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Applications of operator separation in reflection seismology

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ABSTRACT

The principle of operator separation, a generalization of operator splitting, is applied to some problems in reflection seismology. In particular, the examples of wave-equation migration of seismic data in a three-dimensional medium and accurate depth migration in a laterally varying medium are considered in light of this theory. For the case of a stratified three-dimensional medium, the standard dimensional splitting technique used in the downward-continuation step of the migration process can be replaced with full dimensional separation. The computational implications of this result are that the wave field array need only be transposed once during the downward continuation rather than $2n$ times, where n is the number of finite-difference steps taken in the calculation. For the example of downward continuation in a laterally varying medium, the ideas of operator separation can be used to split the downward continuation operator into two parts, one that looks like the conventional downward-continuation operator for a stratified medium, and a second that represents the correction for the effects of lateral variation.

INTRODUCTION

Three-dimensional (3-D) migration of seismic data is in principle not different from conventional two-dimensional (2-D) migration. Algorithmically it is very much the same as 2-D migration, the only major difference being the addition of another horizontal dimension. Instead of recording seismic data along geophone lines in one dimension, the data are recorded on a grid extending in both horizontal directions. Since this amounts to an order of magnitude increase in the amount of information that must be processed, it is extremely important that the algorithm used for the migration be as efficient as possible. In particular, if the in-core memory size of the data processing computer is of limited size, the major cost of a 3-D migration will typically result from the input and output of information to and from the permanent storage area (disc or tape).

As an example, consider the 3-D migration problem using the wave-equation migration technique. An obvious generalization of Claerbout's 2-D method (Claerbout, 1976) leads to the following formulation of the problem. The reflection seismic data are given as, for example, zero-offset sections which come from data recorded along many parallel lines in the x-direction which are spaced in the y-direction (x and y are the horizontal space coordinates corresponding to the in-line and crossline directions, respectively). I call these data $P_0(x, y, t)$ and assume for the moment that they are known everywhere for $-\infty < x, y < +\infty$ and recorded for all $t \geq 0$. The migration process then consists of a downward continuation of the data using a one-way equation, followed by the application of an appropriate seismic imaging principle to get the earth image, or depth section.

Following Claerbout (1976), the fifteen-degree one-way wave equation for a 3-D vertically stratified medium in a retarded-time coordinate system is given by

$$P_{zt} = \frac{v(z)}{2}(P_{xx} + P_{yy}), \tag{1}$$

where P is the state variable (pressure, for example), $v(z)$ is the velocity function for the medium, z is the depth measured downward from the surface, and subscripts denote partial differentiation. The retarded time variable t is related to true one-way traveltime t' by

$$t = t' + \int_0^z d\zeta / v(\zeta). \tag{2}$$

Now solve equation (1) as an initial value problem in the z-direction with initial data

$$P(x, y, z = 0, t) = P_0(x, y, t) \tag{3}$$

for all x, y, t. The seismic imaging principle for this problem, using the "exploding reflector" model for the seismic experiment, says that the function

$$S(x, y, z) = P[x, y, z, t = \int_0^z d\zeta / v(\zeta)] \tag{4}$$

calculated from the problem above is the migrated depth section.

In principle, one can solve the downward continuation problem given by equations (1), (2), and (3) by Fourier transforming over the variables x, y, and t. The Fourier transform of $P(x, y, z, t)$ over these variables is given by

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$$p(k_x, k_y, z, \omega) = (2\pi)^{-3/2} \iiint P(x, y, z, t)\, e^{-ik_x x - ik_y y + i\omega t}\, dx dy dt$$

$$\equiv \mathbf{F}[P(x, y, z, t)]. \tag{5a}$$

Denote the inverse Fourier transform by

$$P(x, y, z, t) = (2\pi)^{-3/2} \iiint p(k_x, k_y, z, \omega)\, e^{ik_x x + ik_y y - i\omega t}\, dk_x dk_y d\omega$$

$$\equiv \mathbf{F}^{-1}[p(k_x, k_y, z, \omega)]. \tag{5b}$$

Upon Fourier transformation, the one-way equation (1) becomes

$$p_z = \frac{-iv(z)}{2\omega}(k_x^2 + k_y^2)p. \tag{6}$$

The initial data (3) become

$$p(k_x, k_y, z = 0, \omega) = p_0(k_x, k_y, \omega) = \mathbf{F}[P_0(x, y, t)]. \tag{7}$$

The solution of equations (6) and (7) is clearly

$$p(k_x, k_y, z, \omega) = p_0(k_x, k_y, \omega) \exp\left[\frac{-i}{2\omega}(k_x^2 + k_y^2)\int_0^z v(\zeta)\, d\zeta\right]. \tag{8}$$

The imaged depth section is then obtained by inverse Fourier transforming this function $p(k_x, k_y, z, \omega)$ according to equation (5) and replacing t by

$$\int_0^z d\zeta / v(\zeta)$$

in the integral.

