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Applications of Measures of Noncompactness to Infinite System of Fractional Differential Equations

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Abstract. In this paper, we discuss few existence result for solution of an infnite system of fractional differential equations of order α $(1 < \alpha < 2)$, with three point boundary value problem in the interval [0, T]. The problem is studied in the classical Banach sequence spaces c_0 and ℓ_p $(1 \leq p < \infty)$, using Hausdorff measure of noncompactness and Darbo type fixed point theorem. We also illustrate our results through some concrete examples..

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

1.1. Measures of noncompactness

In what follows we will give a brief description of measures of noncompactness and condensing operators which will be used in subsequent sections.

Theorem 1.1. (Schauder [20]) *Let C be a closed and convex subset of a Banach space E. Then every compact and continuous map $F: C \to C$ has at least one fixed point.*

In case of infinite dimensional normed spaces or metric spaces, the notion of measure of noncompactness (MNC) plays an important role. This concept was introduced by Kuratowski ([12], [13]). There are various type of MNCs in metric and linear topological spaces. In 1955, Darbo [8] proved a fixed point theorem, which was a generalized form of the classical Schauder fixed point theorem and Banach contraction principle. For a bounded subset S of a metric space X, the Kuratowski measure of noncompactness [12] is defined as

$$\alpha(S) := \inf\{\delta > 0 \,|\, S = \cup_{i=1}^{n} S_i\,,\ diam(S_i) \leq \delta \text{ for } 1 \leq i \leq n < \infty\} \tag{1}$$

where $diam(S_i)$ denotes the diameter of the set S_i, that is,

$$diam(S_i) = \sup\{d(x,y) | \, x,\ y \in S_i\}.$$

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Another, useful measure of noncompactness is the so called Hausdorff measure of noncompactness defined as

$$\chi(S) = \inf\{\epsilon > 0 \mid S \text{ has finite } \epsilon\text{-net in } X\}. \tag{2}$$

We describe some basic properties of MNC's χ and α in the context of a Banach space. Let $(E, \|.\|)$ be a Banach space [6], $\mathbb{R}_+ = [0, \infty)$, the symbols $\bar{X}$ and $convX$ denote closure of X and convex closure of X, respectively. Let $\mathcal{M}_E$ denote the family of non-empty bounded subsets of E and $\mathcal{N}_E$ denote the family of non-empty and relatively compact subsets of E.

Let $\mu : \mathcal{M}_E \to R_+$, then μ is said to be an axiomatic measure of non-compactness on the space E, if it satisfies the following conditions.

1. $\mu(X) = 0$ for relatively compact subsets of E.
2. $X \subset Y \implies \mu(X) \leq \mu(Y)$. (*monotonicity*)
3. $\mu(\bar{X}) = \mu(X)$. (*invariant under passage to closure*)
4. $\mu(ConvX) = \mu(X)$. (*invariant under passage to convex hull*)
5. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6. If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}_E$, such that, if $X_{n+1} \subset X_n$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \phi$.
7. $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$. (*maximum property*)
8. $\mu(X + Y) \leq \mu(X) + \mu(Y)$. (*subadditive*)
9. $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$. (*semi-homegeniety*)
10. $\mu(X + a) = \mu(X)$ for each $a \in E$. (*invariant under translation*)

Definition 1.2. *Let E_1 and E_2 be two Banach spaces and μ_1 and μ_2 be arbitrary MNCs on E_1 and E_2 respectively. An operator T from E_1 to E_2 is called a $(\mu_1$-$\mu_2)$ condensing operator if it is continuous and $\mu_2(T(\Omega)) < \mu_1(\Omega)$ for every bounded noncompact set $\Omega \subset E_1$.*

Remark 1.3. *If $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$ then T is called μ-condensing operator.*

Theorem 1.4 (Darbo [8]). *Let Ω be a nonempty, closed, bounded and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property $\mu(T(\Omega)) \leq k\mu(\Omega)$, then T has a fixed point in Ω.*

Proposition 1.5 ([4]). *If $W \subset C(I, E)$ is bounded and equicontinuous then the set $\mu(W(t))$ is continuous on I and*

$$\mu(W) = \sup_{t\in I} \mu(W(t)), \quad \mu\left(\int_0^t W(s)ds\right) \leq \int_0^t \mu(W(s))ds.$$

The formula for computing measure of noncompactness for a general MNC in a given metric or normed space is a rigorous task, however in some normed spaces the exact formula is available for Hausdorff MNC. We mention the following result which is used in the subsequent sections.

Theorem 1.6. *[4] Let Q be a bounded subset of the Banach space $X = c_0$. As $(e^{(1)}, e^{(2)}, \dots)$ is a Schauder basis for c_0, the Hausdorff MNC χ for Q is given by*

$$\chi_{c_0}(Q) = \lim_{n\to\infty} \left\{ \sup_{x\in Q} (max_{k\geq n} |x_k|) \right\} \tag{3}$$

Theorem 1.7. *[4] Let Q be a bounded subset of the Banach space $X = \ell_p$ for $1 \leq p < \infty$. As $(e^{(1)}, e^{(2)}, \dots)$ is a Schauder basis for ℓ_p, the Hausdorff MNC χ for Q is given by*

$$\chi_{\ell_p}(Q) = \lim_{n\to\infty} \left\{ \sup_{x\in Q} \left(\sum_{k\geq n} |x_k|^p \right)^{1/p} \right\} \tag{4}$$

