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Applications of Fractional Calculus

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Abstract

Explicit formula and graphs of few special functions are derived in the paper on the basis of various definitions of various fractional derivatives and various fractional integrals. Their applications are also reviewed in the paper.

I. Introduction

Fractional calculus comes under purview of mathematical study that culminates from traditional definitions of calculus integral and derivative operators. Similarly, fractionalexponents areproductsofexponentswithintegervalue.

Fractional derivatives, fractional integrals, and their properties are the subject of study in the field of fractional calculus.

Generally, fractional-order integration and differentiation does not have clear physical and geometric interpretations. Since theorigin of idea of differentiation and integration of arbitrary (not necessaryinteger) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year, until 'Igor Podlubny' shown thatgeometric interpretation of fractional integration is "Shadows on the walls"and its Physical interpretation is "'Shadows of the past"'.And the research continued, which gave demonstrable results inthe last few years where their use has been found in studies of viscoelastic materials, aswell asinmanyfieldsofscienceandengineeringincludingfluidflow, rheology,diffusivetransport, electerical networks, electromagnetic theory and probability.

In this paper we have considered different definitions of fractional derivatives andintegrals (differintegrals), which are basic building blocks of differintegrals. For some elementary functions, explicit formula offractional drevative and integral are presented along with some applications of fractional calculus in science and engineering.

II. Different Definitions

1. L. Euler(1730) generalized the following formula

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}$$

by using of the following property of Gamma function,

$$\Gamma(m+1) = m(m-1)\cdots(m-n+1)\Gamma(m-n+1)$$

to obtain

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Where Gamma function is defined as follows.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad Re(z) > 0$$

2. J. B. J. Fourier by means of integral representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp$$

wrote

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz + n\frac{\pi}{2})dp.$$

3. N. H. Abel considered the following integral representation for arbitrary α

$$\int_0^x \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x)$$

Which he wrote as

$$s(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}$$

4. G. F. B. Riemann'sdefinition of Fractional Integral is

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)dt + \psi(t)$$

5. Riemann-Liouvill'spopular definition of fractional calculus is this which shows joining of G.F.B. Rieman's definition and Definitions of N. YaSonin, A.V. Letnikov, H. Laurent, N. Nekrasove and K. Nishimoto.

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$
$$(n-1 \le \alpha < n)$$

6. Grunwald-Letnikove:

This is another joined definition which many a time is useful.

$$_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh)$$

7. M. Caputo (1967):

The second popular definition is

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 \leq \alpha < n) $$

8. K. S. Miller, B. Ross (1993):
   They used following differential operator D where $D^{\alpha i}$ is Riemann-Liouvill or Caputo definitions.

$$ D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n) $$

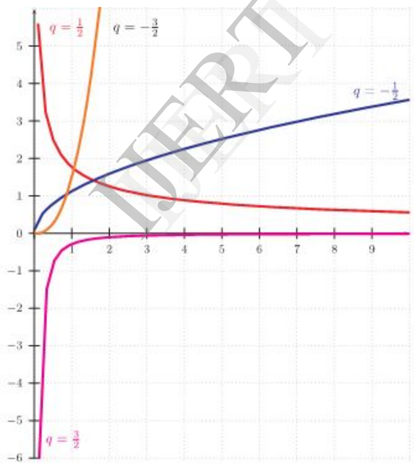
II. Fractional derivative of Some special Functions

Explicit formulas of fractional derivative and integral of some special functions are given here with consideration to their graph.

1. Unit function:

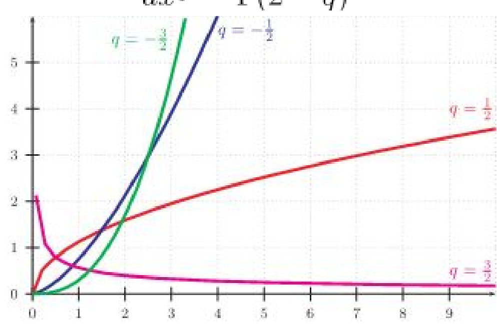
Forf(x) = 1 we have

$$ \frac{d^q 1}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)} \quad \text{for all } q $$