Consider now how the downward continuation step of the migration problem given by equations (1), (2), and (3) would be implemented using a finite-difference technique. The initial data as given, $P_0(x, y, t)$, are a function of the three variables x, y, and t and can be rather unwieldy. It is therefore appropriate to use a "splitting" method to solve the differential equation (1). With such a method, the full 3-D problem is divided up into an alternating sequence of 2-D problems, each of which is easier to solve than the original problem and that in particular reduce the amount of information that must be dealt with at a given time during the computation. The splitting method is illustrated using the differential formulation of the problem. Replace equations (1) and (3) as follows. First solve the 2-D equation

$$\tilde{P}_{zt} = \frac{v(z)}{2}\tilde{P}_{xx} \tag{9}$$

for $0 \le z \le \Delta z$ for each value of y, using as initial data

$$\tilde{P}(x, y, 0, t) = P_0(x, y, t). \tag{10}$$

Then solve

$$\hat{P}_{zt} = \frac{v(z)}{2}\hat{P}_{yy} \tag{11}$$

for $0 \le z \le \Delta z$ for each value of x, using as initial data the result of the first calculation

$$\hat{P}(x, y, 0, t) = \tilde{P}(x, y, \Delta z, t). \tag{12}$$

Then repeat the process to solve from $z = \Delta z$ to $z = 2\Delta z$, replacing equation (10) with $\tilde{P}(x, y, \Delta z, t) = \hat{P}(x, y, \Delta z, t)$ and equation (12) with $\hat{P}(x, y, \Delta z, t) = \tilde{P}(x, y, 2\Delta z, t)$. This process is repeated until the desired value of z is reached. Note that both equations (9) and (11) are 2-D fifteen-degree one-way wave equations, so this process is an alternating sequence of *two*-dimensional downward continuation steps of in-line and crossline sections. For data with N values of x and N values of y, and an implicit finite-difference scheme to solve the equations, each z-step with the full 3-D equation (1) involving the solution of a banded N^2 by N^2 system of equations has been replaced by $2N$ steps using a 2-D equation, each of which involves solving a banded N by N system of equations.

The method of splitting (sometimes referred to as the alternating direction method) and its various advantages are well-known and are discussed in many places throughout the literature (Richtmyer and Morton, 1967; Mitchell, 1969). The usual implementation of splitting methods is to take one step in the extrapolation direction (in this case z) with one equation followed by a second step in the z-direction with the other equation. The computational error associated with the splitting process is typically $O(\Delta z^2)$, where Δz is the z-step being used. This typically is comparable with the error caused by the finite-difference method itself. In the example above, however, it turns out that the splitting procedure is *exact* rather than approximate in the sense that all the computational error results from the finite-difference method used to approximate the derivatives, while none is attributable to the splitting. This is apparent if one writes the solution of the splitting problem explicitly using Fourier transforms. The Fourier transform of the solution of equations (9) and (10) for any depth z is

$$\mathbf{F}[\tilde{P}(x, y, z, t)] = \tilde{p}(k_x, k_y, z, \omega) \tag{13}$$

$$= p_0(k_x, k_y, \omega) \exp -\left[\frac{ik_x^2}{2\omega}\int_0^z v(\zeta)\, d\zeta\right],$$

and the Fourier transform of the solution of equations (11) and (12) is

$$\mathbf{F}[\hat{P}(x, y, z, t)] = \hat{p}(k_x, k_y, \omega) \tag{14}$$

$$= \tilde{p}(k_x, k_y, z, \omega) \exp -\left[\frac{ik_y^2}{2\omega}\int_0^z v(\zeta)\, d\zeta\right].$$

Substituting equation (13) into equation (14),

$$\hat{p}(k_x, k_y, z, \omega) = p_0(k_x, k_y, \omega) \exp\left[\frac{-i}{2\omega}(k_x^2 + k_y^2)\int_0^z v(\zeta)\, d\zeta\right], \tag{15}$$

and hence by comparison with equation (8),

$$\hat{P}(x, y, z, t) \equiv P(x, y, z, t). \tag{16}$$

The computational implications of equation (16) are important. Since the solution of the full problem and the splitting problem are identical, it is not necessary to alternate directions at each z-step. Instead, equation (9) can be integrated numerically to the required depth z; then one can use the resulting solution as initial data for equation (11) and integrate it to that depth. Rather than having to regroup the data in alternating directions twice at every z-step, simply reorganize that data once during the computational procedure. For the downward continuation step in the migration problem, this essentially means a 2-D downward continuation of each of the in-line sections, a regrouping of the result into cross-line sections, and then a 2-D downward continuation of each of these sections. Clearly the computational cost can be reduced drastically using this method.

