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Research Article

Applications of Fractional q-Calculus to Certain Subclass of Analytic p-Valent Functions with Negative Coefficients

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By making use of the concept of fractional q-calculus, we firstly define q-extension of the generalization of the generalized Al-Oboudi differential operator. Then, we introduce new class of q-analogue of p-valently closed-to-convex function, and, consequently, new class by means of this new general differential operator. Our main purpose is to determine the general properties on such class and geometric properties for functions belonging to this class with negative coefficient. Further, the q-extension of interesting properties, such as distortion inequalities, inclusion relations, extreme points, radii of generalized starlikeness, convexity and close-to-convexity, quasi-Hadamard properties, and invariant properties, is obtained. Finally, we briefly indicate the relevant connections of our presented results to the former results.

1. Introduction

The formulation of fractional calculus began shortly after the classical calculus was established. Since its definition is based on the concept of a noninteger order either integral or derivative, the fractional calculus had been considered as a subject in pure mathematics with no real applications for a long time. However, the role of fractional calculus has been changed in recent decades. Its applications take place in many fields of mathematical sciences. Extended from the fractional calculus, the fractional q-calculus is the q-extension of the ordinary fractional calculus. Many results of the study on theory of q-calculus operators in recent decades have been applied in various areas such as problems in the ordinary fractional calculus, optimal control, solutions of q-difference equations, q-differential equations, q-integral equations, and q-transform analysis and also in the geometric function theory of complex analysis.

In the field of geometric function theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. For example, the extension of the theory of univalent functions can be described by using the theory of q-calculus. In [1], Ismail et al. introduced the generalized class of starlike functions by using the q-difference operator and replaced the right-half plane by a suitable domain. In a similar way, Agrawal and Sahoo [2] introduced the generalized class of starlike functions of order α and Raghavendar and Swaminathan [3] also introduced the class of q-analogue to close-to-convex functions. Moreover, the q-calculus operators, such as fractional q-integral and fractional q-derivative operators, are used to construct several subclasses of analytic functions (see, e.g., [4–8]).

In addition, the differential operators have been extensively investigated in the field of geometric function theory. The well-known differential operator defined on the class of analytic functions is introduced by Salagean [9]. This operator was successfully used by many authors and it led to the investigation of several properties of certain known and new classes of analytic functions (see, e.g., [10–14]). However, there are many generalized Salagean operators defined by several authors. In [15], Al-Oboudi defined the generalized Salagean operator by using the technique of convolution structure. In [16], Al-Oboudi and Al-Amoudi used the extension of fractional derivative and fractional integral to define linear multiplier fractional differential operator which yields the Al-Oboudi operator [15] and fractional differential operator. Moreover, Bulut [17] modified the Al-Oboudi and Al-Amoudi operator [16] by introducing nonnegative parameter l in that operator. Recently, Selvakumaran et al. [8] introduced the fractional q-differintegral operator by using the fractional q-calculus operators involving the generalized Al-Oboudi and Al-Amoudi operator [16]. For some recent investigations of these operators on the classes of analytic functions and related topics, such as coefficient estimate, distortion theorem, extreme points, and subordination, we refer to [18–23] and the references cited therein.

This paper is organized as follows. In Section 2, we propose the q-extension of the Bulut operator [17] which generalized Selvakumaran et al. operator [8]. We also define new class to $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ by using this new general differential operator together with q-analogue to p-valent closed-to-convex function. In Section 3, we give linear combination property and coefficient estimate for function belonging to $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. By making use of the coefficient estimate, the q-extension of geometric properties for function with negative coefficients $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ is given in Section 4. Then, we finish our paper by observations and concluding remarks.

2. Preliminaries and Definitions

Let p be a positive integer, and let $\mathscr{A}_p$ be the class of analytic functions and p-valent in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that are of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k. \tag{1}$$

Let $\mathscr{T}_p$ be a subclass of $\mathscr{A}_p$ consisting of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k. \tag{2}$$

In particular, we set $\mathscr{A}_1 \equiv \mathscr{A}$ and $\mathscr{T}_1 \equiv \mathscr{T}$. For $0 \le \alpha < 1$, let $\mathscr{R}(\alpha)$ be the subclass of $\mathscr{A}$ consisting of all functions which satisfy $\operatorname{Re}\{f'(z)\} > \alpha$ in $\mathbb{D}$. The functions in $\mathscr{R}(\alpha)$ are called functions of bounded turning. All of those are univalent and close-to-convex in $\mathbb{D}$ (see [24]). Similarly, we denote by $\mathscr{R}_p(\alpha)$, where $0 \le \alpha < 1$, the class of all functions in $\mathscr{A}_p$ which satisfy $\operatorname{Re}\{f'(z)/pz^{p-1}\} > \alpha$ (see more details in [25, 26]).

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. In the theory of q-calculus, the q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$, $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1, & n = 0; \\ (1-\alpha)(1-\alpha q)\cdots\left(1-\alpha q^{n-1}\right), & n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n > 0), \tag{4}$$

where the q-gamma function [27, 28] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0 < q < 1). \tag{5}$$

We note that if $|q| < 1$, the q-shifted factorial (3) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha; q)_\infty = \prod_{k=0}^{\infty}\left(1 - \alpha q^k\right). \tag{6}$$

Here, we recall the following q-analogue definitions given by Gasper and Rahman [27]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \tag{7}$$

where $[x]_q = (1-q^x)/(1-q)$, and is called q-analogue of x. It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^-$, where $\Gamma(x)$ is the ordinary Euler gamma function.

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{8}$$

we observe that the q-shifted factorial (3) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$.

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subset \mathbb{C}$ is called a μ-geometric set if, for $z \in A$, $\mu z \in A$. Let f be a function defined on a q-geometric set. Jackson's q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [27], pp. 19–22)

$$\begin{gathered} D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0,\ q \neq 0), \\ \int_0^z f(t)\, d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f\left(zq^k\right). \end{gathered} \tag{9}$$

In case $f(z) = z^n$, the q-derivative of $f(z)$, where n is a positive integer, is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q z^{n-1}. \tag{10}$$

As $q \to 1^-$ and $n \in \mathbb{N}$, we have $[n]_q = (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1} \to n$.

We now recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [29].

