Applications of Near Sets

Jim Peters and Som Naimpally

Near sets are disjoint sets that resemble each other. Resemblance is determined by considering set descriptions defined by feature vectors (n-dimensional vectors of numerical features that represent characteristics of objects such as digital image pixels). Near sets are useful in solving problems based on human perception [44, 76, 49, 51, 56] that arise in areas such as image analysis [52, 14, 41, 48, 17, 18], image processing [41], face recognition [13], ethology [63], as well as engineering and science problems [53, 63, 44, 19, 17, 18].

As an illustration of the degree of nearness between two sets, consider an example of the Henry color model for varying degrees of nearness between sets [17, §4.3]. The two pairs of ovals in Figures 1 and 2 contain colored segments. Each segment in the figures corresponds to an equivalence class where all pixels in the class have matching descriptions, i.e., pixels with matching colors. Thus, the ovals in Figure 1 are closer (more near) to each other in terms of their descriptions than the ovals in Figure 2. It is the purpose of this article to give a bird's-eye view of recent developments in the study of the nearness of sets.

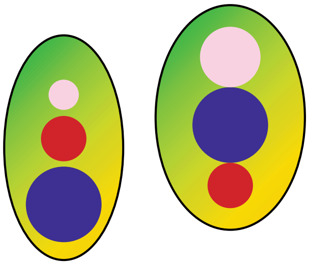

Figure 1. Descriptively, very near sets.

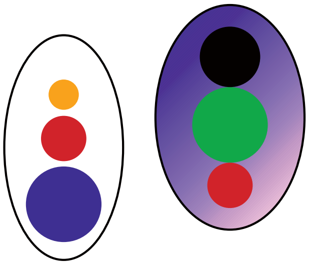

Figure 2. Descriptively, minimally near sets.

Brief History of Nearness

It has been observed that the simple concept of *nearness* unifies various concepts of topological structures [21] inasmuch as the category **Near** of all nearness spaces and nearness-preserving maps contains categories $\mathbf{Top}_s$ (symmetric topological spaces and continuous maps [3]), **Prox** (proximity spaces and δ-maps [8, 66]), **Unif** (uniform spaces and uniformly continuous maps [75, 71]), and **Cont** (contiguity spaces and contiguity maps [23]) as embedded full subcategories [21, 57]. The notion of nearness in mathematics and the more general notion of resemblance can be traced back to J. H. Poincaré, who introduced sets of similar sensations (nascent tolerance classes) to represent the results of G. T. Fechner's sensation sensitivity

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DOI: http://dx.doi.org/10.1090/noti817

experiments [9] and a framework for the study of resemblance in representative spaces as models of what he termed *physical continua* [61, 58, 59].

The elements of a physical continuum (pc) are sets of sensations. The notion of a pc and various representative spaces (tactile, visual, motor spaces) were introduced by Poincaré in an 1894 article on the mathematical continuum [61], an 1895 article on space and geometry [58], and a compendious 1902 book on science and hypothesis [59] followed by a number of elaborations, e.g., [60]. The 1893 and 1895 articles on continua (Pt. 1, ch. II) as well as representative spaces and geometry (Pt. 2, ch. IV) are included as chapters in [59]. Later, F. Riesz introduced the concept of proximity or nearness of pairs of sets at the ICM in Rome in 1908 (ICM 1908) [64].

During the 1960s E. C. Zeeman introduced tolerance spaces in modeling visual perception [78]. A. B. Sossinsky observed in 1986 [67] that the main idea underlying tolerance space theory comes from Poincaré, especially [58] (Poincaré was not mentioned by Zeeman). In 2002, Z. Pawlak and J. Peters considered an informal approach to the perception of the nearness of physical objects, such as snowflakes, that was not limited to spatial nearness [42]. In 2006, a formal approach to the descriptive nearness of objects was considered by J. Peters, A. Skowron, and J. Stepaniuk [54, 55] in the context of proximity spaces [40, 35, 38, 22]. In 2007, descriptively near sets were introduced by J. Peters [46, 45], followed by the introduction of tolerance near sets [43, 47].

Nearness of Sets

The adjective *near* in the context of near sets is used to denote the fact that observed feature value differences of distinct objects are small enough to be considered indistinguishable, i.e., within some tolerance. The exact idea of closeness or "resemblance" or of "being within tolerance" is universal enough to appear, quite naturally, in almost any mathematical setting (see, e.g., [65]). It is especially natural in mathematical applications: practical problems, more often than not, deal with approximate input data and only require viable results with a tolerable level of error [67].

