A_∞ is a nonsingular symmetric matrix, and A_0 is a symmetric matrix,*
4. $\ker(Df(0)) \cap V^{S^1} = \ker(A_0) \cap V^{S^1} = R[k,0]$,
5. *if $V \approx \bigoplus_{i=1}^n R[k_i, j_i]$ and $\ker(A_0) \approx R[k,0] \oplus \bigoplus_{i=1}^{n_0} R[k_i^0, j_i^0]$, then there is $j_{i_0} \in \{j_1, \ldots, j_n\} - \{0\}$ such that*
   (a) $j_{i_0} \neq \gcd\{a_1, \ldots, a_k\}$ *for any* $a_1, \ldots, a_k \in \{j_1^0, \ldots, j_{n_0}^0\}$,
   (b) *if $k = 1$, then*

$$\sum_{\lambda \in \sigma_-((A_\infty)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-((A_0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),$$

   where $\gamma = 0, 1$,
   (c) *if $k = 2$ then*

$$\sum_{\lambda \in \sigma_-((A_\infty)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \frac{\operatorname{sign}(\det((A_0)_{|\operatorname{im}(A_0)}))}{\operatorname{sign}(\det(A_\infty))} \cdot \sum_{\lambda \in \sigma_-((A_0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),$$

   for $\gamma \leq 1$,
   (d) *if $k > 2$, then*

$$\sum_{\lambda \in \sigma_-((A_\infty)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \gamma \cdot \sum_{\lambda \in \sigma_-((A_0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda),$$

   where $\gamma = 0, 1, 2, \ldots$

Then there is a nontrivial (*different from the origin*) *zero of f.*

PROOF. Repeat the proof of Theorem 4.1 using Theorem 2.15 instead of Theorem 2.13. $\square$

THEOREM 4.3. *Let $f : (V,0) \to (V,0)$ be an S^1-equivariant gradient map such that*

1. $f(x) = A_0(x) + o|x|$ *as* $|x| \to 0$,
2. *there is $\beta > 0$ such that there is an S^1-equivariant gradient homotopy*

$$H : (D_\beta(V) \times [0,1], \partial D_\beta(V) \times [0,1]) \to (V, V - \{0\})$$

such that $H(\,\cdot\,,0) = f$ and $H(\,\cdot\,,1) = A_\infty$,

3. A_∞ *is a nonsingular symmetric matrix, and* A_0 *is a symmetric matrix,*
4. $\ker(Df(0)) \cap V^{S^1} = \ker(A_0) \cap V^{S^1} = \{0\}$,
5. $V \approx \bigoplus_{i=1}^n R[k_i, j_i]$ *and* $\ker(A_0) \approx R[1, j_{i_0}]$,
6. $$\sum_{\lambda \in \sigma_-((A_0)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) \neq \frac{\operatorname{sign}(\det((A_0)_{|\operatorname{im}(A_0)}))}{\operatorname{sign}(\det(A_\infty))} \cdot \sum_{\lambda \in \sigma_-((A_\infty)_{|R[k_{i_0}, j_{i_0}]})} \mu(\lambda) + \gamma,$$

   where $\gamma = 2, 0$.

Then there is a nontrivial (*different from the origin*) *zero of* f.

PROOF. Repeat the proof of Theorem 4.1 using Theorem 2.17 instead of Theorem 2.13.

REMARK 4.4. In Theorems 4.1–4.3 we have proved the existence of nontrivial orbits of zeros of S^1-equivariant gradient maps but we have not been able to give a lower estimate of the number of these orbits and their isotropy groups. Under some additional assumptions one can compute these isotropy groups. Namely, assume additionally in Theorems 4.1–4.3 that

1. $(f^{S^1})^{-1}(0) = \{0\}$,
2. $\{j \in \{j_1, \dots, j_n\} : j/j_{i_0} \in \mathbb{N}\} = \{j_{i_0}\}$.