1.2. Fractional differential equations

The theory of fractional calculus is regarded as the natural generalization of the integer order calculus. The subject was first formally presented by eminent mathematicians Liouville and Riemann in nineteenth century. In contemporary study of scientific and engineering problems the theory of fractional differential and integral equations have found novel applications in a large variety of topics such as image processing [7], polymer science [15], control theory [19] etc. Besides, modelling of certain human behavior also leads to formulation of fractional differential or integral equations [10]. The fractional differential equations under various conditions have been studied by [1], [3], [11], [14], *etc.* The three point boundary value problem given by 5 for a coupled system of FDE on the interval $(0,1)$ was studied by Bashir *et. al.* [3]

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), \ t \in (0,1), \\ D^{\beta}v(t) = g(t, u(t), D^{q}v(t)), \ t \in (0,1) \\ u(0) = 0, \ u(1) = au(\xi), \ v(0) = 0, \ v(1) = av(\xi), \end{cases} \tag{5}$$

where $1 < \alpha, \beta < 2$, $p, q, a > 0$, $0 < \xi < 1$, $\alpha - q \geq 1$, $\beta - p \geq 1$, $a\xi^{\alpha-1} < 1$ and $a\xi^{\beta-1} < 1$. D is the standard Riemann-Liouville fractional derivative operator and $f : [0,1] \times E \to E$. We describe briefly certain basic properties of fractional derivative. Let $\alpha > 0$ and $n = [\alpha] + 1 = N + 1$, where $[\alpha]$ denotes the ceiling function (smallest integer greater than or equal to α). For a function $f : (0, \infty) \to \mathbb{R}$, the fractional integral of order α is defined as follows

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha-1}} f(s)ds.$$

provided the integral on the right exists. Similarly the fractional derivative of order α for a function f is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{1}{(t-s)^{\alpha-n+1}} f(s)ds$$

We mention the following properties of the operators I and D, for α, $\beta > 0$, we have

$$I^{\alpha}I^{\beta}f(t) = I^{\alpha+\beta}f(t) \tag{6}$$

$$D^{\alpha}I^{\alpha}f(t) = f(t) \tag{7}$$

For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}u(t) = 0$ with $u \in C(0,T) \cap L^1_{loc}(0,\infty)$ is given by

$$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots N$. Hence $I^{\alpha}D^{\alpha}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}$. Let $C(J)$ be the Banach space of all continuous functions defined on $J = [a,b] \subset \mathbb{R}$ with sup norm $\|u(t)\|_{\infty} = \sup_{t \in J}|u(t)|$.

Proposition 1.8. *Let* $f \in C[0,T]$ *be a given function and* $1 < \alpha < 2$. *Then the unique solution of*

$$D^{\alpha}u(t) = f(t), \ u(0) = 0, \ u(T) = au(\xi) \tag{8}$$

is given by

$$u(t) = \int_0^T K(t,s)f(s)ds \tag{9}$$

where $K(t,s)$ is the Green's function, given by $K(t,s) = \frac{1}{\Gamma(\alpha)(T^{\alpha-1}-a\xi^{\alpha-1})}\begin{cases} K_1(t,s), & 0 \le t \le \xi \\ K_2(t,s), & \xi \le t \le T \end{cases}$

$$K_1(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}]; \ 0 \le s \le t, \\ -t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}]; \ t \le s \le \xi, \\ -(t(T-s)^{\alpha-1}); \ \xi \le s \le T. \end{cases}$$

$$K_2(t,s) = \begin{cases} (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}]; \ 0 \le s \le \xi, \\ (t-s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - (t(T-s))^{\alpha-1}; \ \xi < s \le t, \\ -(t(T-s))^{\alpha-1}; \ t < s \le T. \end{cases}$$

Proof. The general solution of of FDE is
$u(t) = I^\alpha f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$ where C_1, $C_2 \in \mathbb{R}$.
Using $u(0) = 0$ gives $C_2 = 0$. Using the second boundary condition we get

$$C_1 = -\frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})}\left[\int_0^T \frac{f(s)ds}{(T-s)^{1-\alpha}\Gamma(\alpha)} - a\int_0^\xi \frac{f(s)ds}{(\xi-s)^{1-\alpha}\Gamma(\alpha)}\right]$$

$$u(t) = \int_0^t \left[(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})}\right]\frac{f(s)}{\Gamma(\alpha)}ds - \frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_t^T (t(T-s))^{\alpha-1} f(s)ds$$
$$+\frac{a}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (t(\xi-s))^{\alpha-1} f(s)ds$$

which gives the kernel $K_1(t,s)$ and $K_2(t,s)$. □

Remark 1.9. *It can be verified that the Green's function $K(t,s)$ defined on rectangle $[0,T] \times [0,T]$ as $K_1(t,s) : [0,\xi] \times [0,T] \to \mathbb{R}$ and $K_2(t,s) : [\xi,T] \times [0,T] \to \mathbb{R}$ is continuous w.r.t. to t and s.*

1.3. System of fractional differential equations

In this section we describe, what we refer to as an infinite system of fractional differential equation. Infinite systems of ODE's was first studied by Persidskii [18] with the aid of classical tools such as successive approximation and the classical Banach fixed point principle. The infinite systems of differential equations emerge in study of various topics of nonlinear analysis. For example semidiscretization of certain parabolic partial differential equation leads to an infinite system of ODE [21], while modeling certain physical phenomenon in theory of neural sets, branching process and mechanics ([9], [22]).