Frigyes Riesz, 1880–1956

The words *near* and *far* are used in daily life and it was an incisive suggestion of F. Riesz [64] to make these intuitive concepts rigorous. He introduced the concept of nearness of pairs of sets at the ICM 1908. This concept is useful in simplifying teaching calculus and advanced calculus. For example, the passage from an intuitive definition of continuity of a function at a point to its rigorous epsilon-delta definition is sometimes difficult for teachers to explain and for students to understand. Intuitively, continuity can be explained using nearness language, i.e., a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point c, provided points $\{x\}$ near c go into points $\{f(x)\}$ near $f(c)$. Using Riesz's idea, this definition can be made more precise and its contrapositive is the familiar epsilon-delta definition.

Bringing *near* into the discussion of continuity makes the transition simpler from the intuitive level of continuity to the rigorous level of continuity. This approach has been successfully used in the classroom (see, e.g., [6, 22, 38]). This approach can also be used in teaching general topology [35, 39], function spaces, hyperspaces, lattices of closed sets, point-free geometries [22, 5], analysis and topology [38, 35, 5]. Point-free topology focuses on open sets rather than points of a space and deals with lattices of open sets called *frames* and their homomorphisms [57, p. 234ff]. Riesz's idea was to establish a natural framework for defining accumulation points that are derived from enchained sets. To formulate the notion of an accumulation point, Riesz proposed an axiomatization of the closeness between sets called *enchainment*, which is proximity *ante litteram* [5, §I]. Riesz chose enchainment as a vehicle for topology. In doing so, he shifted the focus from the closeness of points and sets (as in F. Hausdorff [15, 16] and K. Kuratowski [27, 28, 29]) to closeness between sets (as in [72, 38]).

Since 1908 a number of mathematicians, such as Efremovič, Smirnov, Leader, Čech, and others, have developed the theory of proximity spaces (see, e.g., [8, 66, 30, 72, 40, 69, 10, 21, 12], nicely summarized in [5]). As per the dictum of A. Einstein, "the significant problems we face cannot be solved by the same level of thinking that created them", problems in topology and analysis can be solved and generalized by the use of proximity that is at a higher level than topology [38]. Here are two examples.

(1) **Proximal Wallman compactification**. Using ultrafilters, one gets Wallman compactification [73], which is usually given as an exercise in topology texts. H. Wallman uses the family wX of *all* closed ultrafilters of X and assigns a topology on wX (known as Wallman topology) that makes wX compact. X is embedded in wX via the map that takes a point $x \in X$ to the closed ultrafilter $\mathcal{L}_x$ containing $\{x\}$ (for details, see [73, Theorem 2, p. 120] and [24, §5.R]). Thus, Wallman gets just one T_1 compactification of any T_1 space. By replacing *intersection* with *near* and using the resulting bunches and clusters of S. Leader, we get infinitely many compactifications, including

all Hausdorff compactifications. For details, see [38, §3.1, p. 39ff] and [39, §9.6].

(2) **Taimanov extension of continuous functions**. A. D. Taimanov [68] gave necessary and sufficient conditions for the existence of extensions of continuous functions from dense subspaces of topological spaces when the range is compact Hausdorff. That led to various generalizations using special techniques (see, e.g., [11, 36]). The generalized Taimanov theorem obtained via a proximity (nearness) relation includes as special cases all results on this topic. For the details, see [38, §1.3, p. 16ff] and [39, §1.16].

Using Mozzochi's results on symmetric generalized uniformity, Gagrat and Naimpally characterized developable spaces as those which have compatible upper semi-continuous semi-metrics [10]. This result was used by Domiaty and Laback in a study of semi-metric spaces in general relativity [7]. This puts S. Hawking's approach to general relativity on a more general mathematical foundation (see [38, §15, p. 163ff] and [35]).

Various Near Sets

From a spatial point of view, nearness (*aka* proximity) is considered a generalization of set intersection. For disjoint sets, a form of nearness set intersection is defined in terms of a set of objects (extracted from disjoint sets) that have similar features within some tolerance (see, e.g., [74, §3]). For example, the ovals in Figure 1 are considered near each other, since these ovals contain pairs of classes that display matching (visually indistinguishable) colors. Next, we give some examples to motivate the theory.

