Combinatorial and Computational Geometry
MSRI Publications
Volume **52**, 2005

Applications of Graph and Hypergraph Theory in Geometry

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Abstract. The aim of this survey is to collect and explain some geometric results whose proof uses graph or hypergraph theory. No attempt has been made to give a complete list of such results. We rather focus on typical and recent examples showing the power and limitations of the method. The topics covered include forbidden configurations, geometric constructions, saturated hypergraphs in geometry, independent sets in graphs, the regularity lemma, and VC-dimension.

1. Introduction

Among n distinct points in the plane the unit distance occurs at most $O(n^{3/2})$ times. The proof of this fact uses two things. The first is a theorem from graph theory saying that a graph on n vertices containing no $K_{2,3}$ can have at most $O(n^{3/2})$ edges. The second is a simple fact from plane geometry: the unit distance graph contains no $K_{2,3}$.

This is the first application of graph theory in geometry, and is contained in a short and extremely influential paper of Paul Erdős [1946]. The first application of hypergraph theory in geometry is even earlier: it is the use of Ramsey's theorem in the famous Erdős and Szekeres result from 1935 (see below in the next section). Actually, Erdős and Szekeres proved Ramsey's theorem (without knowing it had been proved earlier) since they needed it for the geometric result.

The aim of this survey is to collect and explain some geometric results whose proof uses graph or hypergraph theory. Such applications vary in depth and difficulty. Often a very simple geometric statement adds an extra condition to the combinatorial structure at hand, which helps in the proof. At other times, the geometry is not so simple but is dictated by the combinatorics of the objects in question.

I do not attempt to give a complete list of such results, but rather concentrate on typical or recent examples showing the power and limitations of such methods. Instead of presenting complete proofs I have tried to give a sketch emphasizing the interaction between geometry and (hyper)graph theory. To fill in the details the reader is advised to consult the original papers and the excellent books [Matoušek 2002] and [Pach and Agarwal 1995]. Although I've tried to incorporate every important result, the choice of material, of course, reflects my personal preferences. Also, several further examples could have been included: the Lovász Local Lemma, discrepancy results, planar graphs and geometric graphs, etc. But in these cases I felt that either the method is more probabilistic than combinatorial, or the question is not so much geometric.

Some remarks on notation are in place here: b, c, c_i, C denote different constants. The $O(\ )$ and $o(\ )$ notation is often used. $K_{n,m}$ denotes the complete bipartite graph with classes of size n and m. $K^k(t)$ stands for the complete k-partite k-uniform hypergraph with t vertices in each class. The set $\{1, 2, \ldots, n\}$ will be denoted simply by $[n]$. A graph is denoted by $G = (V, E)$ where V is the set of vertices, and E the set of edges. The independence number $\alpha(G)$ of a graph G is the maximum size independent set in G, and a subset $W \subset V$ is independent if there are no edges between vertices of W. A hypergraph, or set system, is usually denoted by H, its ground set (or vertex set) by V, its (hyper)edges are $e \in H$, or sometimes $E \in H$. A transversal of H is a set $T \subset V$ intersecting every edge in H.

2. Forbidden Configurations

This method is typically used for counting geometric objects. It is usually based on a simple geometric fact (showing that some configuration cannot occur) combined with a graph or hypergraph theorem saying that, if certain configuration is forbidden, then the number of edges is bounded. The case of the unit distance graph in the introduction illustrates the method quite clearly; this section gives a few more examples. We mention in passing that the unit distance problem is still wide open: the maximal number of unit distances among n points is somewhere between $n^{1+(c/\ln\ln n)}$ and $cn^{4/3}$.

The first example is counting point-line incidences: Given a set of lines, L, and a set of points, P, both of them finite, how many incidences can there be? We only assume that two lines have at most one point in common and there is at most one line passing through two points. (So we are not working in the Euclidean plane.) The setting immediately defines a bipartite graph with bipartition classes L and P, with $(\ell, p) \in L \times P$ forming an edge if they are incident. This is a bipartite graph containing no $K_{2,2}$. Then a theorem of Kővári, T. Sós, and Turán [Kővári et al. 1954] applies. We state the result for the case when $|L| = |P| = n$: such a graph has at most

$$\frac{n}{2}(1 + \sqrt{4n - 3})$$

edges. This bound is asymptotically tight: the example of the projective plane of order q (where q is a prime power) shows $n = q^2 + q + 1$ points and the same number of lines while the number of incidences is exactly

$$(q^2+q+1)(q+1)=\frac{n}{2}(1+\sqrt{4n-3}).$$

A miracle has happened: of the whole point-line structure, only the bipartiteness and the forbidden subgraph $K_{2,2}$ are needed to obtain the exact bound. It is worth mentioning that while this exact bound follows from the forbidden subgraph theorem [Kővári et al. 1954], the sharpness of the forbidden subgraph theorem is implied by the example of the projective plane. So geometry pays back its due to combinatorics.

Remark. The situation is different when the points and lines belong to the Euclidean plane (cf. the Szemerédi–Trotter theorem [1983]) but there, the structure is richer. The actual bound is $O(|P|^{2/3}|L|^{2/3}+|P|+|L|)$ which is tight apart from the implied constant. There are several proofs available now: the simplest is by L. Székely [1997] based on the crossing lemma. The above forbidden subgraph argument, combined with the so-called cutting lemma, also provides a nice proof, for details see [Matoušek 2002].

Remark. The original motivation for bounding the number of edges in a (bipartite) graph with no $K_{2,2}$ comes from number theory, see [Erdős 1938]. Erdős proves the weaker bound $3n^{3/2}$ on the number of edges but gives the example of the finite projective plane (in disguise) to show that the bound is quite good.

Examples of this type abound. Here is a less well known one due to Turán [1970].