Definition 1 (fractional q-integral operator). The fractional q-integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order δ ($\delta > 0$) is defined by

$$I^\delta_{q,z} f(z) = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z - tq)_{1-\delta} f(t)\, d_q t, \tag{11}$$

where $f(z)$ is analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q-binomial function defined by

$$(z-tq)_{\delta-1} = z^{\delta-1}\prod_{k=0}^{\infty}\left[\frac{1-(tq/z)q^{k}}{1-(tq/z)q^{\delta+k-1}}\right] = z^{\delta}\,{}_{1}\Phi_{0}\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right]. \quad (12)$$

According to Gasper and Rahman [27], the series ${}_1\Phi_0[\delta;-;q,z]$ is single-valued when $|\arg(z)|<\pi$. Therefore, the function $(z-tq)_{\delta-1}$ in (12) is single-valued when $|\arg(-tq^{\delta}/z)|<\pi$, $|tq^{\delta/z}|<1$, and $|\arg(z)|<\pi$.

Definition 2 (fractional q-derivative operator). The fractional q-derivative operator $D^{\delta}_{q,z}$ of a function $f(z)$ of order δ $(0\le\delta<1)$ is defined by

$$D^{\delta}_{q,z}f(z) = D_{q,z}I^{1-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z-tq)_{-\delta}f(t)\,d_qt, \quad (13)$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above.

Definition 3 (extended fractional q-derivative operator). Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z}f(z) = D^{m}_{q,z}I^{m-\delta}_{q,z}f(z), \quad (14)$$

where $m-1\le\delta<m$, $m\in\mathbb{N}_0$.

In addition, the extension of q-differintegral operator $\Omega^{\delta}_{q}:\mathcal{A}_p\to\mathcal{A}_p$, for $\delta<p+1$, $0<q<1$, and $n\in\mathbb{N}$, is defined by

$$\Omega^{\delta}_{q}f(z) = \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}z^{\delta}D^{\delta}_{q,z}f(z) = z^{p}+\sum_{k=p+1}^{\infty}\frac{\Gamma_q(k+1)\Gamma_q(p-\delta+1)}{\Gamma_q(p+1)\Gamma_q(k-\delta+1)}a_kz^{k}, \quad (15)$$

where $D^{\delta}_{q,z}$ in (15) represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty<\delta<0$ and a fractional q-derivative of $f(z)$ of order δ when $0\le\delta<p+1$. We note that when $q\to1^{-}$, the operator Ω^{δ}_{q} reduces to the operator Ω^{δ} introduced by Owa and Srivastava [30].

Now, we define the q-extension of Al-Oboudi type differential operator $\mathcal{D}^{\delta,m}_{q,\lambda,l,p}:\mathcal{A}_p\to\mathcal{A}_p$, for $l,\lambda\ge0$, $\delta<p+1$, and $m\in\mathbb{N}_0$, which is defined by

$$\begin{aligned}
\mathcal{D}^{\delta,0}_{q,\lambda,l,p}f(z) &= f(z),\\
\mathcal{D}^{\delta,1}_{q,\lambda,l,p}f(z) &= \frac{[p]_q-\lambda[p]_q+l}{[p]_q+l}\Omega^{\delta}_{q}f(z) + \frac{\lambda}{[p]_q+l}zD_q\left(\Omega^{\delta}_{q}f(z)\right),\\
\mathcal{D}^{\delta,2}_{q,\lambda,l,p}f(z) &= \mathcal{D}^{\delta,1}_{q,\lambda,l,p}\left(\mathcal{D}^{\delta,1}_{q,\lambda,l,p}f(z)\right),\\
&\vdots\\
\mathcal{D}^{\delta,m}_{q,\lambda,l,p}f(z) &= \mathcal{D}^{\delta,1}_{q,\lambda,l,p}\left(\mathcal{D}^{\delta,m-1}_{q,\lambda,l,p}f(z)\right).
\end{aligned} \quad (16)$$

We note that if $f\in\mathcal{A}$ is given by (1), then by (16) we have

$$\mathcal{D}^{\delta,m}_{q,\lambda,l,p}f(z) = z^{p}+\sum_{k=p+1}^{\infty}\Psi^{\delta,m}_{q,\lambda,l,p}(k)a_kz^{k}, \quad (17)$$

where

$$\Psi^{\delta,m}_{q,\lambda,l,p}(k) = \left[\frac{\Gamma_q(k+1)\Gamma_q(p-\delta+1)}{\Gamma_q(p+1)\Gamma_q(k-\delta+1)}\frac{[p]_q+([k]_q-[p]_q)\lambda+l}{[p]_q+l}\right]^{m}. \quad (18)$$

We note that, by setting appropriated values for the parameters in the operator $\mathcal{D}^{\delta,m}_{q,\lambda,l,p}$, this operator reduces to many known differential operators. For example, in case $l=0$ the operator $\mathcal{D}^{\delta,m}_{q,\lambda,0,p}$ is exactly the Selvakumaran et al. operator $\mathcal{D}^{\delta,m}_{q,\lambda,p}$ in [8]. Also, when $q\to1^{-}$ the operator $\mathcal{D}^{\delta,m}_{\lambda,l,p}$ reduces to the operator introduced by Bulut [17]. Moreover Bulut [17] noticed that, for suitable parameters l,λ,δ,p, and m, the operator $\mathcal{D}^{\delta,m}_{\lambda,l,p}$ generalizes many operators introduced by several authors, for instance, Salagean [9], Al-Oboudi [15], Al-Oboudi and Al-Amoudi [16], Acu and Owa [31], Acu et al. [32], Cătaş [33], Cho and Srivastava [34], Cho and Kim [35], Kumar et al. [36], Owa and Srivastava [30], and Uralegaddi and Somanatha [37].

Next, we define the q-analogous to the function class $\mathcal{R}_p(\alpha)$ by $\mathcal{R}_{q,p}(\alpha)$. A function $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{R}_{q,p}(\alpha)$ of p-valently closed-to-convex with respect to q-differentiation if and only if

$$\operatorname{Re}\left\{\frac{D_qf(z)}{[p]_qz^{p-1}}\right\}>\alpha,\quad z\in\mathbb{D}, \quad (19)$$

where $0<\alpha<1$. In particular, we set $\mathcal{R}_{q,1}(\alpha)\equiv\mathcal{R}_q(\alpha)$. Note that the class $\mathcal{R}_q(\alpha)$ generalizes the class $\mathcal{K}_q$ (with the function $g(z)=z$) which was introduced by

Raghavendar and Swaminathan [3]. Moreover, we see that $D_q f(z) \to f'(z)$, as $q \to 1^-$. This implies that an inequality $\operatorname{Re}\{D_q f(z)/[p]_q z^{p-1}\} > \alpha$ becomes $\operatorname{Re}\{f'(z)/pz^{p-1}\} > \alpha$. Hence, the class $\mathscr{R}_{q,p}(\alpha)$ clearly reduces to $\mathscr{R}_p(\alpha)$ and satisfies

$$\bigcap_{0<q<1} \mathscr{R}_{q,p}(\alpha) \subset \mathscr{R}_p(\alpha) \subset \mathscr{R}_p. \tag{20}$$

Furthermore, by using the operator $\mathscr{D}^{\delta,m}_{q,\lambda,l,p}$ defined by (16) and q-differentiation, we introduce a new class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ as follows.