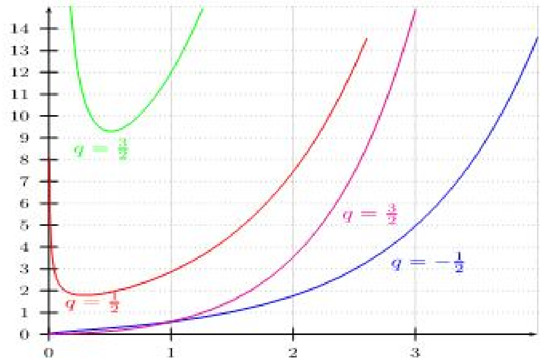
2. Identity function:Forf(x) =xwe have

$$ \frac{d^q x}{dx^q} = \frac{x^{1-q}}{\Gamma(2-q)} $$



3. Exponential function: Fractional differintegral of the function f(x) =

$$e^x \text{ is } \frac{d^q e^{\pm x}}{dx^q} = \sum_{k=0}^{\infty} \frac{x^{k-q}}{\Gamma(k-q+1)}$$



III. Applications of Fractional Calculus

It were renowned mathematicians like Leibniz (1695), Liouville (1834), Riemann (1892) and others who developed the basic mathematical ideas of fractional calculus (integral and differential operations of non integer order). However,recent monographs and symposia proceedings have also exhibited the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. Here we some of applications are brought out.

1. First application of fractionalcalculus was made by Abelin the solution of an integralequation that arises in the formulation of the autochronous problem.This problem deals with the determination of the shape of a frictionlessplane curve through the origin in a vertical plane along which a particleof mass m can fall in a time that is independent of the starting position.If the sliding time is constant T, then the Abel integral equationis

$$\sqrt{2g}T = \int_0^{\eta} (\eta - y)^{-\frac{1}{2}} f'(y)dy,$$

where g is the acceleration due to gravity, (ξ,η) is the initial position and s = f(y) is the equation of the sliding curve. It turns out that this equation is equivalent to the fractional integral equation.

$$T\sqrt{2g} = \Gamma(\frac{1}{2})_0 D_{\eta}^{-\frac{1}{2}} f'(\eta)$$

2. Electric transmission lines

Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator form solution of heat equation due to Gregory(1846), Heaviside introduced the letter p for the differential operator d/dt and gave the solution of the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 p$$

for the temperature distribution u(x,t) in the symbolic form

$$u(x,t) = A\exp(ax\sqrt{p}) + B\exp(-ax\sqrt{p})$$

in which p ≡ d/ dx was treated as constant, where a, A and B are also constant.

3. Ultrasonicwavepropagation in human cancellous bone

Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot’s theory. Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples

4. Modeling of speech signals using fractional calculus

In this paper, a novel approach for speech signal modeling using fractional calculus is presented. This approach is contrasted with the celebrated Linear Predictive Coding (LPC) approach which is based on integer order models. It is demonstrated via numerical simulations that by using a few integrals of fractional orders as basis functions, the speech signal can be modeled accurately.

5. Modeling the Cardiac Tissue Electrode Interface Using Fractional Calculus

The tissue electrode interface is common to all forms of biopotential recording (e.g., ECG, EMG, EEG) and functional electrical stimulation (e.g., pacemaker, cochlear implant, deep brain stimulation). Conventional lumped element circuit models of electrodes can be extended by generalization of the order of differentiation through modification of the defining current-voltage relationships. Such fractional order models provide an improved description of observed bioelectrodebehaviour, but recent experimental studies of cardiac tissue suggest that additional mathematical tools may be needed to describe this complex system.

6. Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials

The observation that the asymptotic expressions of stiffness and damping inporous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behavior of soundwaves in this kind of materials, including relaxation and frequency dependence.

7. Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles

Here it is presented the use of Fractional Order Controllers (FOC) applied to the path tracking problem in an autonomous electric vehicle. A lateral dynamic model of a industrial vehicle has been taken into account to implement conventional and Fractional Order Controllers. Several control schemes with these controllers have been simulated and compared.

8. Application of fractional calculus in the theory of viscoelasticity

The advantage of the method of fractional derivatives in theory of viscoelasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in studies of the complex moduli and impedances for various models of viscoelastic substances.

9. Fractional differentiation for edge detection

In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the Laplacian. This paper demonstrates how introducing an edge detector based onnon-integer (fractional)differentiation canimprove the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

10. Application of Fractional Calculus to Fluid Mechanics

Application of fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems are presented. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shear stress and fluid speed anywhere in the domain. Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results for the first and second Stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful thanexisting techniques.

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