THEOREM 1. *If $X\subset\mathbb{R}^2$ has n elements and is of diameter one, then there are at least $n^2/6-O(n)$ pairs $x,y\in X$ whose distance is at most $1/\sqrt{2}$.*

The proof is simple. First a little geometry: Among any four points of X there are two that are at distance $1/\sqrt{2}$ or closer. (One cannot give a bound smaller than $1/\sqrt{2}$: see the square of diameter one.) So the graph $G(X,E)$, whose edges are the pairs with distance larger than $1/\sqrt{2}$, contains no K_4. By Turán's theorem [1941] the complementary graph has at least $n^2/6-O(n)$ edges. This proof also indicates which set of n points shows that the bound $n^2/6-O(n)$ is tight.

The classical Erdős–Szekeres theorem [1935] uses, in its proof, a certain forbidden configuration. We say that n points in the plane are *in convex position* if they form the vertices of a convex n-gon. We now state the Erdős–Szekeres theorem:

THEOREM 2. *For every $n\geq 3$ there is $N=N(n)$ such that every point set $X\subset R^2$ in general position with $|X|\geq N$ contains a subset of size n that is in convex position.*

For the proof one checks that $N(4)=5$, that is, among 5 points in the plane there are 4 in convex position. Now set $N(n)=R_4(5,n)$, the Ramsey number, which means that in every red-blue colouring of all quadruples of an $R_4(5,n)$-set either there are 5 points whose all quadruples are red or there are n points whose all quadruples are blue. This number is finite (by the Ramsey theorem, [Ramsey 1930]). Now let $X \subset R^2$ contain N or more elements. Colour its quadruples in convex position Blue, and colour the rest Red. There are no 5 points whose all quadruples are Red (since $N(4) = 5$), so there are n points in X with all of their quadruples in convex position. It is very simple to see now that these n points are also in convex position. Here the forbidden configuration was 5 points with all of its quadruples nonconvex.

Our examples so far have shown forbidden subgraphs. Often other structures are forbidden. Here comes the beautiful case of *lower envelope of segments* in R^2. The setting is this: given n line segments in the plane, none of them vertical, what is the complexity of their lower envelope? That is, consider the segments as linear functions, each defined on some interval, take the pointwise minimum, f, of these functions. How many segments make up the graph of this minimum? The answer is $cn\alpha(n)$, where $\alpha(n)$ is a very slowly increasing function, the inverse of the Ackerman function. Without going into the details (which can be found in [Hart and Sharir 1986] and [Matoušek 2002]), I explain what kind of forbidden structure appears here.

Index the segments by $1,\dots,n$. The function $f(x)$ is piecewise linear. Assume $I_1, I_2, \dots, I_t$ are the intervals (in this order on the horizontal axis) where f is linear. (So we want to estimate t, the number segments on the graph of f.) Attach index i to the interval I_k if the graph of f coincides with the ith segment on I_k. Writing the various indexes, as they appear on the horizontal axis from left to right, we get a sequence $a_1, a_2, \dots, a_t$ of numbers from $[n]$ that has the following properties:

- $a_i \neq a_{i+1}$,
- there are no indices $i_1 < i_2 < i_3 < i_4 < i_5$ such that $a_{i_1} = a_{i_3} = a_{i_5} \neq a_{i_2} = a_{i_4}$.

Only the second property (saying that a, b, a, b, a cannot be a subsequence of our sequence) needs a proof, and we leave it to the reader. This is a *forbidden subsequence* condition. Sequences with these properties are called Davenport–Schinzel sequences of order 3. Determining the maximal length of such a sequence on $[n]$ had been an open problem from 1965 until Hart and Sharir [1986] proved, by combinatorial methods, that the maximal length is $O(n\alpha(n))$. That this bound is sharp was shown later (by Peter Shor; see [Matoušek 2002]). The ingenious construction gives n segments whose lower envelope has $cn\alpha(n)$ segments. Once again, combinatorics gives the upper bound in a geometric problem, and a geometric construction shows that this bound is precise.

Further examples of forbidden configurations can be found in the books [Pach and Agarwal 1995] and [Matoušek 2002].

3. Constructions

Any hypergraph H on n vertices gives rise, in a natural way, to a point set $X(H)$ in R^n. Simply represent each $S \in H$ by its characteristic vector $x(S)$ whose ith component is one if the ith element of the ground set is in S and is zero otherwise. This set $X(H)$ is, in fact, a subset of the vertices of the unit cube. The properties of the hypergraph are reflected in the properties of $X(H)$ and vice versa. This simple connection, combined with powerful results from extremal set theory, can have amazing results, like the counterexample to Borsuk's conjecture.

In 1933 Borsuk asked whether every set of diameter one in R^d can be partitioned into $d+1$ sets of diameter smaller than one. One may immediately assume that the sets in question are convex since taking convex hull does not increase the diameter. Among convex sets, the regular simplex and the unit ball can indeed be partitioned into $d+1$ sets of smaller diameter (but not into fewer sets). This had been known for smooth convex bodies as well (with a fairly simple proof), but for polytopes, despite many efforts, there had been no proof in sight. Then, in 1992, an ingenious construction was found by Kahn and Kalai [1993] showing that the conjecture is far from being true: the smallest number of sets in a suitable partition must be at least $2^{c\sqrt{d}}$ for some small positive c. Their construction is based on the following, equally beautiful, result of Frankl and Wilson [1981]:

THEOREM 3. *Let q be a prime power. Let F be a family of $2q$-subsets of $[4q]$ so that no two sets in F have intersection of size q. Then*

$$|F| \le 2\binom{4q-1}{q-1}.$$

How does one use this result to produce a counterexample? Consider the edges of the complete graph $K(V,E)$ whose vertex set is $V=[4q]$. For every partition $P=\{A,B\}$ of V let $S(A,B)$ be the set of edges connecting a vertex in A to one in B. Now define H to be the family of sets $S(A,B)$ where $|A|=|B|=2q$. So H is a $4q^2$-uniform hypergraph on the set E, $|E|=2q(4q-1)$, which gives rise to a point set $X(H)$ in $R^{|E|}$. As is easy to see, the smallest intersection between $S_1=S(A_1,B_1)\in H$ and $S_2=S(A_2,B_2)\in H$ occurs when $|A_1\cap A_2|=q$. It follows that the Euclidean distance between $x(S_1)$ and $x(S_2)$ is the largest when $|A_1\cap A_2|=q$. By the Frankl–Wilson theorem every subfamily of H with more than $2\binom{4q-1}{q-1}$ sets contains two sets, S_1 and S_2 with $|A_1\cap A_2|=q$. That is, when partitioning H into fewer than

$$h(q)=\frac{\frac{1}{2}\binom{4q}{2q}}{2\binom{4q-1}{q-1}}$$

subfamilies, one of them contains a pair S_1 and S_2 with $|A_1\cap A_2|=q$. The same applies to $X(H)$ which sits in $d=2q(4q-1)$-dimensional space: in any partition of $X(H)$ into fewer than $h(q)$ sets one of the sets has the same diameter as $X(H)$. It is easy to see that $h(q)$ grows faster than $1.2^{\sqrt{d}} > d+1$ if d is large enough. This is the first counterexample to Borsuk's conjecture. Several others with direct proofs and better estimates are available now. For a comprehensive survey, see [Raĭgorodskiĭ 2001].

The morale is that geometric intuition can be misleading in higher dimension. Taking convex hulls may not help at all and the discrete structure of the point set can be more important.

The Frankl–Wilson theorem has further geometric applications, many of them given in the original paper [Frankl and Wilson 1981]. They show for instance that the chromatic number, $g(d)$, of R^d is exponential: $g(d) > (1+o(1))1.2^d$. Here $g(d)$ is defined as the smallest number n such that R^d can be coloured by n colours so that no two points of the same colour are distance one apart. The question of estimating $g(2)$ and more generally $g(d)$ goes back to E. Nelson, J. Isbell, and P. Erdős; see [Hadwiger 1961]. Determining $g(d)$ has turned out to be hard. For instance, the value of $g(2)$ is known to be either 4,5,6, or 7, but which of these numbers it is remains a mystery, after 60 years. Larman and Rogers [1972] proved that $g(d) \leq 3^d$. This, together with the Frankl–Wilson theorem shows that the chromatic number of R^d is exponential in d.

Geometric intuition did not help in the following construction, which is based on extremal hypergraph theory. Danzer and Grünbaum [1962] showed that among 2^d+1 points in R^d there are three that form an acute triangle. (The proof is beautiful!) This raised the question to determine the smallest N such that among any set of N points in R^d, there are three that form an angle $\geq \pi/2$. It was conjectured that the smallest such N is $2d-1$. But this was soundly refuted by Erdős and Füredi [1983] with the following example, which is quite natural once you have seen it. Consider the vertices of the unit cube. Clearly, no angle is larger than $\pi/2$. Three vertices a, b, c give angle $\pi/2$ at b if and only if the vectors $a-b$ and $c-b$ are orthogonal. As a, b, c are 0-1 vectors, they are characteristic vectors of sets $A, B, C \subset [d]$. The condition $(a-b)(c-b) = 0$ translates directly to $A \cap C \subset B \subset A \cup C$. Thus the target is to construct a *large* family H of sets on the ground set $[d]$ with no three sets $A, B, C \in H$ satisfying $B \subset A \cup C$ (a slightly weaker yet sufficient condition). A quite natural random hypergraph with 1.13^d edges has this property. In the corresponding set in R^d, with 1.13^d points, all angles are smaller than $\pi/2$. For details see [Erdős and Füredi 1983], where the authors also prove, with similar methods, the existence of a set in R^d of size exponential in d such that all distances between two points of the set are between .99 and 1.01.

4. Saturated Hypergraphs

The saturated hypergraph theorem of Erdős and Simonovits [1983] says the following:

Theorem 4. *For every positive integer k and t and every $p > 0$ there exists $\delta > 0$ with the following property. Let H be a k-uniform hypergraph on n vertices and with at least $p\binom{n}{k}$ edges. Then H contains at least*

$$\lfloor \delta n^{kt} \rfloor$$

copies (not necessarily induced) of $K^k(t)$.

One way to remember the statement is to assume that H is a random k-uniform hypergraph with edge-probability p. Then the expected number of copies of $K^k(t)$ is $p^{t^k}\binom{n}{t,\dots,t} \geq \text{const}\, n^{kt}$. The saturated hypergraph theorem says that a hypergraph with positive edge density behaves like a "random hypergraph" of the same edge density. It is not surprising then that the proof of Theorem 4 goes by averaging.

This theorem is very useful when one has a family F of geometric objects and happens to know that a positive fraction of the k element subfamilies of F have a certain property, and one wants to show that, say, F has a large subfamily with some other property. Our example is the following point-selection theorem of Alon et al. [1992], a similar and earlier example is in [Bárány et al. 1990].

Theorem 5. *Let $X \subset \mathbb{R}^d$ be an n-point set and let F be a family of some $(d+1)$-tuples of X with $|F| = \alpha\binom{n}{d+1}$, where $\alpha \in (0,1]$. Then F contains a subfamily F' of size*

$$c_d \alpha^{s_d} \binom{n}{d+1}$$

(where $c_d > 0$ and s_d are constants) such that $\bigcap_{S \in F'} \operatorname{conv} S$ is nonempty.

In this theorem α may even depend on n, a case which is needed when bounding the number of halving hyperplanes of a given n-set in R^d (see [Bárány et al. 1990] and [Alon et al. 1992]).