Metric Proximity

In a metric space with a metric d, the metric proximity (denoted δ) is defined as follows. Two sets A and B are near (i.e., $A\ \delta\ B$) if and only if $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\} = 0$. This form of metric proximity was introduced by E. Čech [72, §18.A.2], which Čech writes in terms of a *proximity* induced by d [72, §25.A.4] in a seminar on topology given in Brno between May 1936 and November 1939. Metric proximity provides a motivation for the axioms of a proximity space, where there may not even be a metric.

Topological or Fine Proximity

Let cl$E$ denote the *closure* of A such that

$$x \text{ is in the closure of } E \Leftrightarrow \{x\}\ \delta\ E.$$

Put another way, x lies in the closure of a set E, provided that there are points of E as near as we please to x [4, §1.6, p. 22]. In any topological space, there is an associated fine proximity (denoted δ_0). Two sets A and B are *finely* near if and only if their closures intersect, i.e.,

$$A\ \delta_0\ B \ \Leftrightarrow \mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset.$$

It is easy to see that if δ is a metric proximity, then

$$A\ \delta_0\ B \Leftrightarrow \mathrm{cl}A \cap \mathrm{cl}B \neq \emptyset \Rightarrow \mathrm{cl}A\ \delta\ \mathrm{cl}B \Rightarrow A\ \delta\ B.$$

Proximity Space Axioms

Every proximity induces a unique topology that arises from the nearness of points to sets. On the other hand, a topology may have many associated proximities. Metric proximity and fine proximity in a topological space provide a motivation for the following axioms satisfied by all proximity spaces:

(**Prox**. 1) A and B are near sets implies they are not empty,

(**Prox**. 2) A is near B implies B is near A (symmetry),

(**Prox**. 3) A and B intersect implies A and B are near sets,

(**Prox**. 4) A is near $(B \cup C)$ if and only if A is near B or A is near C.

Most of the literature in topology uses an additional axiom that is a vestigial form of the triangle inequality [34, 40]:

(**Prox**. 5) If A is far from B, there is an $E \subset X$ such that A is far from E and B is far from $X - E$.

Quasi-Proximity

Dropping (**Prox**. 2) (symmetry) gives rise to a quasi-proximity relation [26, §2.5, p. 262ff] and (**Prox**. 4) becomes

(**qProx**. 4) $(B \cup C)$ near A if and only if B is near A or C is near A and A is near $(B \cup C)$ implies A is near B or A is near C.

It has been observed by H.-P. Künzi [26] that the topology $\tau(\delta)$ induced by the quasi-proximity δ on X arises from the closure $\mathrm{cl}_{\tau(\delta)}$ defined by

$$x \in \mathrm{cl}_{\tau(\delta)}A \Leftrightarrow x\ \delta\ A.$$

Alexandroff Spaces and Quasi-Proximity

An Alexandroff space is a topological space such that every point has a minimal neighborhood or, equivalently, an Alexandroff space has a unique minimal base [2]. A topological space is Alexandroff if and only if the intersection of every family of open sets is open. These spaces were introduced by P. Alexandroff in 1937 [1]. Let X be a topological space, and let $A, B \in \mathcal{P}(X)$. In terms of quasi-proximity δ, F. G. Arenas obtained the following result. $A\ \delta\ B$ if and only if $A \cap \mathrm{cl}(B) \neq \emptyset$ is a quasi-proximity compatible with the topology [2, Theorem 4.3]. Also observe that every finite topological space is Alexandroff. Starting in the 1990s, Alexandroff spaces were found to be important in the study of digital topology (see, e.g., [20, 25]).

For Alexandroff spaces considered in the context of near sets, see [76, 77].

Adding Proximity Space Axioms

In the study or use of a proximity space in a problem, an additional appropriate axiom is added relative to the application. For example, in studying the nearness of digital images, one can view an image X as a set of points with distinguishing features such as entropy or color or gray level intensity and introduce some form of tolerance relation that determines image tolerance classes. Let X, Y denote a pair of images and let $\phi : X \to \mathfrak{R}$ be a real-valued function representing an image feature such as an average gray level of subimages. Put $\varepsilon \in [0, \infty)$, and let $x \in X, y \in Y$ denote subimages. Then introduce the description-based tolerance relation $\simeq_{\phi,\varepsilon}$, i.e.,

$$\simeq_{\phi,\varepsilon} = \{(x, y) \in X \times Y : \left|\phi(x) - \phi(y)\right| < \varepsilon\}.$$

This leads to

(**Prox**. 6) A and B are near sets $\iff$ there are $x \in A, y \in B$, such that $x \simeq_{\phi,\varepsilon} y$.