The theory of infinite systems of differential equations can be regarded as a particular case of differential equations in Banach spaces, where the infinite system can be represented as an ordinary differential equation. Consider the following infinite system of fractional differential equations

$$\begin{cases} D^\alpha u_i(t) = f_i(t, u(t)), \ t \in (0,T) \\ u_i(0) = u_i^0 = 0, \ u_i(T) = au_i(\xi); \quad i = 1,2,3\ldots \\ 1 < \alpha < 2, \ a\xi^{\alpha-1} < T^{\alpha-1}. \end{cases} \tag{10}$$

where each $u_i(t)$ is a differentiable function of class $C^{[\alpha]+1}$. We will denote the sequence $\{u_i(t)\}_{i=1}^\infty = u(t)$, $\{u_i(0)\}_{i=1}^\infty = u_0$, $\{u_i(\xi)\}_{i=1}^\infty = u(\xi)$ and $\{f_i(t,u(t))\}_{i=1}^\infty = f(t,u(t))$ which is an element of some Banach sequence space $(E, \|.\|)$. We rewrite the above system as follows

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), \ t \in (0,T) \\ u(0) = u_0, \ u(T) = au(\xi). \end{cases} \tag{11}$$

where $f : I \times E \to E$ and $u_0, \ u(\xi) \in E$. As in Banach sequence space (in general in any infinite dimensional linear space) a closed and bounded set is not necessarily compact set, mere continuity of the function f doesn't guarantee the existence of a solution of differential equation. We will use the tools such as measure of noncompactness(MNC) and condensing operators to establish the existence of solution for 11. For each $i \in \mathbb{N}$, fractional differential equation 10 has a solution if and only if the integral equation $u_i(t) = \int_0^t K_i(t,s) f_i(s, u(s))ds$ has a solution, for each $i \in \mathbb{N}$, $K_i(t,s) = K(t,s)$ described in proposition 1.8 .

2. Solution in Sequence Space c_0

In this section we investigate the solution of infinite system 10 in the Banach sequence space c_0, the space of sequences convergent to 0, equipped with the norm $\|x\| = \sup\{|x_i| : i = 1, 2, 3, \dots\}$. The function $f(t, u(t)) = (f_1(t, u(t)), f_2(t, u(t)), f_3(t, u(t)), \dots)$ is defined on $I \times c_0 \to c_0$ and each f_i is a real valued function. We have the following assumptions:

(A1) $\{u_i^0\}_{i=1}^{\infty}$ and $\{u_i(\xi)\}_{i=1}^{\infty}$ belong to c_0.

(A2) $f(., u)$ is measurable for each fixed u.

(A3) For any $t \in I$ and $u \in c_0$ and $n = 1, 2, 3, \dots$

$$|f_n(t, u(t))| \leq p_n(t) + q_n(t) \sup\{|u_i| : i \geq n\}.$$

where $p_i(t)$ and $q_i(t)$ are real valued functions and continuous on I such that sequence $(p_i(t))$ converges uniformly on I to the zero function identically and the sequence $(q_i(t))$ is equibounded on I.

(A4) The family of functions $\{(fu)(t)\}_{t\in I}$ is equicontinuous at each point of the space c_0.

Theorem 2.1. *If the assumptions **A1-A4** are satisfied by the system 10, then if $QMT < 1$, it admits at least one solution $u(t)$, such that $u(t) = \{u_i(t)\}_1^{\infty} \in c_0$ for each $t \in [0, T]$, where $M = \max_{t,s\in I} |K(t,s)|$, $\sup_i \sup_{t\in I} |q_i(t)| \leq Q$.*

Proof. Let $u(t) = \{u_i(t)\}_{i=1}^{\infty}$ be function which satisfies the boundary conditions of the problem 10, and each $u_i(t)$ is continuous on I. Define the operator $\mathcal{F} : C(I, c_0) \to C(I, c_0)$ as

$$(\mathcal{F}u)(t) = \int_0^T K(t,s) f(s, u(s))ds \tag{12}$$

By assumption **A2**, $\mathcal{F}$ is well-defined, we show that $\mathcal{F}$ is bounded w.r.t the classical norm on $C(I, c_0)$, which is given by $\|u\| = max\{\, \|u(t)\|_{c_0} : t \in I\,\}$

$$\begin{aligned}
\|(\mathcal{F}u)(t)\|_{c_0} &= \left\| \int_0^T K(t,s) f(s, u(s))ds \right\|_{c_0} \\
&= \sup_{n\geq 1} \left| \int_0^T K(t,s) f_n(s, u(s))ds \right| \\
&\leq \sup_{n\geq 1} \int_0^T |K(t,s)|\, |f_n(s, u(s))|ds \\
&\leq \sup_{n\geq 1} \int_0^T |K(t,s)|\, (p_n(s) + q_n(s) \sup\{|u_i(s)| : i \geq n\})ds \\
&\leq \sup_{n\geq 1} \int_0^T |K(t,s)|\, p_n(s)ds + \sup_{n\geq 1} \int_0^T |K(t,s)|\, q_n(s) \sup\{|u_i(s)| : i \geq n\}ds \\
max_{t\in I}\|(\mathcal{F}u)(t)\|_{c_0} &\leq max_{t\in I} \left\{ \sup_{n\geq 1} \int_0^T |K(t,s)|\; q_n(s) \sup\{|u_i(s)| : i \geq n\}ds \right\} \\
\|\mathcal{F}u\| &\leq QMT\,.\|u\|
\end{aligned}$$