What is the way of proving such a result? The first (geometric) idea is to use the fractional Helly theorem of [Katchalski and Liu 1979]. It says that if in a family of N convex sets (in R^d) a positive fraction of the $(d+1)$-tuples are intersecting, then the family has a large, cN size intersecting subfamily. So we call the convex hull of an edge in F a *simplex* of F, and try to show that a positive fraction of the $(d+1)$-tuples of the simplices of F are intersecting. Then comes the second (combinatorial) idea: F is a $(d+1)$-uniform hypergraph with positive edge density, thus the saturated hypergraph theorem stated above ensures that there are many copies of $K^{d+1}(t)$ for any fixed number t. So the next target is to prove that such a $K^{d+1}(t)$ contains $(d+1)$ vertex-disjoint simplices that intersect, provided t is large enough. Actually one has the freedom of choosing t as large as needed provided it depends only on d. Once this is proved, a routine double-counting argument shows that a positive fraction of the $(d+1)$-tuples of simplices of F are intersecting. So what remains to be shown is a geometric statement, called the Coloured Tverberg Theorem:

THEOREM 6. *Given pairwise disjoint sets $C_1, \dots, C_{d+1} \subset R^d$ each with $|C_i| = 4d+3$, there are pairwise disjoint sets $S_1, \dots, S_{d+1} \subset R^d$, each with $|S_j| = d+1$, such that $|C_i \cap S_j| = 1$ for all i, j and*

$$\bigcap_{i}^{d+1} \operatorname{conv} S_j \neq \varnothing.$$

Here the C_i are the classes (called colours) of $K^{d+1}(t)$, the convex hull of each edge of $K^{d+1}(t)$ is a simplex of F, and the S_j are what we are after: an intersecting $(d+1)$-tuple of pairwise vertex-disjoint simplices of F. The proof of this theorem, which is due to Živaljević and Vrećica [1992], is difficult and unusual since it is based on equivariant algebraic topology, although the statement is from convex geometry, or linear algebra, if you wish. In fact, all proofs for $d > 2$ use algebraic topology.

Another example of this kind is a lattice-point version of the fractional Helly theorem, due to Bárány and Matoušek [2003]. Assume that in a finite family F of convex sets in R^d the intersection of every $(d+1)$ sets contains a lattice point, i.e., a point all of whose coordinates are integral. Helly's theorem says that all the sets have a common point. But this may not be a lattice point: take, for instance, the convex hull of all but one vertices of the unit cube in R^d, this is one convex set for each (missing) vertex of the cube. They form a family F where every $2^d - 1$ sets share a lattice point, but $\bigcap F$ contains no lattice point whatsoever. However, it is known (see [Doignon 1973] or [Scarf 1977]) that the Helly number of lattice convex sets in R^d is 2^d, that is, if in a finite family F of convex sets in R^d every 2^d or fewer sets have a lattice point in common, then $\bigcap F$ contains a lattice point. In the given case this implies that the fractional Helly number of lattice convex sets in R^d is (at most) 2^d. (This fact is proved in [Alon et al. 2002].) So what is the precise value of this number? The answer is $d+1$:

THEOREM 7. *For every $d \geq 1$ and every $\alpha \in (0,1]$ there is a $\beta > 0$ with the following property. Let $K_1, \dots, K_N$ be convex sets in R^d such that $\bigcap_{i \in I} K_i$ contains a lattice point for at least $\alpha\binom{N}{d+1}$ index sets $I \subset [N]$ of size $(d+1)$. Then there is a lattice point common to at least βN sets among the K_i.*

In the proof the application of the saturated hypergraph theorem leads to what we call the coloured Helly theorem for convex lattice sets:

THEOREM 8. *For every integer d and r, there is an integer t such that the following holds. Assume that for each vertex v of $K^{d+1}(t)$ there is a convex set $K_v \in R^d$, such that for each edge e of $K^{d+1}(t)$, the intersection $\bigcap_{v \in e} K_v$ contains a lattice point. Then there is a set R, of size r, in one of the classes of $K^{d+1}(t)$ such that the intersection $\bigcap_{v \in R} K_v$ contains a lattice point.*

This is only needed for $r = 2^d$, but that does not seem to make any difference in the proof, which, besides using two distinct pieces of geometry, is technical, difficult, and combinatorial in nature. The method can be developed further and, when combined with the Alon–Kleitman technique [1992], it shows what can be saved from Helly's theorem when every $(d+1)$ of the sets have a lattice point in common:

THEOREM 9. *For every integer $d \geq 2$ there is an integer $H(d)$ such that the following holds. Let F be a finite family of convex sets in R^d. Assume that the intersection of every $(d+1)$ sets from F contains a lattice point. Then there is a set S of lattice points with $|S| \leq H(d)$ such that S intersects every set in F.*

For $d = 2$ this was proved by T. Hausel [1995] with $H(2) = 2$.

The applications of the saturated hypergraph theorem always lead to new, and often difficult, problems in geometry. In such problems the vertices of a $K^{d+1}(t)$ are some geometric objects, the objects in each edge satisfy a certain property, and one wants to find a special subfamily of these objects, like in Theorem 9 or in the Coloured Tverberg Theorem.

5. Independent Sets in Graphs

Given a graph $G(V, E)$ on n vertices and maximum degree d, the simplest possible greedy algorithm produces an independent set W of size $n/(d+1)$. (An equally simple random choice gives an independent set of size $n/4d$.) In a seminal paper [Ajtai et al. 1981], Ajtai, Komlós, and Szemerédi showed that this can be improved for triangle-free graphs: if G is triangle free, then

$$\alpha(G) \geq \frac{cn \log d}{d}$$

with some universal constant $c > 0$. Subsequently $c = 1 + o(1)$ was shown by Shearer [1983]. Here d is fixed and n goes to infinity. The original proof goes via sequential random choices, and the difficulty is to ensure that after each iteration, the remaining structure is still random, or behaves as if it were random. According to his coauthors, Szemerédi's philosophy, that random subgraphs of a graph behave very regularly, and his vision that such a proof should work, proved decisive. Since then, the method has been applied several times and with great success.