Then we can show that there is a nontrivial orbit of zeros of f whose isotropy group is equal to $Z_{j_{i_0}}$. We distinguish these orbits by their isotropy groups. □

REMARK 4.5. Notice that instead of assumption 2 in Theorems 4.1–4.3 one can assume that f is asymptotically linear, i.e.

$$f(x) = A_\infty(x) + o|x| \quad \text{as } |x| \to \infty.$$

This assumption is more natural but more restrictive than those in Theorems 4.1–4.3. □

Similarly to Section 3 we try to answer the following question.

> *Is it possible to get the results of Section* 4 *working with subspaces fixed by various isotropy groups and using standard topological invariants like the Brouwer degree, the Morse theory and the Conley index?*

In order to answer it we consider some examples.

Example 4.6. Consider an S^1-equivariant C^2-function

$$f : V = R[1,1] \oplus R[1,2] \to R[1,0].$$

Assume additionally that

1. $\nabla f(x) = \nabla^2 f(0) \cdot x + o(|x|)$, $|x| \to 0$,
2. $\nabla f(x) = \nabla^2 f(\infty) \cdot x + o(|x|)$, $|x| \to \infty$,
3. $$\nabla^2 f(0) = \begin{bmatrix} -\operatorname{Id}_2 & 0 \\ 0 & 0 \cdot \operatorname{Id}_2 \end{bmatrix},$$
4. $$\nabla^2 f(\infty) = \begin{bmatrix} +\operatorname{Id}_2 & 0 \\ 0 & -\operatorname{Id}_2 \end{bmatrix}.$$

It is easy to check that for $j_{i_0} = 1$ all the assumptions of Theorem 4.1 are fulfilled. This shows the existence of an orbit of critical points of f. There are three possible isotropy groups of points in the representation V: S^1, $\{e\}$ and Z_2.

Isotropy group S^1. $V^{S^1} = \{0\}$.

Isotropy group $\{e\}$. After restriction we obtain a map $\nabla f : V \to V$. Choose sufficiently small $\alpha > 0$ and sufficiently large $\beta > 0$. It is easy to see that

$$\deg(\nabla f, D_\alpha(V)) = 1 = \deg(\nabla f, D_\beta(V)).$$

This means that we cannot prove the existence of a nontrivial orbit of critical points of f using the Brouwer degree. Since the intersection of the intervals

$$[m^+(\nabla^2 f(\infty)), m^+(\nabla^2 f(\infty)) + m^0(\nabla^2 f(\infty))] = [2,2]$$

and

$$[m^+(\nabla^2 f(0)), m^+(\nabla^2 f(0)) + m^0(\nabla^2 f(0))] = [0,2]$$

is not empty we cannot prove the existence of a nontrivial orbit of critical points of f using the Conley index and Morse theory.

Isotropy group Z_2. After restriction we obtain a map $(\nabla f)^{Z_2} : V^{Z_2} = R[1,2] \to V^{Z_2}$. Choose sufficiently small $\alpha > 0$ and sufficiently large $\beta > 0$. It is easy to see that

$$\deg((\nabla f)^{Z_2}, D_\alpha(V^{Z_2})) = 1 = \deg((\nabla f)^{Z_2}, D_\beta(V^{Z_2})).$$

This means that we cannot prove the existence of a nontrivial orbit of critical points of f using the Brouwer degree. Since the intersection of the intervals

$$[m^+((\nabla^2 f)^{Z_2}(\infty)), m^+((\nabla^2 f)^{Z_2}(\infty)) + m^0((\nabla^2 f)^{Z_2}(\infty))] = [0,0]$$

and

$$[m^+((\nabla^2 f)^{Z_2}(0)), m^+((\nabla^2 f)^{Z_2}(0)) + m^0((\nabla^2 f)^{Z_2}(0))] = [0,2]$$

is not empty we cannot prove the existence of a nontrivial orbit of critical points of f using the Conley index and Morse theory. $\square$