The above inequality reduces to

$$r \leq QMT.r$$

Let r_0 denotes the optimal solution of the inequality. Consider the set $B = B(u_0, r_0) = \{u(t) \in C(I, c_0) : \|u\|_{C(I,c_0)} \leq r_0,\ u(0) = 0,\ u(T) = au(\xi)\}$, which is closed, bounded and convex, clearly $\mathcal{F}$ is bounded on B. Now we show that $\mathcal{F}$ is continuous. Arbitrarily fix $v \in B$,

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_{c_0} &= \sup_{n\geq 1}\left|\int_0^T K(t,s)f_n(s,u(s))ds - \int_0^T K(t,s)f_n(s,v(s))ds\right| \\
&\leq \sup_{n\geq 1}\int_0^T |K(t,s)|\,|f_n(s,u(s))ds - f_n(s,v(s))|ds \\
&\leq \int_0^T |K(t,s)|\,\|f_n(s,u(s))ds - f_n(s,v(s))\|_{c_0}ds
\end{aligned}$$

Now using assumption **A4** for any $v \in B$ and for any arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that $\|(fu)(t) - (fv)(t)\|_{c_0} \leq \frac{\epsilon}{M}$ for each $t \in I$ and for each $u \in B$ such that $\|u - v\| \leq \delta$.

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_{c_0} &\leq \int_0^T |K(t,s)|\,\|(fu)(s) - (fv)(s)\|_{c_0}ds \\
&\leq \frac{\epsilon}{M}\max_{t\in I}\int_0^T |K(t,s)|ds \ < \epsilon.
\end{aligned}$$

thus $\mathcal{F}$ is continuous.
Now we establish the continuity of $(\mathcal{F}u)$ in $(0,T)$. Let $t_0 \in (0,T)$ and $\epsilon > 0$ be arbitrary then, by continuity of $K(t,s)$ w.r.t t we have $\delta(t_0,\epsilon) > 0$ such that for $|t - t_0| < \delta$, $|K(t,s) - K(t_0,s)| < \epsilon/(QT\|u(s)\|_{c_0})$.

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}u)(t_0)\|_{c_0} &= \sup_{n\geq 1}\left|\int_0^T K(t,s)f_n(s,u(s))ds - \int_0^T K(t_0,s)f_n(s,u(s))ds\right| \\
&\leq \int_0^T |K(t,s) - K(t_0,s)| \sup_{n\geq 1}|f_n(s,u(s))|ds \\
&\leq \int_0^T |K(t,s) - K(t_0,s)|\ \sup_{n\geq 1}(p_n(s) + q_n(s)\{|u_i(s)| \ :\ i \geq n\})ds \\
&\leq \int_0^T |K(t,s) - K(t_0,s)|\,q_n(s)\sup_{n\geq 1}\{|u_i(s)| \ :\ i \geq n\})ds \\
&\leq Q\int_0^T |K(t,s) - K(t_0,s)|\,\|u(s)\|_{c_0}ds \ < \epsilon.
\end{aligned}$$

We claim that operator $\mathcal{F}$ is condensing with respect to Hausdorff MNC χ on the space $C(I, c_0)$. Using the formula 3, we conclude that Hausdorff MNC for $B \subset C(I, c_0)$ is defined as

$$\chi_{C(I,c_0)}(B) = \sup_{t\in I}\chi_{c_0}(B(t))$$

$$\begin{aligned}
\chi_{c_0}(\mathcal{F}B) &= \lim_{n\to\infty}\left\{\sup_{u\in B}\left(\max_{i\geq n}|\mathcal{F}u_i(t)|\right)\right\}\\
&\leq \lim_{n\to\infty}\left\{\sup_{u\in B}\left(\max_{i\geq n}\left|\int_0^T K(t,s)f_i(s,u(s))ds\right|\right)\right\}\\
&\leq \lim_{n\to\infty}\left\{\sup_{u\in B}\left(\max_{i\geq n}\int_0^T |K(t,s)|\left(p_i(s)+q_i(s)\sup\{|u_k(s)|\,:\,k\geq i\}\right)ds\right)\right\}\\
&\leq Q\lim_{n\to\infty}\left\{\sup_{u\in B}\left(\max_{i\geq n}\int_0^T |K(t,s)|\sup\{|u_k(s)|\,:\,k\geq i\}ds\right)\right\}\\
\sup_{t\in I}\chi_{c_0}(\mathcal{F}B) &\leq QMT\sup_{t\in I}\lim_{n\to\infty}\left\{\sup_{u\in B}\left(\max_{i\geq n}|u_i(t)|\right)\right\}\\
\chi_{C(I,c_0)}(\mathcal{F}B) &\leq QMT\chi_{C(I,c_0)}(B).
\end{aligned}$$

As $QMT < 1$, implying $\mathcal{F}$ is a Darbo condensing operator with darbo constant QMT, thus by Theorem1.4 $\mathcal{F}$ admits at least one fixed point in B, which is a solution for 10 in the space $C(I, c_0)$. Moreover for each $t \in [0, T]$, $u(t) \in ker_{\chi_{C(I,c_0)}}$. □

Example 2.2. *Consider the following system of FDE in* c_0

$$\begin{cases} D^{4/3}u_n(t) = \frac{t\exp(-nt)}{(n+1)^2} + \sum_{m=n}^{\infty}\frac{u_m(t)}{(1+m^2)(n^2)} \quad t\in(0,T)\\ u_n(0)=0,\ u_n(T)=\sqrt[3]{4}\,u_n(T/2);\quad n=1,2,3\ldots. \end{cases} \tag{13}$$

$u(T/2) = \{u_n(T/2)\}_{n=1}^{\infty} \in c_0$.