This lower bound on $\alpha(G)$ has the immediate corollary (see [Ajtai et al. 1980]) that the Ramsey number $R_2(n, 3)$ is $O(n^2/\ln n)$ which turned out to be the right order of magnitude (see [Kim 1995]). The result on $\alpha(G)$ has been generalized from triangle-free graphs to "locally sparse" graphs and hypergraphs in various ways. Locally sparse here means, for instance, that there are few edges connecting the neighbours of every vertex, or that two vertices don't have too many common neighbours. We are going to explain two such cases: the problems come from geometry and the solution, or the crucial step of the solution, from hypergraph theory.

The first concerns Heilbronn's conjecture which says that every set of N points in the unit disk B contains three points such that the triangle spanned by them has area less then $const/N^2$. In 1982 Komlós, Pintz, and Szemerédi [Komlós et al. 1982] constructed a counterexample to this conjecture. In the next few paragraphs I describe their construction, starting with the geometric part which is simpler and perhaps more probabilistic than geometric.

Choose first n points randomly, independently, and uniformly from B, set $t = n^{0.1}$ and $\triangle = \frac{t^2}{100n^2}$. ($N$ is going to be smaller than n.) Write V for the set of these points and call a triangle with vertices from V *small* if its area less than $\triangle$. The small triangles define a hypergraph H on V. The target is to show that H contains a large independent set $W \subset V$. The probability that three random points span a small triangle is less than

$$\int_0^2 \frac{8\triangle}{r} 2r\pi dr = 32\pi\triangle < \frac{t^2}{n^2}.$$

This can be seen by fixing two points at distance r, and then averaging over r. The expected size of H is less than $nt^2/6$. Hence by Markov's inequality,

$$|H| < nt^2/3$$

with probability at least 1/2.

A 2-cycle in H is $e_1, e_2 \in H$ with $|e_1 \cap e_2| = 2$, a 3-cycle is $e_1, e_2, e_3 \in H$ with $|e_i \cap e_j| = 1$ for all distinct i, j, and a 4-cycle is $e_1, e_2, e_3, e_4 \in H$ with $|e_i \cap e_j| = 1$ if $j = i + 1 \mod 4$ and 0 if $j = i + 2 \mod 4$. The following facts are checked easily: with high probability

- the number of 2-cycles is less than $n^{0.1}$,
- the number of 3-cycles is less than $n^{0.7}$,
- the number of 4-cycles is less than $n^{0.7}$.

Thus deleting all vertices in 2-,3-, or 4-cycles you get, with positive probability, a new 3-uniform hypergraph H^* on ground set V^* where $|V^*| = n(1 - o(1))$. The next, and crucial, step is plain hypergraph theory.

Lemma 10. *Assume H is a 3-uniform hypergraph on $[n]$ with at most $nt^2/3$ edges, without cycles of length $2, 3, 4$, and let $t \le n^{0.1}$. Then H contains an independent set W with*

$$|W| > const\frac{n}{t}\sqrt{\ln t}.$$

Setting $N = const\frac{n}{t}\sqrt{\ln t}$ we have a point set W in the unit disk, of N points, without small triangles; moreover $\triangle = ct^2/n^2 = c(\ln N)/N^2$. This is the counterexample to Heilbronn's conjecture.

The crucial Lemma 10 is an improvement over the simple estimate $\alpha(H) > n/(3t)$ which is true even if short cycles are not excluded. The proof is by sequential random choices, validating, once more, Szemerédi's philosophy. The short cycle condition guarantees that the hypergraph is locally sparse, in the sense that the neighbourhoods of two distinct vertices are "independent".

Although Lemma 10 is often very useful, it typically improves an existing estimate by a log-factor. In the Heilbronn case, for instance, it is not at all clear where the truth lies. To decide where it lies, most probably, quite different methods will be needed.

In contrast with Heilbronn's problem, the next application of the improved independence number method gives an almost precise answer to a geometric problem. It is a recent result of Kim and Vu [2004]. We need to introduce some terminology.

A graph $G(V, E)$ is (d, ε)-regular if its degrees are between $d(1-\varepsilon)$ and d. The *codegree* of a the graph, $D = D(G)$ is the maximum number of common neighbours of $x, y \in V$, $x \neq y$. An independent set $W \subset V$ is called *maximal* if it is not contained in a larger independent set. In the following theorem, which is from [Kim and Vu 2004], the asymptotics is understood with $d \to \infty$ and $\omega(d)$ denotes a function that tends to infinity as $d \to \infty$.

THEOREM 11. *Let G be a (d, ε)-regular graph on n vertices, where*

$$\varepsilon = (\omega(d) \ln d)^{-1}.$$

If

$$D(G) \leq \frac{d}{\omega(d) \ln^2 d},$$

then G contains a maximal independent set W with

$$(1+o(1)) \frac{n}{d} \ln \frac{d}{D} \leq |W| \leq (1+o(1)) \frac{n}{d} \ln \frac{d}{D} + \omega(d) \frac{n}{d} D \ln^2 D.$$

The error term $\omega(d) \frac{n}{d} D \ln^2 D$ is dominating if $\omega(d) D \ln^2 D$ is larger than $\ln \frac{d}{D}$. Otherwise, that is, when $\omega(d) D \ln^2 D = o(\ln \frac{d}{D})$, G contains a maximal independent set of size $(1+o(1)) \frac{n}{d} \ln \frac{d}{D}$. The method is, again, a sequential random choice of vertices but the remainder term has to be estimated precisely which makes the proof hard.

This result is used in [Kim and Vu 2004] to answer a question of Segre from 1959 (see [Szőnyi 1997]) on arcs in projective planes. An *arc* in a projective plane P of order q is a set $A \subset P$ containing no three points on a line. An arc is *complete* if it is not contained in a larger arc. Segre's question is this: What are the possible sizes of complete arcs in P? Simple counting arguments, using properties of the projective plane, show that the size of a complete arc is always between $\sqrt{2q}$ and $q+2$. Szőnyi [1997] showed that almost all values in the interval $[cq^{3/4}, q]$ can be the size of a complete arc. Kim and Vu [2003] showed the existence of complete arcs whose size is $\sqrt{q}(\ln q)^b$ with some universal constant b. This is close to the lower bound $\sqrt{2q}$. Further, it is proved in [Kim and Vu 2004] that sizes of complete arcs in P are almost dense in the interval $[\sqrt{2q}, q]$.