If a pair of nonempty sets A, B satisfy (**Prox**. 6), then A and B are termed *tolerance near sets*. Such sets provide a basis for a quantitative approach to evaluating the similarity of objects without requiring object descriptions to be exact (see, e.g., [17]).

Nearness of Pictures

The concept of *nearness* enters as soon as one starts studying digital images (see, e.g., [62, 37, 52, 41, 47]). The digital image of a photograph should resemble, as accurately as possible, the original subject, i.e., an image should be globally close to its source. Since proximity deals with global properties, it is appropriate for this study. The quality of a digital image depends on proximity and this proximity is more general than the one obtained from a metric.

We note here that a digital image of a landscape is made up of a very limited number of points (depending on the sensory array of a camera), whereas the original landscape in a visual field contains many more points than its corresponding digital image. However, from the point of view of *perception*, they are near, depending on the tolerance we choose rather crudely in comparing visual field segments of a real scene with digital image patches (sets of scattered pixels). To make such comparisons work, the requirement that the image should appear as precise as possible as the original is relaxed. And the *precise-match* requirement is replaced by a *similarity* requirement so that a digital image should only remind us (within some tolerance) of the original scene.

Then, for example, a cartoon in a newspaper of a person may be considered near, if parts of a cartoon are similar or if it resembles an original scene. For instance, in Figure 3, if the feature we consider is behavior (e.g., braiding hair), the mother is perceptually near the daughter, since both are braiding hair. And, in Figure 3, the drawing of the mother braiding her daughter's hair is near a familiar scene where a mother is caring for her daughter's hair. This suggests that whether two objects are near or not depends on what is needed.

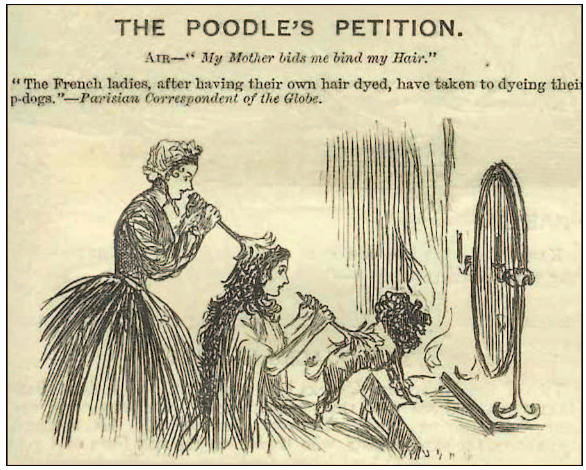

Figure 3. Chain Reaction, Punch, 1869.

Sufficient Nearness of Sets

The notion of *sufficiently near* appears in N. Bourbaki [4, §2, p. 19] in defining an open set, i.e., a set A is open if and only if for each $x \in A$, all points *sufficiently near* x belong to A.

Moreover, a property holds for all points sufficiently near $x \in A$, provided the property holds for all points in the neighborhood of x. Set F_1 in Figure 4 is an example of an open set represented by a dotted boundary. In fact, sets F_2, F_3 are also examples of open sets (i.e., open neighborhoods of the point x). Bourbaki's original view of *sufficiently near* (denoted δ_ε) is now extended to a relaxed view of the nearness of nonempty sets. This form of proximity relation is useful in considering, for example, a relaxed form of metric proximity in relation to ε-collars of sets [5, §2.2], especially in approach space theory [31, 33, 32, 70, 51, 56], as well as in considering the nearness of pictures.

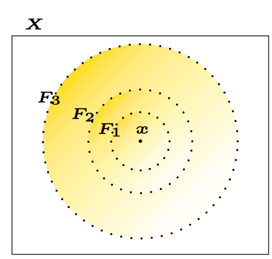

Figure 4. Near open sets.