Here $\xi = T/2$ and $a = \sqrt[3]{4}$, and $f_n(t,u(t)) = \frac{t\exp(-nt)}{(n+1)^2} + \sum_{m=n}^{\infty}\frac{u_m(t)}{(1+m^2)(n^2)}$. Here kernel $K_1(t,s)$ and $K_2(t,s)$ are given as $K(t,s) = \frac{1}{\Gamma(4/3)(\sqrt[3]{T}-\sqrt[3]{2T})}\begin{cases}K_1(t,s),\ 0\leq t\leq T/2,\\ K_2(t,s),\ T/2\leq t\leq T.\end{cases}$

$$K_1(t,s)=\begin{cases}(t-s)^{1/3}(\sqrt[3]{T}-\sqrt[3]{2T})-t^{1/3}[(T-s)^{1/3}-\sqrt[3]{2}(T-2s)^{1/3}];\ 0\leq s\leq t,\\ -t^{1/3}[(T-s)^{1/3}-\sqrt[3]{2}(T-2s)^{1/3}];\ t\leq s\leq \frac{T}{2},\\ -(t(T-s)^{1/3});\ \frac{T}{2}\leq s\leq T.\end{cases}$$

$$K_2(t,s)=\begin{cases}(t-s)^{1/3}(\sqrt[3]{T}-\sqrt[3]{2T})-t^{1/3}[(T-s)^{1/3}-\sqrt[3]{2}(T-2s)^{1/3}];\ 0\leq s\leq \frac{T}{2},\\ (t-s)^{1/3}(\sqrt[3]{T}-\sqrt[3]{2T})-(t(T-s))^{1/3},\ \frac{T}{2}< s\leq t, -(t(T-s))^{1/3};\ t<s\leq T.\end{cases}$$

Assumption (A1)and (A2) are satisfied. Moreover $|f_n(t,u(t))| \leq p_n(t) + q_n(t)\sup\{|u_i(t)|\,:\,i\geq n\}$ where

$$p_n(t)=\frac{t\exp(-nt)}{(n+1)^2},\ q_n(t)=\frac{1}{n^2}\sum_{m=n}^{\infty}\frac{1}{1+m^2}.$$

We first show that $f(t,u(t)) \in c_0$. For any arbitrary $t \in [0,T]$ and $u \in c_0$ we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(t,u(t)) &= \lim_{n\to\infty}\left(\frac{t\exp(-nt)}{(n+1)^2}+\sum_{m\geq n}\frac{|u_m(t)|}{(1+m^2)(n^2)}\right)\\
&\leq \lim_{n\to\infty}\left(\frac{T}{(n+1)^2}+\sup_{m\geq n}|u_m(t)|\frac{\pi^2}{6n^2}\right)\\
&< \lim_{n\to\infty}\left(\sup_{m\geq n}|u_m(t)|\frac{\pi^2}{6n^2}\right)=0.
\end{aligned}$$

It can be seen that assumption A4 is satisfied by functions $p_n(t)$ and $q_n(t)$. $p_n(t)$ converges uniformly to zero and $q_n(t)$ is equibounded by $\frac{\pi^2}{6} = Q$. Now we show that assumption A4 is also satisfied. Let $t \in I$, $v \in c_0$ be arbitrarily fixed, take any $\epsilon > 0$,

$$\begin{aligned}
\|(fu)(t) - (fv)(t)\|_{c_0} &= \sup_{n\geq 1} |(fu)_n(t) - (fv)_n(t)| \\
&= \sup_{n\geq 1} |f_n(t, u(t)) - f_n(t, v(t))| \\
&= \sup_{n\geq 1} \left| \sum_{m=n}^{\infty} \frac{u_m(t)}{(1+m^2)(n^2)} - \sum_{m=n}^{\infty} \frac{v_m(t)}{(1+m^2)(n^2)} \right| \\
&\leq \sup_{n\geq 1} \sum_{m\geq n} \left| \frac{u_m(t) - v_m(t)}{(1+m^2)(n^2)} \right| \\
&\leq \sup_{n\geq 1} |u_n(t) - v_n(t)| \frac{\pi^2}{6} \\
&\leq \|u(t) - v(t)\|_{c_0} \frac{\pi^2}{6} < \epsilon. \text{ when } \|u(t) - v(t)\|_{c_0} < \delta = \epsilon \frac{6}{\pi^2}.
\end{aligned}$$

Thus the system of FDE satisfies the hypotheses of the Theorem2.1, hence it has at least one solution in $C(I, c_0)$. The interval of solution is $[0, T]$ where T is choosen such that $T < \frac{6M}{\pi^2}$.

