APPLICATIONS OF THE GENERAL SIMILARITY SOLUTION OF THE HEAT EQUATION TO BOUNDARY-VALUE PROBLEMS*

By

GEORGE W. BLUMAN

University of British Columbia

1. Introduction. Using Lie theory, Bluman and Cole [2] derive the general similarity solution of the heat equation. In this paper we consider the application of group methods to boundary-value problems for the one-dimensional heat equation.

It will be shown that the initial-value problem for the well-known fundamental solution of the heat equation for an infinite bar is invariant under a three-parameter Lie group. This leads to the solution in an elegant fashion.

An inverse Stefan problem for the melting of a finite bar is considered. Analytical solutions are obtained for a two-parameter class of moving boundaries, extending the previous work of Sanders [5] and Langford [4]. A new solution expressible in terms of a Fourier series is derived for a phase change boundary moving at a constant velocity.

2. Group of the heat equation. Consider the one-dimensional heat equation for a material with constant thermal properties

$$c\rho(\partial\Theta/\partial t) - k(\partial^2\Theta/\partial x^2) = 0 \tag{1}$$

where c = specific heat, ρ = density, and k = thermal conductivity are parameters and $\Theta(x, t)$ is the temperature at position x at time t. For a given problem, by a suitable scaling of the variables Θ, x, and t, (1) is equivalent to the partial differential equation

$$(\partial u/\partial\tau) - (\partial^2 u/\partial y^2) = 0. \tag{2}$$

In [2] it was shown how to find the Lie group leaving invariant (2) by the use of infinitesimal transformations. If

$$\begin{aligned} u^* &= U^*(y, \tau, u; \epsilon) = u + \epsilon\eta(y, \tau, u) + O(\epsilon^2), \\ y^* &= Y^*(y, \tau, u; \epsilon) = y + \epsilon Y(y, \tau, u) + O(\epsilon^2), \\ \tau^* &= T^*(y, \tau, u; \epsilon) = \tau + \epsilon T(y, \tau, u) + O(\epsilon^2), \end{aligned} \tag{3}$$

is a one-parameter Lie group leaving invariant (2), then

$$\eta(y, \tau, u) = f(y, \tau)u + g(y, \tau), \tag{4}$$

where $g(y, \tau)$ is any solution of (2) and

$$\begin{aligned} Y(y, \tau, u) &= Y(y, \tau) = \kappa + \delta\tau - \beta y - \gamma y\tau, \\ T(y, \tau, u) &= T(\tau) = \alpha - 2\beta\tau - \gamma\tau^2, \\ f(y, \tau) &= -\tfrac{1}{2}\,\delta y + \tfrac{1}{4}\gamma[y^2 + 2\tau] + \lambda. \end{aligned} \tag{5}$$

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We consider the case where $g(y, \tau) \equiv 0$.

Effectively (3), is a six-parameter Lie group of transformations leaving invariant (2). Each parameter (or combination of parameters) can in turn be used to generate a similarity solution.

The corresponding general similarity solution generated by this six-parameter family of transformations is derived by integrating out the characteristic equations

$$dy/Y(y, \tau) = d\tau/T(\tau) = du/f(y, \tau)u \tag{6}$$

to obtain a similarity form

$$u(y, \tau) = F(\zeta)G(y, \tau) \tag{7}$$

where $\zeta(y, \tau) =$ const. is the integral of the first equality of (6) and $G(y, \tau)$ is known explicitly. Substitution of (7) into the partial differential equation (2) leads to an ordinary differential equation for $F(\zeta)$.

For the application of similarity solutions to a boundary-value problem it is necessary that the given problem have a unique solution. For a direct application it is necessary that a subgroup of (3) leaves invariant each of the boundary conditions and corresponding boundary curves. Each parameter of the subgroup leads to a similarity form (7) for the solution.

In Sec. 6 an application is considered in which one of the boundary conditions is not invariant. A superposition of similarity solutions satisfies the "non-invariant" boundary condition.

3. Group properties of the fundamental solution. As an application of the group (5) we derive the fundamental solution of the heat equation in an infinite bar.

The governing equation is (1) subject to the boundary conditions

$$\Theta(x, 0) = Q\delta(x)/\rho c, \qquad \lim_{x\to\pm\infty} \Theta(x, t) = 0, \tag{8}$$

where $\delta(x)$ is the Dirac delta function and Q measures the strength of the source located at $x = 0$ at time $t = 0$. The solution to the boundary-value problem (1), (8) is unique.