Theorem 12. *There are positive constants b, c and Q such that the following holds. For every plane P of order $q \geq Q$ and every $q* \in [\sqrt{q}\ln^4 q, q]$, P contains a complete arc A with*

$$cq^* \leq |A| \leq q^* \ln^b q.$$

The proof uses the fact that the conic $C = \{(x, x^2) : x \in GF(q)\}$ is an arc in P whose secants cover every point of $P \setminus C$ (except the one at infinity) $q/2 - O(1)$ times. Set $D = P \setminus C$ and $\varepsilon = (\sqrt{q}\ln q)/q^*$, so ε is small when q is large. Given an arc $A \subset D$ one defines a graph $G_A(V, E)$ as follows: V is the set of points $v \in C$ not covered by secants from A, and $u, v \in V$ form an edge in E if there is $a \in A$ with a, u, v collinear. One has to show next (the proof is hard and probabilistic) that there is an arc $A \subset D$, of size at most $2\varepsilon\sqrt{q}$, such that G_A satisfies the conditions of Theorem 11. Then one applies Theorem 11 and an additional argument to show that G_A contains a maximal independent set of the desired size such that its secants cover $D \setminus A$. Further details of the proof (that are even less geometric) can be found in the forthcoming [Kim and Vu 2004].

Results like Lemma 10 and Theorem 11 have been used to find a large matching in a hypergraph: Given a hypergraph H, a matching M is a collection of pairwise disjoint edges. Define the *intersection graph*, $G(H)$ of H as follows: its vertex set is H, and two vertices, $e, f \in H$ form an edge in $G(H)$ if $e \cap f = \varnothing$. So a matching in H corresponds to an independent set in $G(H)$, and a large independent set corresponds to many pairwise disjoint edges. Further, using such a matching one can find an economic cover of the ground set by edges. This happens if the set of vertices left uncovered by the matching is small. In other words, if the estimate of error term is precise. This is a very promising area with plenty of results and conjectures. Their geometric applications are waiting to be discovered.

6. The Regularity Lemma

Szemerédi's famous regularity lemma is one of the most important and useful results in combinatorics, it has millions of applications in discrete mathematics, but surprisingly few in geometry. Here is a remarkably elegant one, due to János Pach [1998].

Theorem 13. *For every $d \geq 2$ there is a positive constant c_d with the following property. Given sets $X_1, \dots, X_{d+1} \subset R^d$, each of size n, there are subsets $Z_i \subset X_i$, $(i \in [d+1])$, each of size at least $c_d n$ such that*

$$\bigcap \mathrm{conv}\{z_1, \dots, z_{d+1}\} \neq \varnothing,$$

where the intersection is taken over all transversals $z_i \in Z_i$, $i \in [d+1]$.

The proof uses several ingredients: the point selection theorem (Theorem 5), a weak form of the regularity lemma for hypergraphs, and the so-called same-type lemma from [Bárány and Valtr 1998]. To state the last one we say that the sets $Z_1, \dots, Z_k$ in R^d have *same type transversals* if there is no hyperplane intersecting the convex hull of any $d+1$ of them. (For various equivalent definitions see [Bárány and Valtr 1998] or [Matoušek 2002].) What we will need is the following fact. If $Z_1, \dots, Z_{d+2}$ have same type transversals, and if some $z_1 \in Z_1, \dots, z_{d+2} \in Z_{d+2}$ satisfies $z_{d+2} \in \operatorname{conv}\{z_1, \dots, z_{d+1}\}$, then $w_{d+2} \in \operatorname{conv}\{w_1, \dots, w_{d+1}\}$ holds for all $w_1 \in Z_1, \dots, w_{d+2} \in Z_{d+2}$. (Hopefully, this also explains the meaning of "same type".)

LEMMA 14. *For every $d \geq 2$ and every $k \geq d+1$ there is a positive constant $b(d,k)$ with the following property. Given nonempty sets $X_1, \dots, X_k \subset R^d$ in general position, there are subsets $Z_i \subset X_i$, ($i \in [k]$), each with $|Z_i| \geq b(d,k)|X_i|$ such that $Z_1, \dots, Z_k$ have the same type transversals.*

Remark. Ramsey's theorem guarantees the existence of sets Z_i with this property but their size is much smaller than cn. Here geometry is needed to guarantee linear size.

The proof of Theorem 13 begins by forming the $(d+1)$-uniform hypergraph H whose edges are the sets $\{x_1, \dots, x_{d+1}\}$ with $x_i \in X_i$. H has $(d+1)n$ vertices and n^{d+1} edges, so Theorem 5 gives a subhypergraph $H^* \subset H$ and a point $z \in R^d$ such that $|H^*| \geq \beta n^{d+1}$ and $z \in \operatorname{conv} e$ for each edge $e \in H^*$, where $\beta > 0$ depends only on d.

Next, a weak form of the regularity lemma for hypergraph (see [Pach 1998]) is needed. Without stating it we just claim that it ensures the existence of $Y_i \subset X_i$, $|Y_i| \geq \gamma |X_i|$ such that for every subset $Z_1 \subset Y_1, \dots, Z_{d+1} \subset Y_{d+1}$ with $|Z_i| \geq b(d, d+2)|Y_i|$ there are vertices $z_i \in Z_i$ $i \in [d+1]$ such that $\{z_1, \dots, z_{d+1}\}$ is an edge of H^*. Here $\gamma > 0$ depends only on d.