CONDITION FOR SEPARABILITY OF AN OPERATOR

In the simple example above, the full problem given by equations (1) and (3) has been split into two simpler problems that are solved independently but which yield the same final result. Since

this amounts to separating the operator on the right-hand side of equation (1) into two parts, this procedure is called "operator separation." This full separation of the downward continuation procedure into two parts cannot always be theoretically justified. The Appendix establishes that the following easily checked conditions on an operator will allow that operator to be separated in such a way that the corresponding problem can be separated into two independent parts.

Consider the initial-value problem in z for a general linear differential equation of the form

$$\frac{\partial P(x, y, z)}{\partial z} = \left[\mathbf{A}\left(x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + \mathbf{B}\left(x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right] P(x, y, z). \quad (17)$$

A solution of equation (17) is sought for $0 \le z \le z_0$, $-\infty < x, y < \infty$ with initial data

$$P(x, y, 0) = P_0(x, y). \quad (18)$$

Here $\mathbf{A}(x, y, z, \partial/\partial x, \partial/\partial y)$ and $\mathbf{B}(x, y, z, \partial/\partial x, \partial/\partial y)$ are linear differential operators. A separation procedure for equations (17) and (18) follows. First solve

$$\frac{d\hat{P}}{dz} = \mathbf{A}\hat{P} \quad (19)$$

for $z \ge 0$ with initial data

$$\hat{P}(x, y, 0) = P_0(x, y). \quad (20)$$

The solution of equations (19) and (20) at some depth z_0 is then used as initial data for

$$\frac{d\hat{P}}{dz} = \mathbf{B}\hat{P}, \quad (21)$$

i.e., take

$$\hat{P}(x, y, 0) = \hat{P}(x, y, z_0). \quad (22)$$

If the solution at depth z_0 of the problem given by equations (17) and (18) is identical to the solution at depth z_0 of the problem given by equations (19) through (22), i.e., if $\hat{P}(z_0) = P(z_0)$, then the operator $\mathbf{A} + \mathbf{B}$ is *separable*. The condition for this is that operators $\mathbf{A}(x, y, z_1, \partial/\partial x, \partial/\partial y)$ and $\mathbf{B}(x, y, z_2, \partial/\partial x, \partial/\partial y)$ *commute* (i.e., $\mathbf{AB} = \mathbf{BA}$) for all values of z_1 and z_2 (keeping x and y fixed). In general, of course, this condition will not hold. However, there are special problems of interest in reflection seismology where it does hold. The example in the Introduction is one such case.

In most cases, $\mathbf{A}$ and $\mathbf{B}$ will not commute and so $\mathbf{A} + \mathbf{B}$ will not be a separable operator. In this case, however, the problem given by equations (17) and (18) can always be solved by using splitting, i.e., by solving the finite-differenced versions of equations (19) and (21) alternately at each z-step. The error in the solution at each z-step due to the splitting procedure is $O(\Delta z^2)$, where Δz is the length of the finite-difference step in the z-direction. This error is typically comparable to the error from other parts of the computation and so gives a result that is of similar accuracy to that which one would get using the usual finite-difference approach without splitting.

In the next two sections some further examples of applications of the operator separability theory are discussed. While the 3-D migration equation operator is separable into a pair of 2-D operators for a vertically stratified medium (demonstrated above), it turns out that it is not separable in a laterally inhomogeneous medium. In the next section I continue the discussion of 3-D migration to consider both this problem and to find 45-degree type separable one-way wave equations.

The usefulness of the theory for operator separation is not limited to problems in which the number of dimensions in the problem can be reduced. It is also useful as a theoretical tool for understanding how different facets of reflection seismic processing can be separated from each other. For example, Yilmaz (1979) used operator separation theory to motivate the finding of separable approximations to the "double square-root equation" proposed by Claerbout. The essential element of Yilmaz' prestack partial migration theory is the approximate separation of the double square-root opeator into the conventional downward continuation operator, a "stacking" operator, and a "deviation" operator. The deviation operator is applied to unstacked common-offset sections to correct partially the effects of wide offsets and dipping events. Operator separation theory can also be applied to the theory of migration in a laterally varying medium. This is discussed in the last section.

DOWNWARD CONTINUATION IN THREE DIMENSIONS

Two difficulties may arise in separating an operator into two parts. The first is the problem discussed above: the two parts of the operator to be separated may not commute. Another possibility is that the operator simply cannot be written as a sum of two operators at all. An example of the first case shows up in the downward continuation problem in a medium in which the velocity function varies laterally. The square-root dispersion relation associated with upward-traveling waves in a 3-D (non time-retarded) medium is given by

$$k_z = \frac{-\omega}{v(x, y, z)}\left[1 - \frac{v(x, y, z)^2}{\omega^2}(k_x^2 + k_y^2)\right]^{1/2}, \quad (23)$$

where k_x, k_y, k_z are the x, y, and z components, respectively, of the wavenumber and ω is the temporal frequency. I assume here that the velocity function v depends upon all three space dimensions. [This assumption can be considered valid in a WKB or asymptotic sense. A similar approximation was made by Whitham (1974, section 11.8) in his treatment of dispersive waves in an inhomogeneous medium.]