Let $\delta < p+1$, $\lambda, l \geq 0$, $0 \leq \alpha < 1$, and $m \in \mathbb{N}_0$. Denote by $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ the class of all functions $f \in \mathscr{A}_p$ satisfying the condition

$$\operatorname{Re}\left\{\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p} f(z)\right)}{[p]_q z^{p-1}}\right\} > \alpha, \quad z \in \mathbb{D}. \tag{21}$$

Denote by $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ the class obtained by taking intersection of the class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ with the class $\mathscr{T}_p$. That is,

$$\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha) \equiv \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha) \cap \mathscr{T}_p. \tag{22}$$

In particular, we set $\mathscr{T}\mathscr{R}^{\delta,0}_{q,\lambda,l,p}(\alpha) \equiv \mathscr{T}\mathscr{R}_q(\alpha)$. The special cases of the class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, as $q \to 1^-$, have been studied by Bulut [38], Al-Oboudi in [15], and Tăut et al. [39] and the special cases of the class $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, as $q \to 1^-$, have been proved by Altintas [40].

3. Main Results

3.1. General Properties. We begin to derive the linear combination property on $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ in the following result.

Theorem 4. *The class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ is convex.*

Proof. Let $f, g \in \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ of the form

$$\begin{aligned} f(z) &= z^p + \sum_{k=p+1}^{\infty} a_k z^k, \\ g(z) &= z^p + \sum_{k=p+1}^{\infty} b_k z^k. \end{aligned} \tag{23}$$

It is sufficient to show that the function $h(z) = \mu f(z) + (1-\mu)g(z)$, where $0 \leq \mu \leq 1$, is in the class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. By (23), we see that

$$h(z) = z^p + \sum_{k=p+1}^{\infty} \left[\mu a_k + (1-\mu) b_k\right] z^k. \tag{24}$$

Hence

$$\begin{aligned} &\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p} h(z)\right)}{[p]_q z^{p-1}} \\ &= 1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) \left[\mu a_k + (1-\mu) b_k\right] z^{k-p} \\ &= \mu \operatorname{Re}\left\{1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) a_k z^{k-p}\right\} \\ &\quad + (1-\mu) \operatorname{Re}\left\{1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) a_k z^{k-p}\right\}, \end{aligned} \tag{25}$$

where $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ is defined by (18). Since $f, g \in \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, we have

$$\begin{aligned} &\operatorname{Re}\left\{\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p} f(z)\right)}{[p]_q z^{p-1}}\right\} \\ &\quad = \operatorname{Re}\left\{1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) a_k z^{k-p}\right\} > \alpha, \\ &\operatorname{Re}\left\{\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p} g(z)\right)}{[p]_q z^{p-1}}\right\} \\ &\quad = \operatorname{Re}\left\{1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) b_k z^{k-p}\right\} > \alpha. \end{aligned} \tag{26}$$

Applying (26) to (25), we obtain

$$\operatorname{Re}\left\{\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p} h(z)\right)}{[p]_q z^{p-1}}\right\} > \mu\alpha + (1-\mu)\alpha = \alpha. \tag{27}$$

Now, the proof is completed. □

Remark 5. In case $p = 1$, by letting $q \to 1^-$, we obtain Theorem 4.1 in [38]. For $q \to 1^-$ with $\delta = 0$, we obtain Theorem 2.11 in [15]. Moreover, for $\delta = 0$, $\lambda = 1$, and $q \to 1^-$, we obtain Theorem 2.1 in [39].

Next, we derive some sharp coefficient inequalities contained in the following theorem that are useful in the main results.

Theorem 6. *Let $f \in \mathscr{A}_p$ be defined by (1) and satisfy the inequality*

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) \left|a_k\right| \leq 1 - \alpha, \tag{28}$$

where $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ is defined in (18). Then, $f \in \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. Moreover, the converse also holds if $f \in \mathscr{T}_p$. The result is sharp.

Proof. Let the function $f \in \mathscr{A}_p$ be defined by (1). To prove this, we consider

$$\left|\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p}f(z)\right)}{[p]_q z^{p-1}}-1\right| = \left|\sum_{k=p+1}^{\infty}\frac{[k]_q}{[p]_q}\Psi^{\delta,m}_{q,\lambda,l,p}(k)\,a_k z^{k-p}\right| \le \sum_{k=p+1}^{\infty}\frac{[k]_q}{[p]_q}\Psi^{\delta,m}_{q,\lambda,l,p}(k)\left|a_k\right|\left|z^{k-p}\right|. \tag{29}$$

By assumption (28), (29) can be rewritten as

$$\left|\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p}f(z)\right)}{[p]_q z^{p-1}}-1\right| \le \sum_{k=p+1}^{\infty}\frac{[k]_q}{[p]_q}\Psi^{\delta,m}_{q,\lambda,l,p}(k)\left|a_k\right| \le 1-\alpha. \tag{30}$$

Therefore, we infer that $f \in \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. To prove the converse, we let function $f \in \mathscr{T}_p$ be defined by (2) and belong to the class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. Then, we have

$$\Re\left\{\frac{D_q\left(\mathscr{D}^{\delta,m}_{q,\lambda,l,p}f(z)\right)}{[p]_q z^{p-1}}\right\} = \Re\left\{1-\sum_{k=p+1}^{\infty}\frac{[k]_q}{[p]_q}\Psi^{\delta,m}_{q,\lambda,l,p}(k)\left|a_k\right| z^{k-p}\right\} > \alpha. \tag{31}$$

Or, equivalently,

$$\Re\left\{\sum_{k=p+1}^{\infty}\frac{[k]_q}{[p]_q}\Psi^{\delta,m}_{q,\lambda,l,p}(k)\left|a_k\right| z^{k-p}\right\} < 1-\alpha. \tag{32}$$

In (32), by letting $z \to 1^-$ on the real axis, we obtain inequality (28) as desired. Finally, we note that assertion (28) is sharp, the extremal function being

$$f(z) = z^p - \frac{[p]_q(1-\alpha)}{[p+1]_q\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)}z^{p+1}. \tag{33}$$

Now, the proof is completed. □

Corollary 7. *If $f \in \mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then for $k = p+1, p+2, \ldots$,*

$$\left|a_k\right| \le \frac{1-\alpha}{[k]_q\Psi^{\delta,m}_{q,\lambda,l,p}(k)}, \tag{34}$$

where $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ is defined in (18).