Finally, one applies the same type lemma for the sets $Y_1, \dots, Y_{d+1}$ and $Y_{d+2} = \{z\}$. This gives sets $Z_i \subset Y_i$ ($i \in [d+1]$), each of size at least $b(d, d+2)|Y_I|$, and $Z_{d+2} = \{z\}$ with same type transversals. By the weak regularity lemma, there is at least one simplex with vertices $z_i \in Z_i$, $i \in [d+1]$ that contains z. Then, by the same type lemma, all such simplices contain z. This finishes the proof.

It is high time to state the original regularity lemma now. We need some terminology: Given a graph $G(V,E)$, and disjoint sets $X, Y \subset V$, their *density* is defined as

$$d(X,Y) = \frac{|E(X,Y)|}{|X| \cdot |Y|},$$

where $E(X,Y)$ is the set of edges between X and Y. Given some $\delta > 0$, and disjoint $A, B \subset V$, the pair (A,B) is called *δ-regular* if, for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \delta |A|$ and $|Y| > \delta |B|$ we have

$$|d(X,Y) - d(A,B)| < \delta.$$

Now we state the regularity lemma of Szemerédi [1978] in the hope that it will find further geometric applications.

Theorem 15. *Given $\delta > 0$ and an integer m, there is an $M = M(\delta, m)$ such that the vertex set of every graph $G(V, E)$ with $|V| > m$ can be partitioned into classes $V_0, V_1, \ldots, V_k$, where $m \leq k \leq M$, such that $|V_0| \leq |V_1| = \ldots = |V_k|$ and all but at most δk^2 of the pairs (V_i, V_j), $i, j \in [k]$ are δ-regular.*

A proper illustration of the use of this lemma is a very recent result of Pach, Pinchasi, and Vondrák (manuscript, 2004). This result answers a question of Erdős in the following form: Assume $\varepsilon > 0$, X is a set of n points in R^3, and every two points in X are at distance one at least. If there are εn^2 pairs in X whose distance is between t and $t + 1$ for some $t > 0$, then the diameter of X is at least cn where c only depends on ε.

The conditions immediately cry out for the regularity lemma. In the graph $G(X, E)$, $x, y \in X$ form an edge if $\|x - y\| \in [t, t + 1]$. One obtains two disjoint sets $A, B \subset X$ of size $c_1 n$ with (A, B) ε-regular. This is a very strong condition on the point sets A, B. Using geometry one can find subsets $X \subset A$ and $Y \subset B$, each of size $c_2 n$ and such that $\|x - y\| \in [t, t{+}1]$ for *every* $x \in X, y \in Y$. Here $c_2 > 0$ depends on ε only. The rest of the proof is 3-dimensional geometry.

Szemerédi's regularity lemma has recently been generalized for hypergraphs by Gowers and by Rödl et al. (unpublished yet) with the potential of having further geometric applications. The regularity lemma is extremely useful in discrete mathematics, but, so far, it has not been applied in geometry very often.

7. VC-Dimension and ε-Nets

Given a hypergraph H with vertex set V, and ε-net (where $\varepsilon \in (0, 1]$) is a subset $N \subset V$ that intersects each edge $E \in H$ with $|E| \geq \varepsilon|V|$. In other words, N is an ε-net for H if it is a transversal for the edges with at least $\varepsilon|V|$ elements. This definition can be extended to "infinite hypergraphs": Assume V is a set, μ is a probability measure on V, and H is a system of μ-measurable sets. Then $N \subset V$ is called an ε-net for H with respect to μ if it intersects every set $E \in H$ whose measure is at least ε.

There is a special condition, of combinatorial nature, that ensures the existence of "very finite" ε-nets. Given a set system H on a finite or infinite ground set V, a set $A \subset V$ is *shattered by* H if each subset of A can be produced as $A \cap E$ for a suitable $E \in H$. The VC-dimension of the set system H, denoted by $\dim H$, is the maximum of the sizes of all finite shattered subsets of V, or ∞ if there are arbitrarily large shattered subsets. The VC-dimension, introduced by Vapnik and Chervonenkis in [1971] has turned out to be a very powerful tool everywhere: in statistics (the original motivation for the VC-dimension), discrete geometry, computational geometry, combinatorics of hypergraphs, and discrepancy theory. The terminology is sometimes different, for instance in computational geometry, the set system H is called *range space* and its edges *ranges*.

A simple example is a set of points V in R^d for which H is formed by the sets of type $V \cap h$ where h is a half-space. The VC-dimension of H is then $d+1$ since, by Radon's theorem, no $(d+2)$-set is shattered by half-spaces in R^d. Another example with finite VC-dimension, on the same ground set V, is the collection of all Euclidean balls.

The reason for the wide range of applications of VC-dimension lies in the very general setting and in the so called ε-net theorem (see [Haussler and Welzl 1987]) and the ε-approximation theorem (introduced in to [Vapnik and Chervonenkis 1971]).

THEOREM 16. *Let V be a set, and μ be a probability measure on V, H a system of μ-measurable subsets of V, and $\varepsilon \in (0,1]$. If $\dim H \leq d$ where $d \geq 2$, then there exists an ε-net for H of size at most $\frac{4d}{\varepsilon} \ln \frac{1}{\varepsilon}$.*

While an ε-net intersects each (large enough) set in H in at least one point, an ε-approximation $M \subset V$ provides a "proportional representation" of each set in H: for each $E \in H$

$$\left| \mu(E) - \frac{|M \cap E|}{|M|} \right| < \varepsilon.$$

THEOREM 17. *Let V be a set, and μ be a probability measure on V, H a system of μ-measurable subsets of V, and $\varepsilon \in (0,1]$. If $\dim H \leq d$ where $d \geq 2$, then there exists an ε-approximation for H, of size at most*

$$\frac{Cd}{\varepsilon^2} \ln \frac{1}{\varepsilon}.$$

The ε-net theorem is more often used in geometry. The following application to an *art gallery* problem is due to Kalai and Matoušek [1997]. An art gallery is a simply connected compact set T in the plane, and the set of points visible from $x \in T$ is, by definition,

$$V(x) = \{y \in T : [x,y] \subset T\}.$$

In other words, x sees or guards the points in $V(x)$.