3. Solution in Sequence Space ℓ_p

Various types of infinite systems of ordinary differential equations have been studied by several authors, such as Cauchy initial value problem in sequence spaces ℓ_1 by Banaś et. al [5], and similar problem in sequence space ℓ_p was studied by Mursaleen et. al. [16]. The second order boundary value problem, for ODE in space ℓ_1 was investigated by Aghajani et. al. [2] and [17]. In this section we consider the infinite system11 of fractional differential equation in the sequence space ℓ_p for $1 \leq p < \infty$.
We will investigate the solution under the following assumptions:

(B1) $u(T) \in \ell_p$.

(B2) $f = (f_1, f_2, \dots)$ continuously transforms the set $I \times \ell_p$ to ℓ_p.

(B3) There exist nonnegative functions $q_i(t)$ and $r_i(t)$ such that for any $t \in I$ and $u(t) \in \ell_p$.

$$|f_i(t, u)|^p \leq q_i(t) + r_i(t)|u_i(t)|^p$$

(B4) $q_i(t)$ are continuous and the series $\sum_{i=1}^{\infty} q_i(t)$ converges uniformly on I.

(B5) The function sequence $r_i(t)$ is equibounded on I, and $\lim_{i\to\infty} \sup r_i(t)$ is integrable over I.

(B6) The sequence of function $\{(fu)(t)\}_{t\in I}$ is equibounded at each point of ℓ_p.

Theorem 3.1. *If the system 11 satisfies the above assumptions B1-B6 and* $MT^{\frac{2-p}{p}} R^{1/p} < 1$, *then it has at least one solution* $u(t)$ *such that* $u(t) = \{u_i(t)\}_{i=1}^{\infty} \in \ell_p (p \geq 1)$ *for each* $t \in [0, T]$, *where* $M = \max_{t,s\in[0,T]} K(t, s)$, $r_i(t)$ *is equibounded by* R *and* $Q = \sup_{t\in I} |q(t)|$, $q(t) = \sum_{i=1}^{\infty} q_i(t)$.

Proof. Let $u(t) = \{u_i(t)\}_{i=1}^{\infty}$ be function which satisfies the boundary conditions of the problem 11, and each $u_i(t)$ is continuous on I. Define the operator $\mathcal{F} : B \subset C(I, \ell_p) \to C(I, \ell_p)$ as

$$(\mathcal{F}u)(t) = \int_0^T K(t, s) f(s, u(s)) ds \tag{14}$$

By assumption **B2**, $\mathcal{F}$ is well defined on $C(I,\ell_p)$. We show that $\mathcal{F}$ is bounded in the classical supremum norm on $C(I,\ell_p)$, given by $\|u\| = \sup_{t\in I}\|u(t)\|_{\ell_p}$.

$$\begin{aligned}
\|(\mathcal{F}u)(t)\|^p_{\ell_p} &= \left\|\int_0^T K(t,s)f(s,u(s))ds\right\|^p_{\ell_p}\\
&\leq T^{\frac{p-1}{p}}M^p\sum_{n\geq 1}\left|\int_0^T f_n(s,u(s))ds\right|^p\\
&\leq T^{\frac{p-1}{p}}M^p\sum_{n\geq 1}\int_0^T |f_n(s,u(s))|^p ds\\
&\leq T^{\frac{p-1}{p}}M^p\sum_{n\geq 1}\int_0^T (q_n(s)+r_n(s)|u_n(s)|^p)ds\\
&\leq T^{\frac{p-1}{p}}M^p\int_0^T\sum_{n\geq 1}q_n(s)ds + T^{\frac{p-1}{p}}R\sum_{n\geq 1}\int_0^T |u_n(s)|^p ds\\
\|(\mathcal{F}u)\|^p &\leq T^{\frac{2p-1}{p}}M^pQ + \sup_{t\in I}T^{\frac{2p-1}{p}}R\sum_{n\geq 1}|u_n(t)|^p\\
\|(\mathcal{F}u)\|^p &\leq T^{\frac{2p-1}{p}}M^pQ + T^{\frac{2p-1}{p}}R\|u\|^p_p
\end{aligned}$$

Above inequality can be written as

$$r^p \leq T^{\frac{2p-1}{p}}(M^pQ + Rr^p)$$

Let r_0 denotes the optimal solution of the inequality. Now consider the set $B = B(u_0,r_0) = \{\, u(t)\in C(I,\ell_p) : \|u\|_{C(I,\ell_p)} \leq r,\ u(0)=0,\ u(T) = au(\xi)\}$, which is closed, bounded and convex. Now we show that $\mathcal{F}$ is continuous. Arbitrarily fix $v\in B$,

$$\begin{aligned}
\sum_{n\geq 1}|(\mathcal{F}u)_n(t)-(\mathcal{F}v)_n(t)|^p &= \sum_{n\geq 1}\left|\int_0^T K(t,s)f_n(s,u(s))ds - \int_0^T K(t,s)f_n(s,v(s))ds\right|^p\\
&\leq T^{p-1}\sum_{n\geq 1}\int_0^T |K(t,s)|^p\,|f_n(s,u(s))ds - f_n(s,v(s))|^p ds\\
&\leq T^{p-1}M^p\int_0^T\sum_{n\geq 1}|f_n(s,u(s))ds - f_n(s,v(s))|^p ds
\end{aligned}$$