4. Geometric Properties for the Class $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$

By observation, Theorem 6 gives the necessary and sufficient conditions via coefficient bounded for functions to be in the multivalently analytic function class $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. Using this result, we will discuss standard properties for that class in sense of q-theory, such as distortion inequalities, inclusion relations, extreme points, radii of close-to-convexity, starlikeness and convexity, quasi-Hadamard property, and invariant properties. However, some of the mentioned properties can be obtained only in case $0 \le \delta < p+1$ because the monotonicity of the sequence $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ is required to prove those results. The following lemmas guarantee the monotone increasing property for the sequence $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ in case $0 \le \delta \le p+1$ and monotone decreasing in case $\delta < 0$.

Lemma 8. *Let the sequence $(a_k)_{k=p}^{+\infty}$ be defined by*

$$a_k = \frac{\Gamma_q(k+1)\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}. \tag{35}$$

(i) *If $0 \le \delta < p+1$, then $(a_k)_{k=p}^{+\infty}$ is a nondecreasing sequence and $1 \le a_k$ for $k \ge p$.*

(ii) *If $\delta < 0$, then $(a_k)_{k=p}^{+\infty}$ is a decreasing sequence and $a_k \le 1$ for $k \ge p$.*

Proof. It is clear that the sequence $(a_k)_{k=p}^{+\infty}$ is nonnegative for $\delta < p+1$. We have that

$$\frac{a_{k+1}}{a_k} = \frac{\Gamma_q(k+2)\Gamma_q(k+1-\delta)}{\Gamma_q(k+1)\Gamma_q(k+2-\delta)}. \tag{36}$$

So, by using (7), we get

$$\frac{a_{k+1}}{a_k} = \frac{[k+1]_q}{[k+1-\delta]_q} = \frac{1-q^{k+1}}{1-q^{k+1-\delta}}. \tag{37}$$

Since $0 < q < 1$, we see that

$$\frac{1-q^{k+1}}{1-q^{k+1-\delta}} < 1, \quad \text{for } \delta < 0, \qquad \frac{1-q^{k+1}}{1-q^{k+1-\delta}} \ge 1, \quad \text{for } 0 \le \delta < p+1. \tag{38}$$

Then, for $0 \le \delta < p+1$, we conclude that $(a_k)_{k=p}^{+\infty}$ is a nondecreasing sequence and satisfying $1 = a_p \le a_k$ for all $k \ge p$. Also, for $\delta < 0$, the sequence $(a_k)_{k=p}^{+\infty}$ is a decreasing sequence and satisfying $a_k \le a_p = 1$ for all $k \ge p$. □

Lemma 9. *If $0 \le \delta < p+1$, then the sequence $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ defined in (18) is an increasing sequence and satisfying $1 \le \Psi^{\delta,m}_{q,\lambda,l,p}(k)$ for all $k \ge p$.*

Proof. The result is directly obtained by Lemma 8 and the following inequality:

$$1 \leq \frac{[p]_q + \left([k]_q - [p]_q\right)\lambda + l}{[p]_q + l}, \quad (39)$$

where $l, \lambda \geq 0$ and $k \geq p$. □

4.1. Distortion Inequalities. Next, we derive the distortion inequalities for functions in the multivalently analytic functions class $\mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ that will be given by the following results.

Theorem 10. *For $0 \leq \delta < p+1$, suppose that $f \in \mathcal{T}_p$ is defined by (2). If $f \in \mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then*

$$\begin{aligned} |f(z)| &\leq |z|^p + \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)} |z|^{p+1}, \\ |f(z)| &\geq |z|^p - \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)} |z|^{p+1}. \end{aligned} \quad (40)$$

Generally,

$$\begin{aligned} &\left|D^n_q f(z)\right| \\ &\quad \leq \frac{[p]_q!}{[p-n]_q!} |z|^{p-n} \\ &\qquad + [p]_q \frac{[p]_q!}{[p-n+1]_q!} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)} |z|^{p-n+1}, \end{aligned} \quad (41)$$

$$\begin{aligned} &\left|D^n_q f(z)\right| \\ &\quad \geq \frac{[p]_q!}{[p-n]_q!} |z|^{p-n} \\ &\qquad - [p]_q \frac{[p]_q!}{[p-n+1]_q!} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)} |z|^{p-n+1}, \end{aligned} \quad (42)$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. The estimations in (40)–(42) are sharp.

Proof. Let the function $f \in \mathcal{T}_p$ be defined by (2) and belong to the class $\mathcal{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. In virtue of Theorem 6 and Lemma 9, we have

$$\begin{aligned} &\frac{[p+1]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(p+1) \sum_{k=p+1}^{\infty} |a_k| \\ &\quad \leq \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda}(k) |a_k| \leq 1-\alpha. \end{aligned} \quad (43)$$

From (43), the consequence is that

$$\sum_{k=p+1}^{\infty} |a_k| \leq (1-\alpha) \frac{[p]_q}{[p+1]_q} \frac{1}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)}. \quad (44)$$

Since $f \in \mathcal{T}_p$, it is easy to see that

$$\begin{aligned} &|z|^p - |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \\ &\quad \leq |f(z)| \leq |z|^p + |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k|. \end{aligned} \quad (45)$$

The conjunction of (44) and (45) yields assertions (40) of Theorem 10. Hence, (41) and (42) follow from

$$\begin{aligned} &\left|D^n_q f(z)\right| \\ &\quad \leq \frac{[p]_q!}{[p-n]_q!} |z|^{p-n} + \frac{[p+1]_q!}{[p-n+1]_q!} |z|^{p-n+1} \sum_{k=p+1}^{\infty} |a_k|, \\ &\left|D^n_q f(z)\right| \\ &\quad \geq \frac{[p]_q!}{[p-n]_q!} |z|^{p-n} - \frac{[p+1]_q!}{[p-n+1]_q!} |z|^{p-n+1} \sum_{k=p+1}^{\infty} |a_k|. \end{aligned} \quad (46)$$

Finally, we note that assertions (40)–(42) are sharp, since equalities are attained by the function

$$f(z) = z^p - \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)} z^{p+1}. \quad (47)$$

Now, the proof is completed. □

Remark 11. By letting $z \to 1^-$, Theorem 10 demonstrates that the disk $|z| < 1$ is mapped onto a domain that contains the disk

$$|w| < 1 - \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda,l,p}(p+1)}, \quad (48)$$

under any multivalently analytic function $f \in \mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, and onto a domain that contains the disk

$$|w| < 1 - (1-\alpha) \frac{1-q^p}{1-q^{p+1}}, \quad (49)$$

by any $f \in \mathcal{T}\mathcal{R}_{q,p}(\alpha)$.