Let $\varepsilon \in (0, \infty]$. Nonempty sets A, B are considered *sufficiently near* each other if and only if

$$A \ \delta_\varepsilon \ B \Leftrightarrow \inf\{d(a, b) : a \in A, b \in B\} < \varepsilon.$$

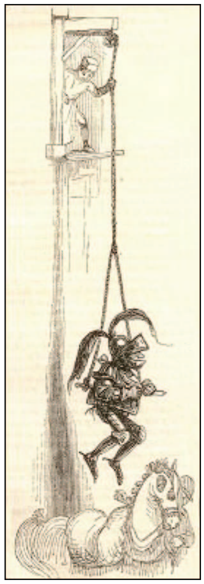

Figure 5. A Bit Far, *Punch*, 1845.

Otherwise, sets A, B are *remote* (denoted $\underline{\delta}_{\varepsilon}$), i.e., *sufficiently apart* or *far* from each other, provided

$$A\ \underline{\delta}_{\varepsilon}\ B \Leftrightarrow \inf\{d(a,b) : a \in A, b \in B\} \geq \varepsilon.$$

In keeping with the proximity space approach, an axiom is added to cover sufficient nearness. This leads to axiom (**Prox**. 7).

(**Prox**. 7) A and B are sufficiently near sets $\iff \inf\{d(a,b) : a \in A, b \in B\} < \varepsilon$.

In a more general setting, nearness and apartness are considered relative to the gap between collections $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ in an approach space [50, 51, 56]. The choice of a particular value of ε is application dependent and is typically determined by a domain expert.

Far Apart and Near Pictures

Apart from the fact that the knight is far from his horse[1] in Figure 5 and the picture of the hairdressers in Figure 3 can be viewed as either near (if we consider the gray-level intensities of the pixels in the two pictures) or far apart (if we consider the behaviors represented by the two pictures). Description-based nearness or apartness between sets depends on the features we select for comparison.

Let $\phi : X \to \mathbb{R}$ be a probe function that extracts a feature value from a picture element. Let $A_i, i \in \{1,2,3,4,5\}$ denote sets of picture elements in A_1 (Figure 1 (ovals)), A_2 (Figure 2 (ovals)), A_3 (Figure 3 (hairdressers)), A_4 (Figure 4 (concentric neighborhoods of point x)), and A_5 (Figure 5 (knight)), respectively, define the description-based sufficient nearness relation $\delta_{\varepsilon,\phi}$ by

$$A\ \delta_{\varepsilon,\phi}\ B \Leftrightarrow \inf\{d(\phi(a), \phi(b)) : a \in A, b \in B\} < \varepsilon,$$

where $d(\phi(a), \phi(b))$ is the standard distance between feature values $\phi(a), \phi(b)$. For Examples 1 and 2 below assume $\phi(x)$ returns the gray level intensity of picture element x and assume $\varepsilon = 25$ (almost black, i.e., almost zero light intensity on a scale from 0 (black) to 255 (white)). It is easy to verify the following examples of near and far sets.

(**Ex**. 1) $A_1\ \delta_{\varepsilon,\phi}\ A_2$, i.e., A_1 is near A_2, since the greatest lower bound of the differences will be close to zero because the intensities of the darker oval pixels are almost equal.

(**Ex**. 2) $F_1\ \delta_{\varepsilon,\phi}\ F_2$, i.e., neighborhood F_1 is near neighborhood F_2 in A_4, since the pixels in F_1 are common to both neighborhoods.

(**Ex**. 3) For this example, let $X = A_3 \cup A_5, \varepsilon = 0.5$ and define

$$\phi(x) = \begin{cases} 1, & \text{if } x \in X \text{ portrays a hairdressing behavior,} \\ 0, & \text{otherwise.} \end{cases}$$

$A_3\ \underline{\delta}_{\varepsilon,\phi}\ A_5$, i.e., A_3 (hairdressers) is far from A_5 (knight), since the behaviors represented by the two pictures are different. If we assume $A = A_3, B = A_5$, then $\inf\{d(\phi(a), \phi(b)) : a \in A, b \in B\} = 1$.

[1] *The picture of the knight being lowered onto his horse appears in vol. IX, 1845,* Punch.