Now using assumption **B6** for any arbitrarily fixed $v\in B$ and $\epsilon > 0$, there exists $\delta > 0$ such that $\sum_{n\geq 1}|(fu)(t) - (fv)(t)|^p \leq \epsilon^p/(TM)^p$ for each $t\in I$ and for each $u\in B$ such that $\|u-v\|_{\ell_p}\leq\delta$.

$$\begin{aligned}
\left(\sum_{n\geq 1}|(\mathcal{F}u)_n(t)-(\mathcal{F}v)_n(t)|^p\right)^{1/p} &\leq T^{\frac{p-1}{p}}M\left(\int_0^T\sum_{n\geq 1}|f_n(s,u(s))ds - f_n(s,v(s))|^p ds\right)^{1/p}\\
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_{\ell_p} &\leq T^{\frac{p-1}{p}}M\left(\int_0^T\frac{\epsilon^p}{(TM)^p}ds\right)^{1/p} < \epsilon.
\end{aligned}$$

thus $\mathcal{F}$ is continuous.
Now we establish the continuity of $(\mathcal{F}u)$ in $(0,T)$. Let $t_0\in(0,T)$ and $\epsilon > 0$ be arbitrary then, by continuity of $K(t,s)$, there exists $\delta = \delta(t_0,\epsilon) > 0$ such that, for $|t-t_0| < \delta$, we have $|K(t,s)-K(t_0,s)| < T^{1-p}\epsilon^p/(QT + r\tilde{R})$,

where $\tilde{R} = \int_0^T \lim_{n\to\infty}\sup r_n(s)ds$.

$$
\begin{aligned}
\sum_{n\geq1} |(\mathcal{F}u)(t) - (\mathcal{F}u)(t_0)|^p &= \sum_{n\geq1}\left|\int_0^T K(t,s)f_n(s,u(s))ds - \int_0^T K(t_0,s)f_n(s,u(s))ds\right|^p \\
&\leq T^{p-1}\int_0^T |K(t,s)-K(t_0,s)|^p \sum_{n\geq1}|f_n(s,u(s))|^p ds \\
&\leq T^{p-1}\int_0^T |K(t,s)-K(t_0,s)| \sum_{n\geq1}(q_n(s)+r_n(s)|u_i(s)|^p)ds \\
&\leq T^{p-1}\left(\frac{T^{1-p}\epsilon^p}{QT+r\tilde{R}}\right)\int_0^T\left(Q(s)+\lim_{n\to\infty}\sup r_n(s)\sum_{n\geq1}|u_i(s)|^p\right)ds \\
\|(\mathcal{F}u)(t)-(\mathcal{F}u)(t_0)\|_{\ell_p} &< \epsilon.
\end{aligned}
$$

We proceed to show that operator $\mathcal{F}$ is condensing with respect to Hausdorff MNC χ on the space $C(I,\ell_p)$, using formula 4 we define

$$\chi_{C(I,\ell_p)}(B) = \sup_{t\in I}\chi_{\ell_p}(B(t)).$$

$$
\begin{aligned}
\chi_{\ell_p}[(\mathcal{F}B)(t)] &= \lim_{n\to\infty}\left\{\sup_{u\in B}\left(\sum_{i\geq n}|\mathcal{F}u_i(t)|^p\right)^{1/p}\right\} \\
&\leq \lim_{n\to\infty}\left\{\sup_{u\in B}\left(\sum_{i\geq n}\left|\int_0^T K(t,s)f_i(s,u(s))ds\right|^p\right)^{1/p}\right\} \\
&\leq T^{\frac{1-p}{p}}\lim_{n\to\infty}\left\{\sup_{u\in B}\left(\sum_{i\geq n}\int_0^T |K(t,s)|^p(q_i(s)+r_i(s)|u_i(s)|^p)ds\right)^{1/p}\right\} \\
&\leq T^{\frac{1-p}{p}}M\lim_{n\to\infty}\left\{\sup_{u\in B}\left(\sum_{i\geq n}\int_0^T (q_i(s)+r_i(s)|u_i(s)|^p)ds\right)^{1/p}\right\} \\
&\leq T^{\frac{1-p}{p}}M\lim_{n\to\infty}\left\{\sup_{u\in B}\left(\int_0^T\sum_{i\geq n}q_i(s)ds+R\sum_{i\geq n}\int_0^T |u_i(s)|^p ds\right)^{1/p}\right\} \\
\sup_{t\in I}\chi_{\ell_p}[(\mathcal{F}B)(t)] &\leq \sup_{t\in I}T^{\frac{1-p}{p}}M\lim_{n\to\infty}\left\{\sup_{u\in B}\left(RT\sum_{i\geq n}|u_i(t)|^p\right)^{1/p}\right\} \leq MT^{\frac{2-p}{p}}R^{1/p}\chi(B).
\end{aligned}
$$

As $MT^{\frac{2-p}{p}}R^{1/p} < 1$ implies $\mathcal{F}$ is a Darbo condensing operator, thus by Theorem 1.4 $\mathcal{F}$ has a fixed point in B, which is a solution of 11 in space $C(I,\ell_p)$, $p\geq1$. Moreover for each $t\in[0,T]$, $u(t)\in \ker\chi_{C(I,\ell_p)}$. □

Example 3.2. *Consider the following system of FDE in the space* ℓ_2

$$
\begin{cases}
D^{5/4}u_n(t) = \frac{\sqrt{t}\sin(-nt)}{n^2} + \sum_{k=n}^{\infty}\frac{u_k(t)\ln(1+t)}{k^3(n+1)^3} & t\in(0,T) \\
u_n(0)=0,\ u_n(T)=\sqrt[4]{2}\,u_n(T/3); & n=1,2,3\ldots.
\end{cases}
\tag{15}
$$

where $u(T/3) = \{u_n(T/3)\}_{n=1}^{\infty}\in\ell_2$.

