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Article

Applications of Neutrosophic q-Poisson distribution Series for Subclass of Analytic Functions and Bi-Univalent Functions

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Abstract: By using the generalization of the neutrosophic q-Poisson distribution series, we introduce a new subclass of analytic and bi-univalent functions defined in the open unit disk. We then apply the q-Gegenbauer polynomials to investigate the estimates for the Taylor coefficients and Fekete–Szegö type inequalities of the functions belonging to this new subclass. In addition, we consider several corollaries and the consequences of the results presented in this paper. The neutrosophic q-Poisson distribution is expected to be significant in a number of areas of mathematics, science, and technology.

Keywords: neutrosophic q-Poisson distribution; q-gegenbauer polynomials; bi-univalent functions; analytic functions; Fekete–Szegö problem; q-calculus

MSC: 30C45; 30C50

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1. Introduction

The distributions of random variables have attracted a lot of interest recently. In probability theory and statistics, their probability density functions have been crucial. As a result, distributions have been extensively studied. Many other types of distributions, including the binomial distribution, the Poisson distribution, and the hypergeometric distribution, are taken into account from real-world scenarios.

The probability density function of a Poisson distribution for a discrete random variable x is as follows [1]:

$$f(x) = \frac{e^{-j}}{x!} j^x, \; x = 0, 1, 2, \cdots \qquad (j > 0),$$

where j is the parameter of the distribution.

Legendre [2] first made the discovery of orthogonal polynomials in 1784. Under specific model restrictions, orthogonal polynomials are frequently employed to solve ordinary differential equations. Furthermore, a crucial function in the approximation theory is performed by the orthogonal polynomials [3].

Let P_m and P_n are two polynomials of order m and n, respectively. Then, P_m and P_n are orthogonal polynomial over the interval $[a, b]$ if

$$\langle P_n, \ P_m \rangle = \int_a^b P_n(x) P_m(x) s(x) dx = 0, \qquad \text{for} \qquad m \neq n,$$

where $s(x)$ is a non-negative function in the interval (a, b); therefore, all finite order polynomials $P_n(x)$ have a well-defined integral (for more details see [2,3]).

Gegenbauer polynomials are a specific type of orthogonal polynomials. As found in [4], when traditional algebraic formulations are used, the generating function of Gegenbauer polynomials and the integral representation of typically real functions T_R are related to each other in a symbolic way T_R. This undoubtedly caused a number of helpful inequalities from the world of Gegenbauer polynomials to emerge.

q-Orthogonal polynomials are now of particular relevance in both physics and mathematics due to the development of quantum groups. The q-deformed A harmonic oscillator, for instance, has a group-theoretic setting for the q-Laguerre and q-Hermite polynomials. Jackson's q-exponential plays a crucial part in the mathematical framework required to characterize the properties of these q-polynomials, such as the recurrence relations, generating functions, and orthogonality relations. Jackson's q-exponential has recently been expressed by Quesne [5] as a closed form multiplicative series of regular exponentials with known coefficients. In this case, it is crucial to look into how this discovery might affect the theory of q-orthogonal polynomials. An effort in such regard is made in the current work. To get novel nonlinear connection equations for q-Gegenbauer in terms of their respective classical equivalents, we use the aforementioned result in particular. The orthogonal polynomials are the theoretic basis for solving the simple governing PDE of pressure distributions in fractured media see ([6,7]).

This study analyzes various features of the class under consideration after associating some bi-univalent functions with q-Gegenbauer polynomials. The following part lays the foundation for mathematical notations and definitions.

2. Preliminaries

Let $\mathfrak{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus, each $f \in \mathfrak{A}$ has a Taylor–Maclaurin series expansion of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in \mathbb{U}). \tag{1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathfrak{A}$ which are univalent in $\mathbb{U}$.