References

1. P. Alexandroff, Diskrete räume, *Mat. Sb. (N.S.)* **2** (1937), 501–518.
2. F. G. Arenas, Alexandroff spaces, *Acta Math. Univ. Comenianae* **1** (1999), 17–25.
3. H. L. Bentley, E. Colebunders, and E. Vandermissen, *A Convenient Setting for Completions and Function Spaces*, Contemporary Mathematics (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., Providence, RI, 2009, pp. 37–88.
4. N. Bourbaki, *Elements of Mathematics. General Topology, Part 1*, Hermann & Addison-Wesley, Paris & Reading, MA, 1966, i-vii, 437 pp.
5. A. Di Concilio, *Proximity: A Powerful Tool in Extension Theory, Function Spaces, Hyperspaces, Boolean Algebras and Point-free Geometry*, Contemporary Mathematics (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., Providence, RI, 2009, pp. 89–114.
6. K. Devlin, *Will the real continuous function please stand up?*, Tech. report, 2006, `http://www.maa.org/devlin/devlin_11_06.html`.
7. R. Z. Domiaty and O. Laback, Semimetric spaces in general relativity (on Hawking–King–McCarthy's path topology), *Russian Math. Surveys* **35** (1980), no. 3, 57–69.
8. V. A. Efremovič, The geometry of proximity. I, *Mat. Sb.* **31** (1951), 189–200.
9. G. T. Fechner, *Elemente der Psychophysik*, 2 vols., E. J. Bonset, Amsterdam, 1860.
10. M. Gagrat and S. A. Naimpally, Proximity approach to semi-metric and developable spaces, *Pacific J. Math.* **44** (1973), no. 1, 93–105.