A second-order Taylor expansion of the square root gives the fifteen-degree dispersion relation

$$k_z = \frac{-\omega}{v(x, y, z)}\left[1 - \frac{v(x, y, z)^2}{2\omega}(k_x^2 + k_y^2)\right]. \quad (24)$$

From this dispersion relation one can deduce the 15-degree one-way wave equation,

$$\hat{P}_z = -\left\{\left[\frac{i\omega}{2v(x, y, z)} - \frac{iv(x, y, z)}{2\omega}\frac{\partial^2}{\partial x^2}\right] + \left[\frac{i\omega}{2v(x, y, z)} - \frac{iv(x, y, z)}{2\omega}\frac{\partial^2}{\partial y^2}\right]\right\}\hat{P}, \quad (25)$$

which is written here in such a way as to indicate the proposed separation of the operator. $\hat{P} = \hat{P}(x, y, z)$ is the wave function in the frequency domain. Equation (25) is in the form of equation (17) with $\mathbf{A}$ and $\mathbf{B}$ given by

$$\mathbf{A} = \frac{-i\omega}{2v(x, y, z)} + \frac{iv(x, y, z)}{2\omega}\frac{\partial^2}{\partial x^2},$$

and

$$\mathbf{B} = \frac{-i\omega}{2v(x, y, z)} + \frac{iv(x, y, z)}{2\omega}\frac{\partial^2}{\partial y^2}. \tag{26}$$

By inspection, one can see that the operator $\mathbf{A} + \mathbf{B}$ is not separable because $\mathbf{A}$ and $\mathbf{B}$ do not commute. If, however, the velocity function v depended only on the depth variable z, i.e., $v = v(z)$, then $\mathbf{A}$ and $\mathbf{B}$ *would* commute and hence the operator would be separable into two parts each containing only one kind of derivative. This is essentially the example given in the Introduction.

In (k_x, k_y, k_z) space the dispersion relation given by equation (23) is [for a given location in (x, y, z) space] a hemisphere of radius $\omega/v(x, y, z)$. The fifteen-degree dispersion relation [equation (24)] approximates this hemisphere with a hyperboloid of revolution. The two dispersion relations agree exactly only at $k_x = k_y = 0$. As k_x and k_y increase, the approximation to the hemisphere becomes worse. Since $v\sqrt{k_x^2 + k_y^2}/\omega$ is a measure of the angle from the vertical direction at which the waves propagate in the medium, the fifteen-degree equation approximates waves traveling at high angles rather poorly. The large dispersion error in the fifteen-degree equation makes it desirable to approximate equation (23) more accurately. Dispersion relations based on Padé rational approximations to the square-root that lead to stable differential equations were given by Clayton and Engquist (1977). The second rational approximation to equation (23) is

$$k_z = -\frac{\omega}{v}\left[\frac{1 - \frac{3}{4}v^2(k_x^2 + k_y^2)/\omega^2}{1 - \frac{1}{4}v^2(k_x^2 + k_y^2)/\omega^2}\right]. \tag{27}$$

In order to calculate solutions to the differential equation corresponding to equation (27) at reasonable cost, it is important to be able to split or separate the operator on the right-hand side of equation (27) into two parts each depending only on k_x or k_y. Unfortunately, because the denominator in equation (27) contains both k_x and k_y, this is a case where the differential equation cannot be written in a useful separable form. However, it *is* possible to find dispersion relations that improve upon equation (24). One possibility, suggested by Francis Muir (personal communication, 1977), is first to approximate equation (23) by

$$k_z = -\frac{\omega}{v}\left[\left(1 - \frac{v^2k_x^2}{\omega^2}\right)^{1/2} + \left(1 - \frac{v^2k_y^2}{\omega^2}\right)^{1/2} - 1\right]. \tag{28}$$

When $k_y = 0$ (corresponding to waves traveling in the in-line direction) or when $k_x = 0$ (corresponding to waves traveling in the cross-line direction), equation (28) is an exact approximation to equation (23). Waves traveling at off-line angles other than 90 degrees are not treated exactly. To find the error in this approximation, replace k_x and k_y in equation (28) by new variables k_r and ϕ defined by

$$k_r = \sqrt{k_x^2 + k_y^2}$$

and

$$\phi = \tan^{-1}(k_y/k_x).$$

The variable ϕ can be identified as the off-line angle. Equation (28) becomes

$$k_z = -\frac{\omega}{v}\left[\left(1 - \frac{v^2k_r^2}{\omega^2}\cos^2\phi\right)^{1/2} + \left(1 - \frac{v^2k_r^2}{\omega^2}\sin^2\phi\right)^{1/2} - 1\right]. \tag{29}$$