Let the functions f and g be analytic in $\mathbb{U}$. We say that the function f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(\xi)| < 1 \qquad (\xi \in \mathbb{U}),$$

such that

$$f(\xi) = g(w(\xi)).$$

In addition, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds

$$f(\xi) \prec g(\xi) \quad \text{if and only if} \quad f(0) = g(0),$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(\xi)) = \xi \qquad (\xi \in \mathbb{U}),$$

and

$$f^{-1}(f(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Example of functions in the class Σ are

$$\frac{\xi}{1-\xi}, \qquad \log\sqrt{\frac{1+\xi}{1-\xi}}.$$

In 2021, Mustafa and Nezir [8] introduced certain subclasses of analytic and univalent functions in $\mathbb{U}$. In addition, applications of a q–Poisson distribution series on the analytic functions.

A discrete random variable ϑ is said to have q–Poisson distribution if it takes the values $0, 1, 2, 3, \ldots$ with the probabilities

$$e_q^{-\mu}, \frac{\mu e_q^{-\mu}}{[1]_q!}, \frac{\mu^2 e_q^{-\mu}}{[2]_q!}, \frac{\mu^3 e_q^{-\mu}}{[3]_q!}, \frac{\mu^4 e_q^{-\mu}}{[4]_q!}, \ldots$$

respectively, where μ is called the parameter and

$$e_q^{\vartheta} = 1 + \frac{\vartheta}{[1]_q!} + \frac{\vartheta^2}{[2]_q!} + \frac{\vartheta^3}{[3]_q!} + \cdots + \frac{\vartheta^m}{[m]_q!} + \cdots = \sum_{m=0}^{\infty} \frac{\vartheta^m}{[m]_q!}$$

is the q–analogue of the exponential function e^{ϑ} and

$$[m]_q! = [m]_q.[m-1]_q \cdots [2]_q[1]_q$$

is the q–analogue of the factorial function $m! = m.(m-1)\cdots 3.2.1$, (see [9,10]).

Thus, for q–Poisson distribution, we have

$$\mathcal{P}_q(\vartheta = m) = \frac{\mu^m}{[m]_q!} e_q^{-\mu}, \quad m = 0, 1, 2, \cdots.$$

Meanwhile, q–Poisson distribution series is defined as

$$\xi + \sum_{n=2}^{\infty} \frac{\mu^{m-1} e_q^{-\mu}}{[m-1]_q!} \xi^m, \qquad \xi \in \mathbb{U}. \tag{3}$$

We note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity.

Recently, precisely in 1995, Smarandache introduced the concept of neutrosophic theory. It is a new branch of philosophy as a generalization for the fuzzy logic, also as a generalization of the intrinstic fuzzy logic (see [11]). Neutrosophic q–Poisson distribution of a discrete variable ϑ is a classical q–Poisson distribution of ϑ, but its parameter is imprecise, m can be set with two or more elements. The most common of such distribution is when m is interval. Let

$$\mathcal{N}\mathcal{P}_q(\vartheta = k) = \frac{(m_{\mathcal{N}})^k}{[k]_q!} e_q^{-m_{\mathcal{N}}}, \quad k = 0, 1, 2, \cdots.$$

where $m_{\mathcal{N}}$ is the distribution parameter and is equal to the expected value and the variance, that is,

$$\mathcal{N}E(\vartheta) = \mathcal{N}V(\vartheta) = m_{\mathcal{N}},$$

and $\mathcal{N} = d + I$ is a neutrosophic statistical number (see [11]) and the references therein. Now, we modifiy (3) as follows

$$\mathfrak{M}_q(m_{\mathcal{N}}, \xi) = \xi + \sum_{n=2}^{\infty} \frac{(m_{\mathcal{N}})^{n-1} e_q^{-m_{\mathcal{N}}}}{[n-1]_q!} \xi^n, \qquad \xi \in \mathbb{U}. \tag{4}$$

Now, we consider the linear operator $\mathfrak{B}_{m_{\mathcal{N}}}(\xi) : \mathfrak{A} \longrightarrow \mathfrak{A}$ defined by the convolution or Hadamard product

$$\mathfrak{B}_{m_{\mathcal{N}}} f(\xi) = \mathfrak{M}_q(m_{\mathcal{N}}, \xi) * f(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(m_{\mathcal{N}})^{n-1} e_q^{-m_{\mathcal{N}}}}{[n-1]_q!} a_n \xi^n, \qquad \xi \in \mathbb{U}, \tag{5}$$