4.2. Inclusion Relation. In the following results, we obtain some inclusion relation for the parameters m, λ, and l of the class $\mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$.

Theorem 12. *If $0 \leq l_1 \leq l_2$ and $0 \leq \lambda_1 \leq \lambda_2$, then*

$$\mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda_2,l_2,p}(\alpha) \subset \mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda_1,l_1,p}(\alpha), \quad (50)$$

and $\mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda_2,l_2,p}(\alpha) \neq \mathcal{T}\mathcal{R}^{\delta,m}_{q,\lambda_1,l_1,p}(\alpha)$ if those parameters satisfy either $l_1 \neq l_2$ or $\lambda_1 \neq \lambda_2$.

Proof. The inclusion relation is directly obtained by Theorem 6 and the inequality

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda_1,l_1,p}(k) |a_k| \leq \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda_2,l_2,p}(k) |a_k|. \quad (51)$$

In case $l_1 \neq l_2$ or $\lambda_1 \neq \lambda_2$, we see that

$$f_0(z) = z^p - \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha)}{\Psi^{\delta,m}_{q,\lambda_1,l_1,p}(p+1)} z^{p+1} \quad (52)$$

belongs to the class $\mathcal{TR}^{\delta,m}_{q,\lambda_1,l_1,p}(\alpha)$ but does not belong to the class $\mathcal{TR}^{\delta,m}_{q,\lambda_2,l_2,p}(\alpha)$, which implies that $\mathcal{TR}^{\delta,m}_{q,\lambda_2,l_2,p}(\alpha) \neq \mathcal{TR}^{\delta,m}_{q,\lambda_1,l_1,p}(\alpha)$. Now, the proof is completed. □

Applying Theorem 6 and Lemma 9, we obtain another inclusion relation as follows.

Theorem 13. *If $0 \leq \delta < p+1$, then*

$$\mathcal{TR}^{\delta,m+1}_{q,\lambda,l,p}(\alpha) \subset \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha). \quad (53)$$

4.3. Extreme Points. Now, let us determine extreme points of the class $\mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$.

Theorem 14 (extreme points). *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{[p]_q (1-\alpha)}{[k]_q \Psi^{\delta,m}_{q,\lambda,l,p}(k)} z^k, \quad k \geq p+1. \quad (54)$$

Then $f(z)$ is in the class $\mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \mu_p z^p + \sum_{k=p+1}^{\infty} \mu_k f_k(z), \quad (55)$$

where $\mu_k \geq 0$ and $\sum_{k=p}^{\infty} \mu_k = 1$.

Proof. Let the function $f(z) \in \mathcal{T}_p$ be defined by (2). Since $\sum_{k=p+1}^{\infty} \mu_k = 1$, we then have

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \mu_k \frac{[p]_q (1-\alpha)}{[k]_q \Psi^{\delta,m}_{q,\lambda,l,p}(k)} z^k. \quad (56)$$

Now, we obtain

$$(1-\alpha) \sum_{k=p+1}^{\infty} \frac{[k]_q \Psi^{\delta,m}_{q,\lambda,l,p}(k)}{[p]_q (1-\alpha)} \mu_k \frac{[p]_q (1-\alpha)}{[k]_q \Psi^{\delta,m}_{q,\lambda,l,p}(k)} = (1-\alpha) \sum_{k=p+1}^{\infty} \mu_k \leq 1-\alpha. \quad (57)$$

Thus, $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ by Theorem 6. Conversely, suppose that $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. We may set

$$\mu_k = \frac{[k]_q}{[p]_q} \frac{\Psi^{\delta,m}_{q,\lambda,l,p}(k)}{(1-\alpha)} |a_k|, \quad k \geq p+1 \quad (58)$$

and $\mu_p = 1 - \sum_{k=p+1}^{\infty} \mu_k$. Then we have $f(z) = \mu_p z^p - \sum_{k=p+1}^{\infty} \mu_k f_k(z)$. This completes the proof of Theorem 14. □

4.4. Radii of Generalized Close-to-Convexity, Starlikeness, and Convexity. Now, the discussions on radii of generalized close-to-convexity, starlikeness, and convexity for the class $\mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ are given by the following results. In order to establish, we will also require the use of those classes of functions. First of all, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}^*_{q,p}(\beta)$ of p-valently starlike with respect to q-differentiation of order β $(0 \leq \beta < p)$ if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{zD_q(f(z))}{f(z)}\right\} > \beta, \quad z \in \mathbb{D}. \quad (59)$$

Furthermore, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_{q,p}(\beta)$ of p-valently convex with respect to q-differentiation of order β $(0 \leq \beta < p)$ if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{zD_q^2(f(z))}{D_q(f(z))}\right\} > \beta, \quad z \in \mathbb{D}. \quad (60)$$

Both $\mathcal{S}^*_{q,p}(\beta)$ and $\mathcal{C}_{q,p}(\beta)$ were introduced by Selvakumaran et al. [8]. However, we consider the case $0 \leq \beta < [p]_q$ instead of $0 \leq \beta < p$. The definition of q-analogous of p-valently closed-to-convex was already recalled in (19).

Theorem 15. *For $0 \leq \delta < p+1$, if $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then f is p-valently closed-to-convex with respect to q-differentiation of order α.*

Proof. By Theorem 13, we obtain

$$\mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha) \subset \mathcal{TR}^{\delta,m-1}_{q,\lambda,l,p}(\alpha) \subset \cdots \subset \mathcal{TR}_{q,p}(\alpha). \quad (61)$$

This completes the proof. □

In general, for $0 \leq \alpha < \beta < 1$, the function $f \in \mathcal{TR}_{q,p}(\alpha)$ does not necessarily belong to the class $\mathcal{TR}_{q,p}(\beta)$. We then derive the radii of generalized close-to-convexity order $0 \leq \alpha < \beta < 1$ for the function $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$.