Clearly, equation (29) deviates the most from equation (23) at $\phi = 45$ degrees. For this value of ϕ, equation (29) becomes

$$k_z = -\frac{\omega}{v}\left[-1 + 2\left(1 - \frac{v^2k_r^2}{2\omega^2}\right)^{1/2}\right]. \tag{30}$$

For any reasonable approximation of the square roots in equation (28) (i.e., fifteen-degree or higher rational approximations), it is clear from equation (30) that the resulting dispersion relation will be at least as good as the fifteen-degree dispersion relation (24) in its approximation to equation (23). Thus replacing the square roots in equation (28) with the second rational approximation given by Clayton and Engquist (1977), yields a dispersion relation in which the operator is either separable or splittable, depending upon the spatial dependence of v.

The new 45-degree dispersion relation for three space dimensions is therefore

$$k_z = -\frac{\omega}{v}\left(\frac{1 - \frac{3}{4}\frac{v^2k_x^2}{\omega^2}}{1 - \frac{1}{4}\frac{v^2k_x^2}{\omega^2}} - \frac{1}{2}\right) + \left(\frac{1 - \frac{3}{4}\frac{v^2k_y^2}{\omega^2}}{1 - \frac{1}{4}\frac{v^2k_y^2}{\omega^2}} - \frac{1}{2}\right), \tag{31}$$

where the parentheses emphasize which parts are to be separated. The separated differential equations for equation (31), corresponding to equations (19) and (21) in the general theory, will be

$$P_z + \frac{1}{4}\frac{v^2}{\omega^2}P_{xxz} = -\frac{i\omega}{2v}P - \frac{5iv}{8\omega}P_{xx},$$

and

$$P_z + \frac{1}{4}\frac{v^2}{\omega^2}P_{yyz} = -\frac{i\omega}{2v}P - \frac{5iv}{8\omega}P_{yy}. \tag{32}$$

If $v = v(z)$ then the problem is separable. If $v = v(x, y, z)$ then the two equations (32) are to be used in a splitting procedure.

Another approach to finding accurate separable approximations to the square root in equation (23) is by using least-squares approximations to the operator. This was discussed by Ristow and Houba (1979).

ACCURATE DEPTH MIGRATION IN A LATERALLY VARYING MEDIUM

In a fixed reference frame, the 45-degree one-way wave equation for upgoing waves in two space dimensions (x and z) is given by

$$\left(\partial_t - \frac{v^2}{4}\partial_{xx}\right)P_z = \frac{1}{v}\left(\partial_{tt} - \frac{3v^2}{4}\partial_{xx}\right)P_t, \tag{33}$$

where $v = v(x, z)$ and a subscripted ∂ is used to represent a differential operator, e.g., $\partial_x = \partial/\partial x$, etc. [Compare with the downgoing equation given by Clayton and Engquist (1977), equation (10).] For convenience, write equation (33) as

$$P_z = \frac{\frac{1}{v}\partial_{tt} - \frac{3v}{4}\partial_{xx}}{\partial_t - \frac{v^2}{4}\partial_{xx}}P_t. \tag{33a}$$

(The interpretation of "dividing" by a differential operator is to operate on the equation from the left with the operator in the denominator.) When equation (33) is used for migration, it is conventional first to transform to a time-retarded coordinate system in which vertically traveling waves either appear not to move or travel at only a very small speed. (One motivation for this change of variables is that it reduces the dispersion error due to the finite difference approximation since that error is proportional to the velocity at which waves are traveling.) If one changes to a retarded time variable t' that depends upon a constant or z-variable velocity $\bar{v}(z)$, i.e.,

$$t' = t + \int_0^z d\zeta/\bar{v}(\zeta),$$

equation (33a) becomes (dropping primes on variables)

$$P_z = \left(\frac{\frac{1}{v}\partial_{tt} - \frac{3v}{4}\partial_{xx}}{\partial_{tt} - \frac{v^2}{4}\partial_{xx}} \partial_t - \frac{1}{\bar{v}}\partial_t \right) P. \tag{34}$$

Written in this way, equation (34) can easily be interpreted in terms of operator separation theory. Let **A** be the differential operator on the right-hand side of equation (34) and let $\mathbf{B} = -(1/\bar{v})\partial_t$. It is clear that $\mathbf{A} + \mathbf{B}$ is the operator on the right-hand side of equation (33a). Note also that because $\bar{v}(z)$ depends only upon the depth variable z, the operators **A** and **B** commute. Hence time-retardation is equivalent to separating equation (33a) into the two equations (34) and

$$P_z = \frac{-1}{\bar{v}(z)} P_t, \tag{35}$$

and then neglecting equation (35). The general solutions of equation (35) are functions of the form

$$P = f\left[t - \int_0^z d\zeta/v(\zeta) \right].$$

If the initial data for equation (35) are a single time trace, then integration of equation (35) from 0 to z results in time shifting the trace by

$$t = \int_0^z d\zeta/v(\zeta).$$

In this context, it is clear that $(1/v)\partial_t$ is a purely shifting operator.