In 1983, Askey and Ismail [12] found a class of polynomials which can be interpreted as q–analogues of the Gegenbauer polynomials. These are essentially, the polynomials

$$\mathfrak{G}_q^{(\aleph)}(\varkappa, \xi) = \sum_{n=0}^{\infty} \mathcal{C}_n^{(\aleph)}(\varkappa; q)\, \xi^n, \tag{6}$$

the initial values of $\mathcal{C}_n^{(\aleph)}(\varkappa; q)$ was found by Chakrabarti et al. [13] in 2006, given by

$$\begin{aligned} &\mathcal{C}_0^{(\aleph)}(\varkappa; q) = 1, \qquad \mathcal{C}_1^{(\aleph)}(\varkappa; q) = [\aleph]_q C_1^1(\varkappa) = 2[\aleph]_q \varkappa, \\ &\mathcal{C}_2^{(\aleph)}(\varkappa; q) = [\aleph]_{q^2} C_2^1(\varkappa) - \frac{1}{2}\left([\aleph]_{q^2} - [\aleph]_q^2\right) C_1^2(\varkappa) = 2\left([\aleph]_{q^2} + [\aleph]_q^2\right)\varkappa^2 - [\aleph]_{q^2}. \end{aligned} \tag{7}$$

where $\aleph \in \mathbb{N} = \{1, 2, 3, \cdots\}$, $q \in (0, 1)$ and $C_n^{\alpha}(\varkappa)$ is the classical Gegenbauer polynomial of degree n (for more details, see [14]).

In 2021, Amourah et al. [14,15] considered the classical Gegenbauer polynomials $\mathfrak{G}^{(\lambda)}(x, \xi)$ where $\xi \in \mathbb{U}$ and $x \in [-1, 1]$. For fixed x the function $\mathfrak{G}^{(\lambda)}$ is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$\mathfrak{G}^{(\alpha)}(\varkappa, \xi) = \sum_{n=0}^{\infty} C_n^{\alpha}(\varkappa) \xi^n,$$

where $C_n^{\alpha}(\varkappa)$ is the classical Gegenbauer polynomial of degree n (see, [16]).

In 2022, Amourah et al. [16] three subclasses of analytic and bi-univalent functions are introduced through the use of q-Gegenbauer polynomials. For functions falling within these subclasses, coefficient bounds $|a_2|$ and $|a_3|$ as well as Fekete–Szegö inequalities are derived.

Recently, several authors have begun examining bi-univalent functions connected to orthogonal polynomials, of which a few are worth mentioning ([17–29]). This study can be applied to risk and financial management, and there are several recently published studies ([30,31]).

As far as we are aware, there is no published work on bi-univalent functions for the neutrophilic q-Poisson distribution series subordinate q-Gegenbauer polynomials. The major objective of this work is to start an investigation of the characteristics of bi-univalent functions related to q-Gegenbauer polynomials. In order to accomplish so, we consider the definitions below.

3. The Class $\mathcal{B}_\Sigma(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$

In this section, we introduce a new subclass of Σ involving the new constructed series (4) and q–analogues of the Gegenbauer polynomials.

Definition 1. *For $x \in (\frac{1}{2}, 1]$ and $0 < q < 1$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ if the following subordination is satisfied:*

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} f(\xi)}{\xi} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} f(\xi) \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi), \tag{8}$$

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} g(\omega)}{\omega} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} g(\omega) \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \omega), \tag{9}$$

where $\aleph > 0$, $\delta \geq 0$ and m is a nonzero real constant, and the function $g = f^{-1}$ is given by (1).

Special cases:

(i) Let $\delta = 1$, $x \in (\frac{1}{2}, 1]$ and $0 < q < 1$, A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_{\Sigma}(1, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ if the following subordination is satisfied:

$$\partial_q \mathfrak{B}_{m_{\mathcal{N}}} f(\xi) \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi),$$

and

$$\partial_q \mathfrak{B}_{m_{\mathcal{N}}} g(\omega) \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \omega),$$

where $\aleph > 0$.