Theorem 16. *For $0 \leq \delta < p+1$, if $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then f is p-valently closed-to-convex with respect to q-differentiation of order β in $|z| < r_1$, where*

$$r_1 = \inf_{k \geq p+1} \left(\frac{1-\beta}{1-\alpha} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right)^{1/(k-p)}. \quad (62)$$

Proof. It is sufficient to show that $|D_q f(z)/([p]_q z^p - 1) - 1| < 1 - \beta$. That is,

$$\left|\frac{D_q f(z)}{[p]_q z^{p-1}} - 1\right| \le \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} |a|_k |z|^{k-p} \le 1 - \beta. \quad (63)$$

Since $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ and by application of Theorem 6, we obtain

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) |a|_k \le 1 - \alpha. \quad (64)$$

Hence, (63) is true if

$$|z| \le \left(\frac{1-\beta}{1-\alpha} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right)^{1/(k-p)}, \quad k \ge p+1. \quad (65)$$

This completes the proof. □

Next, we obtain the radii of generalized starlikeness of order β in the following result.

Theorem 17. *For $0 \le \delta < p+1$, if $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then f is p-valently starlike with respect to q-differentiation of order β $(0 \le \beta < [p]_q)$ in $|z| < r_2$ where*

$$r_2 = \inf_{k \ge p+1} \left[\frac{1}{1-\alpha} \cdot \frac{[k]_q ([p]_q - \beta)}{[p]_q ([k]_q - \beta)} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right]^{1/(k-p)}. \quad (66)$$

Proof. We have to show that $|zD_q f(z)/f(z) - [p]_q| < [p]_q - \beta$. That is,

$$\begin{aligned}\left|\frac{zD_q f(z)}{f(z)} - [p]_q\right| &= \left|\frac{z\left([p]_q z^{p-1} - \sum_{k=p+1}^{\infty} [k]_q |a_k| z^{k-1}\right)}{z^p - \sum_{k=p+1}^{\infty} |a_k| z^k} - [p]_q\right| \quad (67)\\ &\le \frac{\sum_{k=p+1}^{\infty} ([k]_q - [p]_q) |a_k| |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} |a_k| |z|^{k-p}} \le [p]_q - \beta.\end{aligned}$$

Hence, (67) is true if

$$\sum_{k=p+1}^{\infty} ([k]_q - \beta) |a_k| |z|^{k-p} \le [p]_q - \beta. \quad (68)$$

By using (64), we can say (68) is true if

$$|z| < \left[\frac{1}{1-\alpha} \frac{[k]_q ([p]_q - \beta)}{[p]_q ([k]_q - \beta)} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right]^{1/(k-p)}, \quad k \ge p+1, \quad (69)$$

which completes the proof. □

Corollary 18. *If $0 \le \delta < p+1$, then*

$$\mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha) \subset \mathcal{S}^*_{q,p}. \quad (70)$$

Proof. By Theorem 17, we see that a function $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ is p-valently starlike with respect to q-differentiation $(\beta = 0)$ in $|z| < r_2$ where

$$\begin{aligned} r_2 &= \inf_{k \ge p+1} \left(\frac{1}{1-\alpha} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right)^{1/(k-p)} \\ &\ge \inf_{k \ge p+1} \left(\frac{1}{1-\alpha}\right)^{1/(k-p)} \cdot \inf_{k \ge p+1} \left(\Psi^{\delta,m}_{q,\lambda,l,p}(k)\right)^{1/(k-p)}. \end{aligned} \quad (71)$$

It is easy to see that $(1/(1-\alpha))^{1/(k-p)}$ is a decreasing sequence and $\lim_{k \to +\infty} (1/(1-\alpha))^{1/(k-p)} = 1$. This implies

$$\inf_{k \ge p+1} \left(\frac{1}{1-\alpha}\right)^{1/(k-p)} = \lim_{k \to +\infty} \left(\frac{1}{1-\alpha}\right)^{1/(k-p)} = 1. \quad (72)$$

Moreover, by using Lemma 9, we obtain

$$\inf_{k \ge p+1} \left(\Psi^{\delta,m}_{q,\lambda,l,p}(k)\right)^{1/(k-p)} \ge 1. \quad (73)$$

Then, we have $r_2 \ge 1$. That is, $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ is p-valently starlike with respect to q-differentiation in $z \in \mathbb{D}$. The proof is completed. □

Next, we obtain the radii of generalized convexity of order β, where $\beta \le 2[p]_q - [p-1]_q - 1$, in the following result.

Theorem 19. *For $0 \le \delta < p+1$ and $\beta \le 2[p]_q - [p-1]_q - 1$, if $f \in \mathcal{TR}^{\delta,m}_{q,\lambda,l,p}(\alpha)$, then f is p-valently convex with respect to q-differentiation of order β $(0 \le \beta < [p]_q)$ in $|z| < r_3$, where*

$$\begin{aligned} r_3 &= \inf_{k \ge p+1} \left[\frac{1}{1-\alpha} \cdot \frac{(2[p]_q - [p-1]_q - \beta - 1)}{(1 + [k-1]_q - \beta)} \Psi^{\delta,m}_{q,\lambda,l,p}(k)\right]^{1/(k-p)}. \end{aligned} \quad (74)$$

Proof. We have to show that $|1 + zD_q^2 f(z)/D_q f(z) - [p]_q| < [p]_q - \beta$. That is,

$$\begin{aligned}\left|1 + \frac{zD_q^2 f(z)}{D_q f(z)} - [p]_q\right| = \Bigg|1 + z\Bigg(& [p]_q [p-1]_q z^{p-2} \\ & - \sum_{k=p+1}^{\infty} [k]_q [k+1]_q |a_k| z^{k-2}\Bigg)\end{aligned}$$

$$\cdot \left([p]_q z^{p-1} - \sum_{k=p+1}^{\infty} [k]_q |a_k| z^{k-1} \right)^{-1}$$
$$- [p]_q \Bigg|$$
$$\le \left([p]_q \left(1 + [p-1]_q - [p]_q\right) + \sum_{k=p+1}^{\infty} [k]_q \left(1 + [k-1]_q - [p]_q\right) |a_k| |z|^{k-p} \right) \cdot \left([p]_q - \sum_{k=p+1}^{\infty} [k]_q |a_k| |z|^{k-p} \right)^{-1}$$
$$\le [p]_q - \beta. \tag{75}$$