Let

$$y = x, \qquad \tau = kt/c\rho, \qquad u = (\rho c/Q)\Theta. \tag{9}$$

Then (1) is equivalent to (2) and the corresponding transformed boundary conditions are:

$$u(y, 0) = \delta(y), \qquad \lim_{y\to\pm\infty} u(y, \tau) = 0. \tag{10}$$

The group (5) leaves invariant (2). However, for direct application to the given boundary-value problem we must leave invariant the given boundaries $\tau = 0$ and $y = \pm\infty$ and the corresponding boundary conditions. Trivially the full group (5) leaves invariant $u(\pm\infty, \tau) = 0$.

Invariance of $\tau = 0$ implies that

$$\tau^* = \tau + \epsilon T(\tau) = 0 \quad \text{when} \quad \tau = 0,$$

i.e., $T(0) = 0$, and hence $\alpha = 0$.

$$u^*(y, 0) = \delta(y^*)$$

implies that

$$u(y, 0) + \epsilon f(y, 0)u(y, 0) = \delta(y) + \epsilon Y(y, 0)\delta'(y). \tag{11}\dagger$$

Thus invariance of the source condition further requires that

$$f(y, 0)\delta(y) = Y(y, 0)\delta'(y). \tag{12}$$

Formally,

$$\begin{aligned} y\delta'(y) &= -\delta(y) \\ \Rightarrow A(y)\delta'(y) &= -\left[\frac{A(y)}{y}\right]\delta(y) \\ &= -A'(0)\delta(y) \quad \text{if} \quad A(0) = 0. \end{aligned}$$

Moreover, if $B(0) = 0$ then $B(y)\delta(y) = 0$. Hence (12) is satisfied if

$$Y(0, 0) = 0, \qquad f(0, 0) = -(\partial Y/\partial y)(0, 0). \tag{13}$$

Thus $\kappa = 0$ and $\lambda = \beta$. Hence the three-parameter group

$$\begin{aligned} Y &= \delta\tau - \beta y - \gamma y\tau, \\ T &= -2\beta\tau - \gamma\tau^2, \\ f &= -\tfrac{1}{2}\,\delta y + \beta + \tfrac{1}{4}\gamma[y^2 + 2\tau] \end{aligned} \tag{14}$$

leaves invariant the differential equation (2) and the boundary conditions (10).

4. Invariance under a multi-parameter group. Each of the parameters (δ, β, γ) in (14) can be taken in turn to generate a similarity form for the fundamental solution. From uniqueness of the solution we can equate the functional forms corresponding to any two of these parameters [1]. Solving the resulting functional equation, we obtain a solution containing some arbitrary constant which is computed from the initial source condition.

THEOREM 1. Let

$$\mathfrak{X} = Y(y, \tau)\frac{\partial}{\partial y} + T(\tau)\frac{\partial}{\partial \tau} + f(y, \tau)u\frac{\partial}{\partial u} \tag{15}$$

represent the infinitesimal operator generating the similarity form

$$u(y, \tau) = F(\zeta)G(y, \tau). \tag{16}$$

If $u - \phi(y, \tau) = 0$ is a similarity solution corresponding to invariance under a one-parameter group whose infinitesimal operator is $\mathfrak{X}$, then

$$\mathfrak{X}\{u - \phi(y, \tau)\} = 0.$$

Proof: First it should be noted that by definition the similarity form (16) contains all similarity solutions corresponding to $\mathfrak{X}$, i.e., $\phi(y, \tau)$ corresponds to a particular choice of $F(\zeta)$.

From the characteristic equations (6) $\zeta(y, \tau)$ and $u/G(y, \tau)$ are independent invariants corresponding to $\mathfrak{X}$, i.e.

$$\mathfrak{X}\mathfrak{F}(u/G(y, \tau), \zeta(y, \tau)) = 0 \tag{17}$$

† $0(\epsilon^2)$ terms can be neglected since we are dealing with a Lie group of transformations.

for any differentiable function $\mathfrak{F}$ of $(u/G(y, \tau))$ and $\zeta(y, \tau)$.

$$\mathfrak{X}\{u - F(\zeta)G(y, \tau)\} = \mathfrak{X}\left\{G\left[\frac{u}{G} - F(\zeta)\right]\right\}$$

$$= \left[\frac{u}{G} - F(\zeta)\right]\mathfrak{X}G + G\mathfrak{X}\left\{\frac{u}{G} - F(\zeta)\right\}.$$

Hence if $u - F(\zeta)G(y, \tau) = 0$ then $\mathfrak{X}\{u - F(\zeta)G(y, \tau)\} = 0$.