(ii) Let $\delta = 0$, $x \in (\frac{1}{2}, 1]$ and $0 < q < 1$, A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_{\Sigma}(0, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ if the following subordination is satisfied:

$$\frac{\mathfrak{B}_{m_{\mathcal{N}}} f(\xi)}{\xi} \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi),$$

and

$$\frac{\mathfrak{B}_{m_{\mathcal{N}}} g(\omega)}{\omega} \prec \mathfrak{G}_q^{(\aleph)}(\varkappa, \omega),$$

where $\aleph > 0$.

4. Coefficient Bounds of the Subclass $\mathcal{B}_{\Sigma}(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$

First, we give the coefficient estimates for the class $\mathcal{B}_{\Sigma}(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ given in Definition 1.

Theorem 1. *Let given by (1) belong to the class $\mathcal{B}_{\Sigma}(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$, then*

$$|a_2| \le \frac{2[\aleph]_q \varkappa e_q^{m_{\mathcal{N}}} \sqrt{2[2]_q [\aleph]_q \varkappa}}{m_{\mathcal{N}} \sqrt{\begin{array}{c} \left(4[\aleph]_q^2 e_q^{m_{\mathcal{N}}}(1 + q(1+q)\delta) - 2[2]_q(1+q\delta)^2\left([\aleph]_{q^2} + [\aleph]_q^2\right)\right)\varkappa^2 \\ +[2]_q(1+q\delta)^2[\aleph]_{q^2} \end{array}}},$$

and

$$|a_3| \le \frac{4[\aleph]_q^2 \varkappa^2 \left(e_q^{m_{\mathcal{N}}}\right)^2}{\left((1+q\delta)m_{\mathcal{N}}\right)^2} + \frac{2[2]_q[\aleph]_q \varkappa e_q^{m_{\mathcal{N}}}}{(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2}.$$

Proof. Let $f \in \mathcal{B}_{\Sigma}(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$. From Definition 1, for some analytical ω, v such that $\omega(0) = v(0) = 0$ and $|\omega(z)| < 1$, $|v(\omega)| < 1$ for all $z, \omega \in \mathbb{U}$, then we can write

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} f(\xi)}{\xi} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} f(\xi) = \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi), \tag{10}$$

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} g(\omega)}{\omega} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} g(\omega) = \mathfrak{G}_q^{(\aleph)}(\varkappa, \omega), \tag{11}$$

From the equalities (10) and (11), we obtain that

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} f(\xi)}{\xi} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} f(\xi) = 1 + C_1^{(\aleph)}(\varkappa;q)c_1 z + \left[C_1^{(\aleph)}(\varkappa;q)c_2 + C_2^{(\aleph)}(\varkappa;q)c_1^2\right]z^2 + \cdots \tag{12}$$

and

$$(1-\delta)\frac{\mathfrak{B}_{m_{\mathcal{N}}} g(\omega)}{\omega} + \delta \partial_q \mathfrak{B}_{m_{\mathcal{N}}} g(\omega) = 1 + C_1^{(\aleph)}(\varkappa;q)d_1 w + \left[C_1^{(\aleph)}(\varkappa;q)d_2 + C_2^{(\aleph)}(\varkappa;q)d_1^2\right])w^2 + \cdots. \tag{13}$$

It is fairly well known that if

$$|w(\xi)| = \left|c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots\right| < 1, \quad (\xi \in \mathbb{U})$$

and
$$|v(w)| = \left|d_1 w + d_2 w^2 + d_3 w^3 + \cdots\right| < 1, \quad (w \in \mathbb{U}),$$
then
$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{14}$$

Thus, upon comparing the corresponding coefficients in (12) and (13), we have
$$\frac{(1 + ([2]_q - 1)\delta)m_{\mathcal{N}}}{e_q^{m_{\mathcal{N}}}} a_2 = C_1^{(\aleph)}(\varkappa; q)c_1, \tag{15}$$
$$\frac{(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2}{[2]_q! e_q^{m_{\mathcal{N}}}} a_3 = C_1^{(\aleph)}(\varkappa; q)c_2 + C_2^{(\aleph)}(\varkappa; q)c_1^2, \tag{16}$$
$$-\frac{(1 + ([2]_q - 1)\delta)m_{\mathcal{N}}}{e_q^{m_{\mathcal{N}}}} a_2 = C_1^{(\aleph)}(\varkappa; q)d_1, \tag{17}$$
and
$$\frac{(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2}{[2]_q! e_q^{m_{\mathcal{N}}}} \left(2a_2^2 - a_3\right) = C_1^{(\aleph)}(\varkappa; q)d_2 + C_2^{(\aleph)}(\varkappa; q)d_1^2. \tag{18}$$