Hence, (75) is true if

$$[p]_q \left(1 + [p-1]_q - [p]_q\right) + \sum_{k=p+1}^{\infty} [k]_q \left(1 + [k-1]_q - [p]_q\right) |a_k| |z|^{k-p} \le \left([p]_q - \beta\right) \left[[p]_q - \sum_{k=p+1}^{\infty} [k]_q |a_k| |z|^{k-p} \right], \tag{76}$$

or equivalently

$$\sum_{k=p+1}^{\infty} [k]_q \left(1 + [k-1]_q - \beta\right) |a_k| |z|^{k-p} \le [p]_q \left(2[p]_q - [p-1]_q - \beta - 1\right). \tag{77}$$

Since $\beta \le 2[p]_q - [p-1]_q - 1$, by using (64), we can say (77) is true if

$$|z| < \left[\frac{1}{1-\alpha} \frac{\left(2[p]_q - [p-1]_q - \beta - 1\right)}{\left(1 + [k-1]_q - \beta\right)} \Psi_{q,\lambda,l,p}^{\delta,m}(k) \right]^{1/(k-p)}, \quad k \ge p+1. \tag{78}$$

The proof is completed. □

4.5. Quasi-Hadamard Properties. In this section, we derive the quasi-Hadamard (convolution) properties. Before we derive the result, we recall the definition of the quasi-Hadamard properties. For any functions $f_j \in \mathcal{T}_p$, $j = 1, 2, 3, \ldots, n$ of the form

$$f_j(z) = z^p - \sum_{k=p+1}^{\infty} \left|a_{k,j}\right| z^k, \tag{79}$$

the quasi-Hadamard product $(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z)$ is defined by

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z) = z^p - \sum_{k=p+1}^{\infty} \left(\prod_{j=1}^{n} \left|a_{k,j}\right| \right) z^k, \quad z \in \mathbb{D}. \tag{80}$$

Next, we derive the quasi-Hadamard properties for the class $\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$. Using the techniques of Schild and Silverman [41] with Theorem 6, we prove the following results.

Theorem 20. *For $0 \le \delta < p+1$, suppose that $f_j \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha_j)$, $j = 1, 2, \ldots, n$; then $(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\gamma)$, where*

$$\gamma = 1 - \left[\frac{[p]_q}{[p+1]_q \Psi_{q,\lambda,l,p}^{\delta,m}(p+1)} \right]^{n-1} \prod_{j=1}^{n} \left(1 - \alpha_j\right). \tag{81}$$

Proof. To prove this theorem, we use the principle of mathematical induction on j. Let the functions $f_j \in \mathcal{T}_p$, for $j = 1, 2$ of the form

$$f_j(z) = z^p - \sum_{k=p+1}^{\infty} \left|a_{k,j}\right| z^k, \tag{82}$$

for $j = 1, 2$ and $k \ge 2$. Since $f_j \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha_j)$ for $j = 1, 2$, by Theorem 6, we see that

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \frac{\Psi_{q,\lambda,l,p}^{\delta,m}(k)}{1-\alpha_j} a_{k,j} \le 1, \quad \text{for } j = 1, 2. \tag{83}$$

According to Theorem 6, it is sufficient to prove that

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \frac{\Psi_{q,\lambda,l,p}^{\delta,m}(k)}{1-\gamma} a_{k,1} a_{k,2} \le 1, \tag{84}$$

where γ is defined in (81). Applying Cauchy-Schwarz's inequality to (83) for $j = 1, 2$, we have the following inequality:

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \frac{\Psi_{q,\lambda,l,p}^{\delta,m}(k)}{\sqrt{(1-\alpha_1)(1-\alpha_2)}} \sqrt{a_{k,1} a_{k,2}} \le \sqrt{\sum_{k=2}^{\infty} \frac{[k]_q}{[p]_q} \frac{\Psi_{q,\lambda,l,p}^{\delta,m}(k)}{1-\alpha_1} a_{k,1}} \sqrt{\sum_{k=2}^{\infty} \frac{[k]_q}{[p]_q} \frac{\Psi_{q,\lambda,l,p}^{\delta,m}(k)}{1-\alpha_2} a_{k,2}} \le 1. \tag{85}$$

From (84) and (85), if the following inequality

$$\frac{\sqrt{(1-\alpha_1)(1-\alpha_2)}}{1-\gamma} \sqrt{a_{k,1} a_{k,2}} \le 1, \tag{86}$$

for all $k \geq p+1$, is satisfied, it can be concluded that $(f_1 \otimes f_2) \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\gamma)$. Now, applying Corollary 7, inequality (86) will be held if

$$\frac{[p]_q}{[k]_q} \frac{(1-\alpha_1)(1-\alpha_2)}{\Psi_{q,\lambda,l,p}^{\delta,m}(k)} \leq 1-\gamma. \tag{87}$$

By Lemma 9, we see that

$$\frac{[p]_q}{[k]_q} \frac{(1-\alpha_1)(1-\alpha_2)}{\Psi_{q,\lambda,l,p}^{\delta,m}(k)} \leq \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha_1)(1-\alpha_2)}{\Psi_{q,\lambda,l,p}^{\delta,m}(p+1)} := 1-\gamma. \tag{88}$$

This yields our desired inequality (86). Now, we have $(f_1 \otimes f_2) \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\gamma)$. Next, we let the functions $f_j \in \mathcal{T}_p$, for $j = 1, 2, \ldots, N+1$ and $f_j \in \mathcal{TR}_{q,\lambda}^{\delta,m}(\alpha_j)$ for $j = 1, 2, \ldots, N+1$. Suppose that

$$f_1 \otimes f_2 \otimes \cdots \otimes f_N \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\gamma), \tag{89}$$

where

$$\gamma := 1 - \left[\frac{[p]_q}{[p+1]_q \Psi_{q,\lambda,l,p}^{\delta,m}(p+1)}\right]^{N-1} \prod_{j=1}^{N}(1-\alpha_j). \tag{90}$$

Then, by means of the above technique, it can be shown that

$$f_1 \otimes f_2 \otimes \cdots \otimes f_N \otimes f_{N+1} \in \mathcal{TR}_{q,\lambda}^{\delta,m}(\gamma'), \tag{91}$$

where

$$\begin{aligned} \gamma' &:= 1 - \frac{[p]_q}{[p+1]_q} \frac{(1-\alpha_{N+1})(1-\gamma)}{\Psi_{q,\lambda,l,p}^{\delta,m}(p+1)} \\ &= 1 - \left[\frac{[p]_q}{[p+1]_q \Psi_{q,\lambda,l,p}^{\delta,m}(p+1)}\right]^{N} \prod_{j=1}^{N+1}(1-\alpha_j). \end{aligned} \tag{92}$$

By Lemma 9, we have $0 < \gamma < 1$. This completes the proof of the theorem. □

4.6. Invariant Properties. In the following results, we discuss invariant properties of the class $\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$ via Theorem 6. We consider the formerly studied operators in terms of the standard convolution formula; we choose g as a fixed function in $\mathcal{A}_p$ such that $(f * g)(z)$ exists for any $f \in \mathcal{A}_p$. For various choices of g we get different linear operators that have been studied in the recent past.