EXAMPLE 4.7. Consider an S^1-equivariant C^2-function

$$f : V = R[1,0] \oplus R[1,1] \oplus R[1,2] \to R[1,0].$$

Assume additionally that

1. $\nabla f(x) = \nabla^2 f(0) \cdot x + o(|x|)$, $|x| \to 0$,
2. $\nabla f(x) = \nabla^2 f(\infty) \cdot x + o(|x|)$, $|x| \to \infty$,
3. $$\nabla^2 f(0) = \begin{bmatrix} 0 \cdot \mathrm{Id}_1 & 0 & 0 \\ 0 & +\mathrm{Id}_2 & 0 \\ 0 & 0 & 0 \cdot \mathrm{Id}_2 \end{bmatrix},$$
4. $$\nabla^2 f(\infty) = \begin{bmatrix} +\mathrm{Id}_1 & 0 & 0 \\ 0 & -\mathrm{Id}_2 & 0 \\ 0 & 0 & +\mathrm{Id}_2 \end{bmatrix}.$$

It is easy to check that for $j_{i_0} = 1$ all the assumptions of Theorem 4.2 are fulfilled. This shows the existence of an orbit of critical points of f.

Reasoning as in Example 3.9 we can show that standard topological invariants do not work in this case. $\square$

EXAMPLE 4.8. Consider an S^1-equivariant C^2-function

$$f : V = R[5,1] \to R[1,0].$$

Assume additionally that

1. $\nabla f(x) = \nabla^2 f(0) \cdot x + o(|x|)$, $|x| \to 0$,
2. $\nabla f(x) = \nabla^2 f(\infty) \cdot x + o(|x|)$, $|x| \to \infty$,
3. $$\nabla^2 f(0) = \begin{bmatrix} 0 \cdot \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_8 \end{bmatrix},$$
4. $$\nabla^2 f(\infty) = [\,+\mathrm{Id}_{10}\,].$$

It is easy to check that for $j_{i_0} = 1$ all the assumptions of Theorem 4.3 are fulfilled. This gives the existence of an orbit of critical points of f. It is easily seen that it is possible to prove the existence of a nontrivial orbit of zeros of f using Morse theory. $\square$

5. Final remarks

In this paper we have prepared a topological tool which we intend to apply to qualitative investigations of elliptic differential equations, Hamiltonian systems, wave equations and second order ODE's. Mainly, we are interested in sufficient conditions for the existence of nontrivial solutions of asymptotically linear equations and in the existence of bifurcation points.

In order to apply our results to differential equations one can use Amann–Zehnder saddle-point reduction. The first step in this direction is done in [58]. The degree for S^1-equivariant gradient maps has also been defined in the infinite-dimensional case, for compact perturbations of the identity. In fact, in order to compute this invariant it is enough to compute the degree of a finite-dimensional map. Therefore the results of Section 2 will be used in infinite-dimensional computations.

We can distinguish orbits of critical points by their isotropy groups. It allows one to prove multiplicity results for differential equations. Moreover, sufficient conditions for local and global bifurcations of solutions of differential equations can be formulated. Additionally, one can prove symmetry-breaking bifurcation results.

In this article we have only considered problems with resonance at the origin, i.e. the linearization $Df(0)$ at the origin was degenerate, while at infinity, $Df(\infty)$ was an isomorphism. Let $f : V \to V$ be an S^1-equivariant, gradient, asymptotically linear map whose "derivative at infinity" $Df(\infty)$ is degenerate and let $D_\beta(V)$ denote an open disc centered at the origin with sufficiently large radius $\beta \gg 0$. A still open and very interesting question is

Is it possible to compute $\mathrm{DEG}(f, D_\beta(V))$ *when* $Df(\infty)$ *is degenerate?*

T. Bartsch and S. Li [9] have recently proved an implicit function theorem at infinity. Using their results one can prove a splitting lemma at infinity. Combining the splitting lemma at the origin (Lemma 2.4), the splitting lemma at infinity and the Cartesian product formula (Theorem 2.11), one can investigate asymptotically linear problems with both kinds of resonance, at the origin and at infinity.

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