It follows from (15) and (17) that
$$c_1 = -d_1 \tag{19}$$
and
$$2\left(\frac{(1 + ([2]_q - 1)\delta)m_{\mathcal{N}}}{e_q^{m_{\mathcal{N}}}}\right)^2 a_2^2 = \left[C_1^{(\aleph)}(\varkappa; q)\right]^2 \left(c_1^2 + d_1^2\right). \tag{20}$$

If we add (16) and (18), we get
$$\frac{2(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2}{[2]_q! e_q^{m_{\mathcal{N}}}} a_2^2 = C_1^{(\aleph)}(\varkappa; q)(c_2 + d_2) + C_2^{(\aleph)}(\varkappa; q)(c_1^2 + d_1^2) \tag{21}$$

Substituting the value of $(c_1^2 + d_1^2)$ from (20) in the right hand side of (21), we deduce that
$$\left[\frac{(1 + ([3]_q - 1)\delta)}{[2]_q!} - \frac{(1 + ([2]_q - 1)\delta)^2}{e_q^{m_{\mathcal{N}}}} \frac{C_2^{(\aleph)}(\varkappa; q)}{\left[C_1^{(\aleph)}(\varkappa; q)\right]^2}\right] \frac{2(m_{\mathcal{N}})^2}{e_q^{m_{\mathcal{N}}}} a_2^2 = C_1^{(\aleph)}(\varkappa; q)(c_2 + d_2) \tag{22}$$

Moreover, using (7) and (22), we find that
$$|a_2| \leq \frac{2[\aleph]_q \varkappa e_q^{m_{\mathcal{N}}} \sqrt{2[2]_q [\aleph]_q \varkappa}}{m_{\mathcal{N}} \sqrt{\left(4[\aleph]_q^2 e_q^{m_{\mathcal{N}}}(1 + q(1 + q)\delta) - 2[2]_q (1 + q\delta)^2 ([\aleph]_{q^2} + [\aleph]_q^2)\right)\varkappa^2 + [2]_q (1 + q\delta)^2 [\aleph]_{q^2}}}.$$

Moreover, if we subtract (18) from (16), we obtain
$$\frac{2(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2}{[2]_q! e_q^{m_{\mathcal{N}}}} \left(a_3 - a_2^2\right) = C_1^{(\aleph)}(\varkappa; q)(c_2 - d_2) + C_2^{(\aleph)}(\varkappa; q)(c_1^2 - d_1^2). \tag{23}$$

Then, in view of (20), (23) becomes
$$a_3 = \frac{\left(e_q^{m_{\mathcal{N}}}\right)^2 \left[C_1^{(\aleph)}(\varkappa; q)\right]^2}{2\left((1 + q\delta)m_{\mathcal{N}}\right)^2} \left(c_1^2 + d_1^2\right) + \frac{[2]_q e_q^{m_{\mathcal{N}}} C_1^{(\aleph)}(\varkappa; q)}{2(1 + ([3]_q - 1)\delta)(m_{\mathcal{N}})^2} (c_2 - d_2).$$

Thus, applying (7) and (14), we conclude that

$$|a_3| \le \frac{4[\aleph]_q^2 \varkappa^2 \left(e_q^{m_{\mathcal{N}}}\right)^2}{((1+q\delta)m_{\mathcal{N}})^2} + \frac{2[2]_q[\aleph]_q \varkappa e_q^{m_{\mathcal{N}}}}{(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}.$$

This completes the proof of Theorem 1. □

Motivated by the result of Zaprawa [32], we discuss the following Fekete–Szegö inequality for functions in the class $\mathcal{B}_\Sigma(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa,\xi))$.