According to Theorem 6, we easily obtain the following properties.

Theorem 21. *For $\delta < p+1$, if the function $g \in \mathcal{A}_p$ is of the form*

$$g(z) = z^p + \sum_{k=p+1}^{\infty} |\mu_k| z^k, \tag{93}$$

where $|\mu_k| \leq 1$ for $k \geq p+1$, then $(f * g) \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$ for each $f \in \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$.

Next, we recall the definition of Bernardi's integral operator. For nonnegative real number γ and $f \in \mathcal{A}_p$, Bernardi-Libera's integral operator $L_\gamma f : \mathcal{A}_p \to \mathcal{A}_p$ is defined as follows:

$$L_\gamma f(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\, dt, \tag{94}$$

which was studied by Bernardi in [42]. Also, their properties for $\gamma = 1$ are reported in [43, 44]. By using the concept of q-calculus, we introduce q-analogous to Bernardi's integral operator defined by

$$L_{q,\gamma} f(z) = \frac{[\gamma+1]_q}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\, d_q t. \tag{95}$$

From (95), we have verified that

$$\begin{aligned} L_{q,\gamma} f(z) &= \frac{[\gamma+p]_q}{z^\gamma} z(1-q) \sum_{j=0}^{\infty} q^j (zq^j)^{\gamma-1} f(zq^j) \\ &= [\gamma+p]_q (1-q) \sum_{j=0}^{\infty} q^{j\gamma} f(zq^j) \\ &= [\gamma+p]_q (1-q) \sum_{j=0}^{\infty} q^{j\gamma} \sum_{k=p}^{\infty} q^{jk} a_k z^k, \quad a_p = 1 \\ &= [\gamma+p]_q \sum_{k=p}^{\infty}\sum_{j=0}^{\infty} (1-q) q^{j(\gamma+k)} a_k z^k \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{[\gamma+p]_q}{[\gamma+k]_q} a_k z^k. \end{aligned} \tag{96}$$

That is,

$$L_{q,\gamma} f(z) := (f * g)(z) = z^p + \sum_{k=p+1}^{\infty} \frac{[\gamma+p]_q}{[\gamma+k]_q} a_k z^k, \tag{97}$$

where $g = z^p + \sum_{k=p+1}^{\infty}([\gamma+p]_q/[\gamma+k]_q) z^k$. It is clear that $[\gamma+p]_q/[\gamma+k]_q \leq 1$ for $k \geq p+1$. Then, we obtain the invariant properties under integral operator $L_{q,\gamma}$ as follows.

Theorem 22. *For $\delta < p+1$, and $\gamma > 0$, the class $\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$ is invariant under the integral operator $L_{q,\gamma}$ defined in (95). That is,*

$$L_{q,\gamma}\left[\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)\right] \subset \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha). \tag{98}$$

Moreover, in the view of the definition of fractional q-integral and Lemma 8, we obtain the invariant properties under fractional q-integral.

Theorem 23. *For $\delta < p+1$, the class $\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)$ is invariant under the integral operator $I_{q,z}^{\delta}$ defined in (11). That is,*

$$I_{q,z}^{\delta}\left[\mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha)\right] \subset \mathcal{TR}_{q,\lambda,l,p}^{\delta,m}(\alpha). \tag{99}$$

5. Observation and Concluding Remark

In this section we briefly point out some consequences of the results derived in the proceeding sections. If we let $q \to 1^-$, we observe that the function classes $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ and $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ provide the q-extensions of both known and newly obtained. By assigning appropriated values to the parameters m, l, γ, δ, and p, we can derive the corresponding results for several simpler subclasses of the class $\mathscr{T}_p$ from each of our theorems, especially as indicated in Altintas [40]. Therefore, it leads to the q-extension of the former results.

Furthermore, we let $f \in \mathscr{T}_{n,p}$, where $\mathscr{T}_{n,p}$ is a subclass of $\mathscr{T}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} |a_k| z^k. \tag{100}$$

Theorem 6 can indeed be generalized further by considering the class of multivalent function $\mathscr{T}_{n,p}$ in place of $\mathscr{T}_p$. That leads to the following corollary.

Corollary 24. *Let $f \in \mathscr{T}_{n,p}$ be defined by (100); then $f \in \mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ if and only if f satisfies the inequality*

$$\sum_{k=n+p}^{\infty} \frac{[k]_q}{[p]_q} \Psi^{\delta,m}_{q,\lambda,l,p}(k) |a_k| \le 1 - \alpha, \tag{101}$$

where $\Psi^{\delta,m}_{q,\lambda,l,p}(k)$ is defined in (18). Moreover, the result is sharp.

By previous argument in this paper, Corollary 24 can fruitfully be used in investigating the geometric properties for several subclasses of $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha) \cap \mathscr{T}_{n,p}$. Then the q-extension of the former results, such as Sarangi and Uralegaddi [45], Aouf et al. [46], and Aouf [47], is obtained.

We conclude this paper by remarking that the results presented in this paper give various q-extension properties of different classes of analytic and multivalent function. Here, our results generalize several formerly known results. We introduce q-extension of the general differential operator, which is generalized from Bulut operator [17], in sense of q-theory. The new subclass of multivalently analytic function is proposed consequently by joining the class of q-analogue to close-to-convexity together with our q-extension of Bulut operator [17]. We discuss the linear combination property and coefficient estimate for the class $\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. By making use of the coefficient estimate and the concept of q-theory, we obtain the q-extension of geometric properties for the class $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$. We give the q-analogue to distortion properties and the radii of q-analogue starlikeness and convexity which were defined in [8]. Moreover, we consider the radii of q-analogue close-to-convexity that replace usual derivative by q-derivative operator. We also use the concept of q-theory to extend the Bernardi integral operator. As a consequence, the invariant property for $\mathscr{T}\mathscr{R}^{\delta,m}_{q,\lambda,l,p}(\alpha)$ under such q-integral operator is obtained. Finally, the presented results can be extended to investigate the function on $\mathscr{T}_{n,p}$ which gives generalized formerly known and newly obtained results.

Conflict of Interests

The authors declare that they have no conflict of interests.

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