Equation (34) is unfortunately not particularly convenient for computation. Unless $\bar{v} = v$, equation (34) when written out contains a triple time derivative term. A much more convenient formulation splits the operator **A** into two parts also. Introduce $\mathbf{C} = (1/v - 1/\bar{v})\partial_t$, a differential shifting operator; then $P_z = (\mathbf{A} - \mathbf{C})P$, when written out, is

$$\left[\partial_{tt} - \frac{v^2(x, z)}{4}\partial_{xx} \right] P_z = -\frac{v(x, z)}{2} P_{xxt}. \tag{36}$$

Unfortunately, the operators **C** and $\mathbf{A} - \mathbf{C}$ do not commute in general. So one cannot dismiss $P_z = \mathbf{C}P$ as being equivalent to a time retardation and drop it altogether. Instead one can use a splitting procedure, solving equation (36) and

$$P_z = \left(\frac{1}{v} - \frac{1}{\bar{v}} \right) P_t \tag{37}$$

alternately at each step in z. Solving equation (37) for one step in z just amounts to shifting each trace slightly. If $v = v(x, z)$, then traces at different locations in x will be shifted by different amounts. The procedure just described is the basis for most modern accurate depth migration procedures for laterally varying media (Judson et al, 1980).

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APPENDIX
JUSTIFICATION OF THE SEPARABILITY CONDITION

The separability condition discussed in the text can be justified by explicit calculation of the solutions of the various problems involved. In this Appendix, the finite differenced versions of the problem given by equations (17) and (18) are shown to be equivalent to the finite differenced version of equations (19) through (22), provided that the finite differenced versions of the operators **A** and **B** commute. For numerical solutions of the downward continuation equations, this result is fully adequate.

First assume that operators **A** and **B** can be taken to be polynomials of order q in $\partial/\partial x$ and $\partial/\partial y$ with variable coefficients, for example,

$$\mathbf{A} = \sum_{\nu_1 + \nu_2 \le q} \alpha_{\nu_1 \nu_2}(x, y, z) \frac{\partial^{\nu_1 + \nu_2}}{\partial x^{\nu_1} \partial y^{\nu_2}}. \tag{A-1}$$

The operator **B** can be defined analogously.

Now consider finite-difference approximations to equation (17) and discretize the problem in x and y, leaving z as a continuous variable. Truncate the region in the $x - y$ plane to $0 \le x \le X$ and $0 \le y \le Y$ and introduce a computational grid which covers this region by dividing the x-axis into $N + 1$ points with equal spacing $\Delta x = X/N$ and dividing the y-axis into $M + 1$ points with equal spacing $\Delta y = Y/M$. The result is a mesh with $(N + 1)$ $(M + 1)$ points located at $(x_\nu, y_\mu) = (\nu\Delta x, \mu\Delta y)$ $\nu = 0, 1, \ldots N$, $\mu = 0, 1, \ldots, M$. The function $P(x, y, z)$ is then approximated at the points $(x_\nu, y_\mu) = (\nu\Delta x, \mu\Delta y)$, $\nu = 0, 1, \ldots, N$, $\mu = 0, 1, \ldots, M$. The function $P(x, y, z)$ is then approximated at the points (x_ν, y_μ) with the grid function $P_{\nu,\mu}(z)$. Approximate the differential operators **A** and **B** using finite-differences. In general at each point (x_ν, y_μ), $\mathbf{A}P(x_\nu, y_\mu, z)$ and $\mathbf{B}P(x_\nu, y_\mu, z)$

are replaced by linear combinations of the values of $P_{\nu,\mu}(z)$ near the point (x_ν, y_μ). This can be represented by

$$\mathbf{A}\left(x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) P(x_\nu, y_\mu, z) \rightarrow \sum_{\nu=-j}^{N-j} \sum_{\mu=-k}^{M-k} a_{jk}(x_\nu, y_\mu, z) P_{\nu+j,\mu+k}(z)$$

and

$$\mathbf{B}\left(x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) P(x_\nu, y_\mu, z) \rightarrow \sum_{\nu=-j}^{N-j} \sum_{\mu=-k}^{M-k} b_{jk}(x_\nu, y_\mu, z) P_{\nu-j,\mu+k}(z). \tag{A-2}$$