Theorem 2. *Let given by (1) belong to the class $\mathcal{B}_\Sigma(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa,\xi))$, then*

$$\left|a_3 - \sigma a_2^2\right| \le \begin{cases} \dfrac{2[2]_q e_q^{m_{\mathcal{N}}} \left|[\aleph]_q\right| \varkappa}{(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}, & |\sigma - 1| \le \varpi(\varkappa) \\[2ex] \dfrac{8\left|[\aleph]_q\right|^3 \varkappa^3 [2]_q \left(e_q^{m_{\mathcal{N}}}\right)^2 |1-\sigma|}{\left|\left(4[\aleph]_q^2(1+([3]_q-1)\delta)e_q^{m_{\mathcal{N}}} - 2[2]_q(1+q\delta)^2\left([\aleph]_{q^2}+[\aleph]_q^2\right)\right)\varkappa^2 + [2]_q(1+q\delta)^2[\aleph]_{q^2}\right|}, & |\sigma - 1| \ge \varpi(\varkappa) \end{cases}$$

where

$$\varpi(\varkappa) = \left|1 - \frac{[2]_q(1+q\delta)^2\left(2\left([\aleph]_{q^2}+[\aleph]_q^2\right)\varkappa^2 - [\aleph]_{q^2}\right)}{(1+([3]_q-1)\delta)e_q^{m_{\mathcal{N}}} 4[\aleph]_q^2 \varkappa^2}\right|$$

Proof. From (22) and (23)

$$\begin{aligned} a_3 - \sigma a_2^2 &= \frac{[2]_q e_q^{m_{\mathcal{N}}} C_1^{(\aleph)}(\varkappa;q)}{2(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}(c_2 - d_2) \\ &+ \frac{(1-\sigma)[2]_q \left(e_q^{m_{\mathcal{N}}}\right)^2 \left[C_1^{(\aleph)}(\varkappa;q)\right]^3}{2(m_{\mathcal{N}})^2\left[(1+([3]_q-1)\delta)e_q^{m_{\mathcal{N}}}\left[C_1^{(\aleph)}(\varkappa;q)\right]^2 + [2]_q(1+q\delta)^2 C_2^{(\aleph)}(\varkappa;q)\right]}(c_2 + d_2) \\ &= C_1^{(\aleph)}(\varkappa;q)\left[\mathfrak{h}(\sigma) + \frac{[2]_q e_q^{m_{\mathcal{N}}}}{2(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}\right]c_2 \\ &\quad + C_1^{(\aleph)}(\varkappa;q)\left[\mathfrak{h}(\sigma) - \frac{[2]_q e_q^{m_{\mathcal{N}}}}{2(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}\right]d_2 \end{aligned}$$

where

$$\mathfrak{h}(\sigma) = \frac{(1-\sigma)[2]_q \left(e_q^{m_{\mathcal{N}}}\right)^2 \left[C_1^{(\aleph)}(\varkappa;q)\right]^2}{2(m_{\mathcal{N}})^2\left[(1+([3]_q-1)\delta)e_q^{m_{\mathcal{N}}}\left[C_1^{(\aleph)}(\varkappa;q)\right]^2 - [2]_q(1+q\delta)^2 C_2^{(\aleph)}(\varkappa;q)\right]}.$$

Then, in view of (7), we conclude that

$$\left|a_3 - \sigma a_2^2\right| \le \begin{cases} \dfrac{[2]_q e_q^{m_{\mathcal{N}}}\left|C_1^{(\aleph)}(\varkappa;q)\right|}{(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}, & |\mathfrak{h}(\sigma)| \le \dfrac{[2]_q e_q^{m_{\mathcal{N}}}}{2(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}, \\[2ex] 2\left|C_1^{(\aleph)}(\varkappa;q)\right||\mathfrak{h}(\sigma)|, & |\mathfrak{h}(\sigma)| \ge \dfrac{[2]_q e_q^{m_{\mathcal{N}}}}{2(1+([3]_q-1)\delta)(m_{\mathcal{N}})^2}. \end{cases}$$

□

5. Corollaries and Consequences

Each new corollary and implication that follows is derived from our key findings in this section.