11. ______, Proximity approach to extension problems, *Fund. Math.* **71** (1971), 63–76.
12. G. DiMaio and S. A. Naimpally, *D*-proximity spaces, *Czech. Math. J.* **41** (1991), no. 116, 232–248.
13. S. Gupta and K. Patnaik, Enhancing performance of face recognition systems by using near set approach for selecting facial features, *J. Theoretical and Applied Information Technology* **4** (2008), no. 5, 433–441.
14. A. E. Hassanien, A. Abraham, J. F. Peters, G. Schaefer, and C. Henry, Rough sets and near sets in medical imaging: A review, *IEEE Trans. Info. Tech. in Biomedicine* **13** (2009), no. 6, 955–968, doi: 10.1109/TITB.2009.2017017.
15. F. Hausdorff, *Grundzüge der Mengenlehre*, Veit and Company, Leipzig, 1914, viii + 476 pp.
16. ______, *Set Theory*, AMS Chelsea Publishing, Providence, RI, 1914, 352 pp.
17. C. J. Henry, *Near Sets: Theory and Applications*, Ph.D. dissertation, supervisor: J. F. Peters, Ph.D. thesis, Department of Electrical & Computer Engineering, 2010.
18. C. J. Henry and J. F. Peters, Arthritic hand-finger movement similarity measurements: Tolerance near set approach, *Comp. & Math. Methods in Medicine* 2011, Article ID 569898 (2011), 1–14, doi:10.1155/2011/569898.
19. C. J. Henry and S. Ramanna, *Parallel Computation with Near Sets*, Rough Sets and Knowledge Technology (RSKT2011), Springer-Verlag, 2011, pp. 523–532.
20. G. T. Herman, On topology as applied to image analysis, *Computer Vision, Graphics, and Image Processing* **52** (1990), 409–415.
21. H. Herrlich, A concept of nearness, *Gen. Top. & Appl.* **4** (1974), 191–212.
22. J. G. Hocking and S. A. Naimpally, *Nearness—A Better Approach to Continuity and Limits*, Allahabad Mathematical Society Lecture Note Series, 3, The Allahabad Math. Soc., Allahabad, 2009, iv+66 pp., ISBN 978-81-908159-1-8.
23. V. M. Ivanova and A. A. Ivanov, Contiguity spaces and bicompact extensions of topological spaces (Russian), *Dokl. Akad. Nauk SSSR* **127** (1959), 20–22.
24. J. L. Kelley, *General Topology*, Springer-Verlag, Berlin, 1955, xiv + 298 pp.
25. E. H. Kronheimer, The topology of digital images, *Top. and its Appl.* **46** (1992), 279–303.
26. H.-P. Künzi, *An introduction to quasi-uniform spaces*, Contemporary Mathematics (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., Providence, RI, 2009, pp. 239–304.
27. C. Kuratowski, *Topologie. I*, Panstwowe Wydawnictwo Naukowe, Warsaw, 1958, xiii + 494pp.
28. ______, *Introduction to Calculus*, Pergamon Press, Oxford, UK, 1961, 316pp.
29. ______, *Introduction to Set Theory and Topology*, 2nd ed., Pergamon Press, Oxford, UK, 1962, 1972, 349pp.
30. S. Leader, On clusters in proximity spaces, *Fund. Math.* **47** (1959), 205–213.
31. R. Lowen, *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press, Oxford, UK, 1997, viii + 253pp.
32. R. Lowen and C. V. Olmen, *Approach Theory*, Contemporary Mathematics (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., Providence, RI, 2009, pp. 305–332.
33. R. Lowen, D. Vaughan, and M. Sioen, Completing quasi metric spaces: An alternative approach, *Houston J. Math.* **29** (2003), no. 1, 113–136.
34. C. J. Mozzochi, M. S. Gagrat, and S. A. Naimpally, *Symmetric Generalized Topological Structures*, Exposition Press, Hicksville, N.Y., 1976, xii+74 pp., ISBN 978-3-486-58917-7.
35. C. J. Mozzochi and S. A. Naimpally, *Uniformity and Proximity*, Allahabad Mathematical Society Lecture Note Series, 2, The Allahabad Math. Soc., Allahabad, 2009, xii+153 pp., ISBN 978-81-908159-1-8.
36. S. A. Naimpally, Reflective functors via nearness, *Fund. Math.* **85** (1974), 245–255.
37. ______, Near and far. A centennial tribute to Frigyes Riesz, *Siberian Electronic Mathematical Reports* **2** (2005), 144–153.
38. ______, *Proximity Approach to Problems in Topology and Analysis*, Oldenburg Verlag, München, 2009, xiv + 204pp, ISBN 978-3-486-58917-7.
39. S. A. Naimpally and J. F. Peters, *Topology with Applications. Topological Spaces via Near and Far*, World Scientific, Singapore, 2012, *to appear.*
40. S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge Tract in Mathematics No. 59, Cambridge University Press, Cambridge, UK, 1970, x+128 pp., paperback (2008).
41. S. K. Pal and J. F. Peters, *Rough Fuzzy Image Analysis. Foundations and Methodologies*, CRC Press, Taylor & Francis Group, Sept., 2010, ISBN 13: 9781439803295 ISBN 10: 1439803293.
42. Z. Pawlak and J. F. Peters, Jak blisko, *Systemy Wspomagania Decyzji* **I** (2007), 57.
43. J. F. Peters, Tolerance near sets and image correspondence, *International Journal of Bio-Inspired Computation* **1** (2009), no. 4, 239–245.
44. J. F. Peters and P. Wasilewski, Foundations of near sets, *Info. Sciences* **179** (2009), no. 18, 3091–3109.
45. J. F. Peters, Near sets. General theory about nearness of objects, *Applied Mathematical Sciences* **1** (2007), no. 53, 2609–2029.
46. ______, Near sets. Special theory about nearness of objects, *Fund. Inform.* **75** (2007), no. 1-4, 407–433.
47. ______, Corrigenda and addenda: Tolerance near sets and image correspondence, *Int. J. Bio-Inspired Computation* **2** (2010), no. 5, 310–319.
48. ______, *How Near are Zdzisław Pawlak's Paintings? Merotopic Distance Between Regions of Interest*, Intelligent Systems Reference Library, volume dedicated to Prof. Zdzisław Pawlak (A. Skowron and S. Suraj, eds.), Springer, Berlin, 2011, pp. 1–19.
49. ______, *Sufficiently Near Sets of Neighbourhoods*, Lecture Notes in Artificial Intelligence 6954 (J. T. Yao, S. Ramanna, G. Wang, and Z. Suraj, eds.), Springer, Berlin, 2011, pp. 17–24.
50. J. F. Peters and M. Borkowski, *ε-Near Collections*, Lecture Notes in Artificial Intelligence 6954 (J. T. Yao, S. Ramanna, G. Wang, and Z. Suraj, eds.), Springer, Berlin, 2011, pp. 533–542.
51. J. F. Peters and S. A. Naimpally, Approach spaces for near families, *Gen. Math. Notes* **2** (2011), no. 1, 159–164.
52. J. F. Peters and L. Puzio, Image analysis with anisotropic wavelet-based nearness measures, *International Journal of Computational*