The right-hand side of equation (A–2) can be written more compactly by introducing vector notation. Let $\mathbf{p}(z)$ be an $(N+1)(M+1)$—dimensional vector with elements $p_i(z)$ given by the relation $p_{(M+1)\nu+\mu}(z) = P_{\nu,\mu}(z)$. Then it is obvious that equation (A–2) can be replaced with

$$\mathbf{A}P \rightarrow \mathbf{A}(z)\mathbf{p}(z)$$

and

$$\mathbf{B}P \rightarrow \mathbf{B}(z)\mathbf{p}(z), \tag{A-3}$$

where coefficients of the $(N+1)(M+1)$ by $(N+1)(M+1)$ matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ clearly depend upon the coefficients $a_{jk}(x_\nu, y_\mu, z)$ and $b_{jk}(x_\nu, y_\mu, z)$ in equation (A–2). Using these approximations, the finite-difference scheme for solving equation (17) can be written as

$$\frac{d\mathbf{p}}{dz} = [\mathbf{A}(z) + \mathbf{B}(z)]\mathbf{p}(z) \tag{A-4}$$

with initial data

$$\mathbf{p}(0) = \mathbf{p}^\circ. \tag{A-5}$$

where the elements p_i° of the vector $\mathbf{p}^\circ$ are given by $p^\circ_{(M+1)\nu+\mu} = P_0(x_\nu, y_\mu)$. Equation (A–4) is a linear ordinary differential equation with matrix coefficients, and its solution can be written explicity if $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are sufficiently smooth functions of z:

$$\mathbf{p}(z) = \sum_{n=0}^{\infty} \left\{\int_0^z [\mathbf{A}(\zeta) + \mathbf{B}(\zeta)]d\zeta\right\}^{\hat{n}} \mathbf{p}^\circ \tag{A-6}$$

(Gantmacher, 1959), where for convenience the notation

$$\left(\int_0^z \cdots d\zeta\right)^{\hat{n}}$$

is interpreted as a multiple integral, i.e.,

$$\left[\int_0^z \mathbf{A}(\zeta)d\zeta\right]^{\hat{n}} \equiv \int_0^z \mathbf{A}(\zeta_1)d\zeta_1 \int_0^{\zeta_1} \mathbf{A}(\zeta_2)d\zeta_2 \cdots \int_0^{\zeta_{n-1}} \mathbf{A}(\zeta_n)d\zeta_n.$$

Note that if $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, equation (A–6) reduces to the more familiar result

$$\mathbf{p}(z) = \mathbf{p}^\circ \exp[(\mathbf{A} + \mathbf{B})z]. \tag{A-7}$$

Now turn to the separability question. The separation procedure for the differential case is given by equations (19) through (22). The finite differenced version of this is given by the following. First solve

$$\frac{d\tilde{\mathbf{p}}(z)}{dz} = \mathbf{A}(z)\tilde{\mathbf{p}}(z) \tag{A-8}$$

for $z \geq 0$ with initial data

$$\tilde{\mathbf{p}}(0) = \mathbf{p}^\circ. \tag{A-9}$$

The solution of equations (A–8) and (A–9) at some depth z_0 is then used as initial data for

$$\frac{d\hat{\mathbf{p}}(z)}{dz} = \mathbf{B}(z)\hat{\mathbf{p}}(z), \tag{A-10}$$

i.e., take

$$\hat{\mathbf{p}}(0) = \tilde{\mathbf{p}}(z_0). \tag{A-11}$$

Using the formulation of equation (A–6), the solution of equations (A–8) through (A–11) is given by

$$\hat{\mathbf{p}}(z_0) = \left\{\sum_{n=0}^{\infty} \left[\int_0^{z_0} \mathbf{B}(\zeta)d\zeta\right]^{\hat{n}}\right\}\left\{\sum_{m=0}^{\infty} \left[\int_0^{z_0} \mathbf{A}(\zeta)d\zeta\right]^{\hat{m}}\right\}\mathbf{p}^\circ. \tag{A-12}$$

Now consider under what conditions the solution of equations (A–8) through (A–11) at depth z_0 is the same as or near the solution of equations (A–4) and (A–5). Comparison of equation (A–6) with equation (A–12) shows that $\hat{\mathbf{p}}(z_0) = \mathbf{p}(z_0)$ only if all the integrals involved commute. In particular, one must have

$$\int_0^z \mathbf{B}\,d\zeta_1 \int_0^z \mathbf{A}d\zeta_2 = \int_0^z \mathbf{A}d\zeta_1 \int_0^{\zeta_1} \mathbf{B}d\zeta_2 + \int_0^z \mathbf{B}d\zeta_1 \int_0^{\zeta_1} \mathbf{A}d\zeta_2. \tag{A-13}$$

Equation (A–13) will hold if (and only if) the matrix operators $\mathbf{A}(z_1)$ and $\mathbf{B}(z_2)$ commute for all values of their arguments z_1 and z_2. If the matrix operators $\mathbf{A}(z)$ and $\mathbf{B}(z)$ do commute for all values of their arguments, then the full problem given by equations (A–4) and (A–5) and the separated problem given by equations (A–8) through (A–11) are exactly equivalent, and I will say that the operator $\mathbf{A} + \mathbf{B}$ is *separable*. One can formally replace the matrix operators $\mathbf{A}$ and $\mathbf{B}$ with the differential operators $\mathbf{A}$ and $\mathbf{B}$ in this argument and say that the differential operator $\mathbf{A} + \mathbf{B}$ is separable if the operators $\mathbf{A}$ and $\mathbf{B}$ commute. In practice, the commutivity of differential operators is easier to check than the commutivity of matrix operators, and this is what was done in some of the examples in the text.