THEOREM 18. *Let $T \subset R^2$ be a simply connected art gallery of Lebesgue measure one. Assume that for some $r \geq 2$ the Lebesgue measure of each $V(x)$ is at least $1/r$. Then T can be guarded by at most $Cr \ln r$ points, that is, there is a set $N \subset T$, having at most $Cr \ln r$ points, with $T = \cup_{x \in N} V(x)$.*

The proof begins by introducing the set system $H = \{V(x) : x \in T\}$ and noting that a set $N \subset T$ guards T iff it intersects each set in H. So we are done if H admits an $(1/r)$-net of the required size. This is guaranteed by the ε-net theorem provided the VC-dimension of H is bounded by some constant independent of T. This can be shown by a geometric argument using the fact that T is simply connected. The details can be found in [Kalai and Matoušek 1997], or in [Valtr 1998] where $\dim H \leq 23$ is shown.

There are several geometric applications of VC-dimension and the ε-net theorem, see for instance the books [Chazelle 2000], [Matoušek 2002], and [Pach and Agarwal 1995]. Since most of them require new concepts and further preparations that go beyond the limits of this survey, I only explain one more case, that of a spanning tree with low crossing number. The setting is this. Given a set X of n points in R^2 in general position, we want to build a spanning tree (with vertex set X and edge set segments connecting certain pairs of X) such that no line meets too many of the edges. The following beautiful theorem is due to Welzl [1988] (the $\ln n$ factor has been since then removed).

THEOREM 19. *Given a set X of n points in R^2 in general position, there is a spanning tree with vertex set X such that no line meets more than $O(\sqrt{n}\ln n)$ edges of the tree.*

For the proof one checks first that the following set system H has finite VC-dimension: The ground set is the collection of all lines in R^2 and H consists of sets of lines L_s that intersect a fixed segment s. To see that $\dim H$ is finite assume an n element set of lines A is shattered by H. These lines divide the plane into $m \le \binom{n}{2}+n+1$ cells, and if s and t are two segments whose endpoints (in pairs) belong to the same cell, then L_s and L_t have the same intersection with A. Consequently there are at most $\binom{m}{2}$ segments s for which $L_s \cap A$ are pairwise distinct, so $2^n \le \binom{m}{2}$ implying that $\dim H \le n \le 12$.

LEMMA 20. *Given a set S of k points in general position, and a set L of m lines in R^2 with no point incident to any of the lines, there exist $x, y \in S$ such that the line segment $[x,y]$ intersects at most $(cm\ln k)/\sqrt{k}$ lines from L.*

For the proof one notes that the set system H has finite VC-dimension, so the ε-net theorem applies: with $\varepsilon = c_1(\ln k)/k$ we get a collection of lines $L' \subset L$ of size $c_2\varepsilon^{-1}\ln\varepsilon^{-1} < \sqrt{k}/2$ such that every open segment crossing

$$\varepsilon m = c\frac{m\ln k}{\sqrt{k}}$$

elements of L crosses some line in L'. The lines in L' divide the plane into less than k cells. Thus one cell contains two points of S; the segment connecting them satisfies the requirements of the lemma.

To finish the proof of the spanning tree theorem one starts with constructing a set, L, of $\binom{n}{2}$ lines that represent all possible partitions of X by lines. Setting $S_0 = X$ and $L_0 = L$ one applies the lemma to S_i, L_i $(i = 0, 1, \dots, n-2)$ to obtain a segment $[x_i, y_i]$ intersecting at most $c\frac{m_i \ln n_i}{\sqrt{n_i}}$ from L_i. For the next iteration $S_{i+1} = S_i \setminus \{x_i\}$ and L_{i+1} is the set of lines consisting of L_i plus one more, slightly perturbed, copy of each line in L_i intersecting $[x_i, y_i]$. The analysis of this algorithm finishes the proof; the details can be found in [Welzl 1988] or [Pach and Agarwal 1995].

8. Epilogue

László Fejes Tóth asked in 1976 whether the densest packing of congruent circles in the plane is unique or not in the following sense: Assume that in a circle packing, $\mathcal{C}$, in the plane, every circle is touched by at least six others. Is it true then, that arbitrarily large or arbitrarily small circles occur in $\mathcal{C}$ unless it is the densest packing of congruent circles. The answer is yes and is the content of [Bárány et al. 1984]:

THEOREM 21. *Under the conditions above arbitrarily small circles occur in* $\mathcal{C}$ *unless* $\mathcal{C}$ *is the densest packing of congruent circles.*

For the proof one defines the graph $G(V,E)$ whose vertices are the circles with two of them forming an edge if the corresponding circles are touching each other. G is a planar graph. Define the function $f : V \to R$ by $f(v) = 1/r$ when r is the radius of the circle corresponding to $v \in V$. Surprisingly, this function is *subharmonic* on G, that is, $f(v)$ is less than or equal to the average of f on the neighbours of v. This is the first geometric component in the proof. Then one uses, or rather proves a theorem saying that, under suitable conditions on the underlying graph, if a subharmonic function is bounded from above, then it is necessarily constant. Finally, the "suitable" condition follows from the planarity of G. I'm sure that, in the world of geometry, there are hundreds of similar proofs waiting to be discovered.

Acknowledgment

The demand for such a survey, and the idea of writing one, emerged in the semester on Discrete and Computational Geometry at MSRI in Berkeley, 2003, autumn term. I thank the organizers and the participants for the excellent atmosphere and working conditions provided. I'm also grateful to Microsoft Research (in Redmond, WA), as this paper was written on a very pleasant and fruitful visit there. My thanks are due to Anders Björner, Jiří Matoušek, and Van Vu Ha for valuable comments on an earlier version of this paper. Partial support from Hungarian National Foundation Grants No. 046246 and 037846 is acknowledged.

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