Intelligence Systems **2** (2009), no. 3, 168–183, doi 10.1016/j.ins.2009.04.018.
53. J. F. Peters, S. Shahfar, S. Ramanna, and T. Szturm, *Biologically-Inspired Adaptive Learning: A Near Set Approach*, Frontiers in the Convergence of Bioscience and Information Technologies (Korea), 2007.
54. J. F. Peters, A. Skowron, and J. Stepaniuk, *Nearness in Approximation Spaces*, Proc. Concurrency, Specification and Programming (CS&P 2006) (Humboldt Universität), 2006, pp. 435–445.
55. ______, Nearness of objects: Extension of approximation space model, *Fund. Inform.* **79** (2007), no. 3-4, 497–512.
56. J. F. Peters and S. Tiwari, Approach merotopies and near filters, *Gen. Math. Notes* **3** (2011), no. 1, 1–15.
57. J. Picado, Weil nearness spaces, *Portugaliae Math.* **55**, no. 2.
58. J. H. Poincaré, L'espace et la géomètrie, *Revue de métaphysique et de morale* **3** (1895), 631–646.
59. ______, Sur certaines surfaces algébriques; troisième complément a l'analysis situs, *Bulletin de la Société de France* **30** (1902), 49–70.
60. ______, *Dernières Pensées*, trans. by J. W. Bolduc as mathematics and science: Last essays, Flammarion & Kessinger Pub., Paris & NY, 1913 & 2009.
61. ______, Sur la nature du raisonnement mathématique, *Revue de métaphysique et de morale* **2** (juil. 1894), 371–384.
62. P. Pták and W. G. Kropatsch, Nearness in digital images and proximity spaces, *LNCS* **1953** (2000), 69–77, In Proc. 9th International Conference on Discrete Geometry for Computer Imagery.
63. S. Ramanna and A. H. Meghdadi, Measuring resemblances between swarm behaviours: A perceptual tolerance near set approach, *Fund. Inform.* **95** (2009), no. 4, 533–552, doi: 10.3233/FI-2009-163.
64. F. Riesz, Stetigkeitsbegriff und abstrakte mengenlehre, *Atti del IV Congresso Internazionale dei Matematici* **II** (1908), 18–24.
65. Ju. A. Shreider, *Equality, Resemblance, and Order*, Mir Publishers, Russia, 1975, 279 pp.
66. J. M. Smirnov, On proximity spaces, *Mat. Sb.* **31** (1952), no. 73, 543–574.
67. A. B. Sossinsky, Tolerance space theory and some applications, *Acta Applicandae Mathematicae: An International Survey Journal on Applying Mathematics and Mathematical Applications* **5** (1986), no. 2, 137–167.
68. A. D. Taimanov, On the extension of continuous mappings of topological spaces, *Mat. Sb.* **31** (1952), 451–463.
69. W. J. Thron, Proximity structures and grills, *Math. Ann.* **206** (1973), 35–62.
70. S. Tiwari, *Some aspects of general topology and applications. Approach merotopic structures and applications*, supervisor: M. Khare, Ph.D. thesis, Department of Mathematics, Allahabad (U.P.), India, Jan. 2010.
71. J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton Univ. Press, Annals of Math. Studies AM-2, Princeton, NJ, 1940, 90pp.
72. E. Čech, *Topological Spaces*, revised ed. by Z. Frolik and M. Katătov, John Wiley & Sons, NY, 1966.
73. H. Wallman, Lattices and topological spaces, *Ann. of Math.* **39** (1938), no. 1, 112–126.
74. P. Wasilewski, J. F. Peters, and S. Ramanna, Perceptual tolerance intersection, *Trans. on Rough Sets* **XIII** (2011), 159–174.
75. A. Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Harmann & cie, Actualités scientifique et industrielles, Paris, 1938.
76. M. Wolski, Perception and classification. A note on near sets and rough sets, *Fund. Inform.* **101** (2010), 143–155.
77. ______, *Gauges, Pregauges and Completions: Some Theoretical Aspects of Near and Rough Set Approaches to Data*, Lecture Notes in Artificial Intelligence 6954 (J. T. Yao, S. Ramanna, G. Wang, and Z. Suraj, eds.), Springer, Berlin, 2011, pp. 559–568.
78. E. C. Zeeman, *The Topology of the Brain and Visual Perception*, University of Georgia Institute Conference Proceedings (1962), Published in M. K. Fort, Jr. (ed.), Topology of 3-Manifolds and Related Topics, Prentice-Hall, Inc., 1962, 240–256.