If $\mathbf{A}$ and $\mathbf{B}$ do not commute, then $\mathbf{A} + \mathbf{B}$ is not a separable operator. In this case, it is easy to show that the problem given by equations (17) and (18) can always be solved by using splitting, i.e., by solving the finite-difference equations (A–8) and (A–10) alternately at each z-step. If the length of the z-step is Δz, then the solution of equations (A–8) through (A–11) after one such step from z to Δz is

$$\begin{aligned}\hat{\mathbf{p}}(z+\Delta z) &= \left\{\sum_{n=0}^{\infty} \left[\int_z^{z+\Delta z} \mathbf{B}(\zeta)d\zeta\right]^{\hat{n}}\right\} \left\{\sum_{m=0}^{\infty} \left[\int_z^{z+\Delta z} \mathbf{A}(\zeta)d\zeta\right]^{\hat{m}}\right\}\hat{\mathbf{p}}(z) \\ &= \left[I + \left(\int_z^{z+\Delta z} \mathbf{B}d\zeta + \int_2^{z+\Delta z} \mathbf{A}d\zeta\right)\right. \\ &\quad + \left(\int_z^{z+\Delta z} \mathbf{B}d\zeta \int_z^{z+\zeta} \mathbf{B}d\zeta_1 + \right. \\ &\quad \left.\left.\int_z^{z+\Delta z} \mathbf{B}d\zeta \int_z^{z+\Delta z} \mathbf{A}d\zeta_1 + \int_z^{z-\Delta z} \mathbf{A}d\zeta \int_z^{z+\zeta} \mathbf{B}d\zeta_1\right)\right.\end{aligned}$$

$$+ \mathbf{O}(\Delta z^2)\Big]\hat{\mathbf{p}}(z). \tag{A-14}$$

Similarly, the solution of equations (A–4) and (A–5) after a step from z to $z + \Delta z$ is

$$\begin{aligned}\mathbf{p}(z+\Delta z) = \Bigg[I + \left(\int_z^{z+\Delta z}\mathbf{B}d\zeta + \int_z^{z+\Delta z}\mathbf{A}d\zeta\right) \\ + \left(\int_z^{z+\Delta z}\mathbf{B}d\zeta\int_z^{z+\zeta}\mathbf{B}d\zeta_1 + \int_z^{z+\Delta z}\mathbf{B}d\zeta\int_z^{z+\zeta}\mathbf{A}d\zeta_1\right. \\ \left. + \int_z^{z+\Delta z}\mathbf{A}d\zeta\int_z^{z+\zeta}\mathbf{B}d\zeta_1 + \int_z^{z+\Delta z}\mathbf{A}d\zeta\int_z^{z+\zeta}\mathbf{B}d\zeta_1\right) \\ + \mathbf{O}(\Delta z^3)\Bigg]\mathbf{p}(z).\end{aligned} \tag{A-15}$$

If $\hat{\mathbf{p}}(z) = \mathbf{p}(z)$, then

$$\begin{aligned}\hat{\mathbf{p}}(z+\Delta z) - \mathbf{p}(z+\Delta z) = \Bigg[\int_z^{z+\Delta z}\mathbf{B}d\zeta\int_z^{z+\Delta z}\mathbf{A}d\zeta_1 \\ - \int_z^{z+\Delta z}\mathbf{B}d\zeta\int_z^{z+\zeta}\mathbf{A}d\zeta_1 - \int_z^{z+\Delta z}\mathbf{A}d\zeta\int_z^{z+\zeta}\mathbf{B}d\zeta_1 \\ + O(\Delta z^3)\Bigg]\mathbf{P}(z) = O(\Delta z^2).\end{aligned} \tag{A-16}$$

Hence the error in the solution at each z-step caused by the splitting procedure is $O(\Delta z^2)$, which is typically comparable to the error from other parts of the computation.

To summarize, the operator $\mathbf{A} + \mathbf{B}$ is separable if and only if $\mathbf{AB} = \mathbf{BA}$. If this is the case, then using equations (A–8) through (A–11) is basically a two-step procedure. If $\mathbf{A}$ and $\mathbf{B}$ do not commute, then a splitting procedure can be expected to work. This means that equations (A–8) and (A–10) must be solved alternately at every z-step, or at least often enough so that the splitting error does not become unacceptably large. In many cases, a separated problem is much cheaper to solve than a problem solved by splitting.