Comparing with our result we have, $\xi = T/3$ and $a = \sqrt[4]{2}$ and $f_n(t, u(t)) = \frac{\sqrt{t}\sin(-nt)}{n^2} + \sum_{k=n}^{\infty} \frac{u_k(t)\ln(1+t)}{k^3(n+1)^3}$. The system of FDE is transformed by the following equation

$$u_n(t) = \frac{1}{\Gamma(5/4)(T^{1/4} - \sqrt[4]{2}(T/3)^{1/4})} \int_0^T K(t,s) f_n(s, u(s))ds$$

where $K(t,s)$ is given by $K(t,s) = \frac{1}{\Gamma(1.25)(\sqrt[4]{T} - \sqrt[4]{2}\sqrt[4]{T/3})} \begin{cases} K_1(t,s), & 0 \le t \le \frac{T}{3} \\ K_2(t,s), & \frac{T}{3} \le t \le T \end{cases}$

$$K_1(t,s) = \begin{cases} (t-s)^{1/4}(T^{1/4} - (\frac{2T}{3})^{1/4}) - t^{1/4}[(T-s)^{1/4} - (\frac{2T}{3} - s)^{1/4}]; \ 0 \le s \le t, \\ -t^{1/4}[(T-s)^{1/4} - (\frac{2T}{3} - 2s)^{1/4}]; \ t \le s \le \frac{T}{3}, \\ -(t(T-s)^{1/4}); \ \frac{T}{3} \le s \le T. \end{cases}$$

$$K_2(t,s) = \begin{cases} (t-s)^{1/4}(T^{1/4} - (\frac{2T}{3})^{1/4}) - t^{1/4}[(T-s)^{1/4} - (\frac{2T}{3} - 2s)^{1/4}]; \ 0 \le s \le \frac{T}{3}, \\ (t-s)^{1/4}(T^{1/4} - (\frac{2T}{3})^{1/4}) - (t(T-s))^{1/4}; \ \frac{T}{3} < s \le t, \\ -(t(T-s))^{1/4}; \ t < s \le T. \end{cases}$$

Assumption B1 is satisfied, moreover $f \in \ell_2$ and f is continuous. We show that (B6) is satisfied i.e. $\{(fu)(t)\}_{t\in I}$ is equicontinuous. Let $v \in \ell_2$ and $t \in [0,T]$ be arbitrary, for $\epsilon > 0$ choose $\delta := \epsilon\frac{\sqrt{945}}{\pi\ln(1+T)}$

$$\begin{aligned} \sum_{n\ge1} \left|f(t,u(t)) - f(t,v(t))\right|^2 &= \sum_{n\ge1} \left| \sum_{k\ge n} \frac{u_k(t)\ln(1+t)}{k^3(n+1)^3} - \sum_{k\ge n} \frac{v_k(t)\ln(1+t)}{k^3(n+1)^3} \right|^2 \\ &\le \sum_{n\ge1} \left| \sum_{k\ge n} \frac{u_k(t) - v_k(t)\ln(1+t)}{k^3(n+1)^3} \right|^2 \\ &\le \sum_{n\ge1} \frac{|\ln(1+t)|^2}{(n+1)^6} \sum_{k\ge n} \frac{|u_k(t) - v_k(t)|^2}{k^6} \\ &< \sum_{n\ge1} \frac{|\ln(1+t)|^2}{(n+1)^6} \|u(t) - v(t)\|_{\ell_2}^2 \\ &< \frac{\pi^2}{945} |\ln(1+T)|^2 \|u(t) - v(t)\|_{\ell_2}^2 \\ \|(fu)(t) - (fv)(t)\|_{\ell_2} &< \epsilon, \text{ since } \|u(t) - v(t)\|_{\ell_2} < \delta. \end{aligned}$$

Now we show that f satisfies (B3)

$$\begin{aligned} |f_n(t,u(t))|^2 &= \left| \frac{\sqrt{t}\sin(-nt)}{n^2} + \sum_{k\ge n} \frac{u_k(t)\ln(1+t)}{k^3(n+1)^3} \right|^2 \\ &\le \left| \frac{\sqrt{t}\sin(-nt)}{n^2} \right|^2 + \sum_{k\ge n} \left| \frac{u_k(t)\ln(1+t)}{k^3(n+1)^3} \right|^2 \\ &\le \frac{|t|}{n^4} + \frac{\pi^2\ln(1+t)}{945n^6} |u_n(t)|^2 \end{aligned}$$

$q_n(t) = |t|/n^4$ and $r_n(t) = \frac{\pi^2\ln(1+t)}{945n^6}$. The functions $q_n(t)$ are continuous and the series $\sum_{n\ge1} q_n(t)$ converges uniformly to $q(t) = |t|\frac{\pi^4}{90}$, satisfying B4, also, $\lim_{n\to\infty} r_n(t) = 0$ which is integrable over I thus assumption B5 is satisfied. Hence by Theorem1.7 the system 15 has at least one solution in ℓ_2.

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