Corollary 1. *Let given by (1) belong to the class $\mathcal{B}_\Sigma(1, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$, then*

$$|a_2| \leq \frac{2[\aleph]_q \varkappa e_q^{m_\mathcal{N}} \sqrt{2[2]_q[\aleph]_q \varkappa}}{m_\mathcal{N} \sqrt{\left(4[\aleph]_q^2 e_q^{m_\mathcal{N}}(1+q(1+q)\delta) - 2[2]_q(1+q)^2\left([\aleph]_{q^2} + [\aleph]_q^2\right)\right)\varkappa^2 + [2]_q(1+q)^2[\aleph]_{q^2}}},$$

$$|a_3| \leq \frac{4[\aleph]_q^2 \varkappa^2 \left(e_q^{m_\mathcal{N}}\right)^2}{\left((1+q)m_\mathcal{N}\right)^2} + \frac{2[2]_q[\aleph]_q \varkappa e_q^{m_\mathcal{N}}}{(1+([3]_q-1))(m_\mathcal{N})^2},$$

and

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{2[2]_q e_q^{m_\mathcal{N}} \left|[\aleph]_q\right| \varkappa}{(1+([3]_q-1))(m_\mathcal{N})^2}, & |\tau - 1| \leq \varpi(\varkappa) \\ \frac{8\left|[\aleph]_q\right|^3 \varkappa^3 [2]_q \left(e_q^{m_\mathcal{N}}\right)^2 |1-\tau|}{\left|\left(4[\aleph]_q^2(1+([3]_q-1))e_q^{m_\mathcal{N}} - 2[2]_q(1+q)^2\left([\aleph]_{q^2} + [\aleph]_q^2\right)\right)\varkappa^2 + [2]_q(1+q)^2[\aleph]_{q^2}\right|}, & |\tau - 1| \geq \varpi(\varkappa) \end{cases}.$$

Corollary 2. *Let given by (1) belong to the class $\mathcal{B}_\Sigma(0, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$, then*

$$|a_2| \leq \frac{2[\aleph]_q \varkappa e_q^{m_\mathcal{N}} \sqrt{2[2]_q[\aleph]_q \varkappa}}{m_\mathcal{N} \sqrt{\left(4[\aleph]_q^2 e_q^{m_\mathcal{N}} - 2[2]_q\left([\aleph]_{q^2} + [\aleph]_q^2\right)\right)\varkappa^2 + [2]_q[\aleph]_{q^2}}},$$

$$|a_3| \leq \frac{4[\aleph]_q^2 \varkappa^2 \left(e_q^{m_\mathcal{N}}\right)^2}{\left(m_\mathcal{N}\right)^2} + \frac{2[2]_q[\aleph]_q \varkappa e_q^{m_\mathcal{N}}}{(m_\mathcal{N})^2},$$

and

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{2[2]_q e_q^{m_\mathcal{N}} \left|[\aleph]_q\right| \varkappa}{(m_\mathcal{N})^2}, & |\rho - 1| \leq \varpi(\varkappa) \\ \frac{8\left|[\aleph]_q\right|^3 \varkappa^3 [2]_q \left(e_q^{m_\mathcal{N}}\right)^2 |1-\rho|}{\left|\left(4[\aleph]_q^2 e_q^{m_\mathcal{N}} - 2[2]_q\left([\aleph]_{q^2} + [\aleph]_q^2\right)\right)\varkappa^2 + [2]_q[\aleph]_{q^2}\right|}, & |\rho - 1| \geq \varpi(\varkappa) \end{cases}$$

6. Concluding Remark

In the current study, we have introduced and examined the coefficient issues related to each of the three new subclasses $\mathcal{B}_\Sigma(\delta, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$, $\mathcal{B}_\Sigma(0, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ and $\mathcal{B}_\Sigma(1, \mathfrak{G}_q^{(\aleph)}(\varkappa, \xi))$ of the class of bi-univalent functions in the open unit disk $\mathbb{U}$. These bi-univalent function classes are described, accordingly, in Definition 1. We have calculated the estimates of the Fekete–Szegö functional problems and the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these three bi-univalent function classes. Several more fresh outcomes are revealed to follow following specializing the parameters involved in our main results. The bi-univalent functions employing the modified Caputo's derivative operator can be used in this investigation. In the future, it is possible to look into the Hankel determinant for this distribution. Caputo's derivative operator is expected to be significant in a number of areas of mathematics, science, and technology.

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