For any values of the parameters δ, β, and γ, (14) leaves invariant (2) and (10). Let

$$\mathfrak{X}_1 \leftrightarrow \delta = 1, \qquad \beta = \gamma = 0,$$

$$\mathfrak{X}_2 \leftrightarrow \beta = 1, \qquad \delta = \gamma = 0,$$

$$\mathfrak{X}_3 \leftrightarrow \gamma = 1, \qquad \beta = \delta = 0.$$

Then

$$\mathfrak{X}_1 = \tau \frac{\partial}{\partial y} - \frac{1}{2} yu \frac{\partial}{\partial u}, \tag{18}$$

$$\mathfrak{X}_2 = -y \frac{\partial}{\partial y} - 2\tau \frac{\partial}{\partial \tau} + u \frac{\partial}{\partial u}, \tag{19}$$

$$\mathfrak{X}_3 = -y\tau \frac{\partial}{\partial y} - \tau^2 \frac{\partial}{\partial \tau} + \frac{1}{4}(y^2 + 2\tau)u \frac{\partial}{\partial u}. \tag{20}$$

Let $\zeta_i(y, \tau)$, $u/G_i(y, \tau) = F_i(\zeta_i)$ be the independent invariants of $\mathfrak{X}_i$, $i = 1, 2, 3$. Then the functional form corresponding to $\mathfrak{X}_i$ is

$$u = F_i(\zeta_i)G_i(y, \tau),$$

where $F_i(\zeta_i)$ is some arbitrary function of ζ_i, $i = 1, 2, 3$.

From uniqueness of the solution the following functional equations are satisfied by the unknown functions $F_1(\zeta_1)$, $F_2(\zeta_2)$ and $F_3(\zeta_3)$:

$$F_1(\zeta_1)G_1(y, \tau) = F_2(\zeta_2)G_2(y, \tau) = F_3(\zeta_3)G_3(y, \tau). \tag{21}$$

Solving the first equality of (21), we determine $F_1(\zeta_1)$ (or equivalently $F_2(\zeta_2)$) explicitly except for an arbitrary constant.

It is easy to show that ζ_2 is functionally independent of ζ_1 iff $\mathfrak{X}_2 \not\equiv \alpha(y, \tau)\mathfrak{X}_1$ for some function $\alpha(y, \tau)$. Hence from (18), (19) we see that ζ_2 is functionally independent of ζ_1.

To solve the functional equation

$$F_1(\zeta_1)G_1(y, \tau) = F_2(\zeta_2)G_2(y, \tau)$$

we let ζ_1 and ζ_2 be the new variables and express y and τ in terms of ζ_1 and ζ_2. Then we differentiate each side of the equality with respect to ζ_1, say. As a result $F_1(\zeta_1)$ satisfies a simple linear homogeneous first-order ordinary differential equation, namely

$$\frac{dF_1}{d\zeta_1} - \left[\frac{\partial}{\partial \zeta_1}\left(\log \frac{G_2}{G_1}\right)\right]F_1 = 0. \tag{22}$$

Alternatively, $G_2/G_1 = A(\zeta_1)B(\zeta_2)$ for some known functions $A(\zeta_1)$ and $B(\zeta_2)$.

$$\Rightarrow \quad F_1(\zeta_1) = cA(\zeta_1)$$

where c is an arbitrary constant.

Hence the solution of the first functional equation of (21) leads to the similarity solution

$$u - c\phi(y, \tau) = 0 \tag{23}$$

of (2), (10) where $\phi(y, \tau)$ is known explicitly and c is determined from the initial source condition.

By Theorem 1, $\mathfrak{X}_1\{u - c\phi(y, \tau)\} = \mathfrak{X}_2\{u - c\phi(y, \tau)\} \equiv 0$.

From uniqueness of the solution $\mathfrak{X}_3\{u - c\phi(y, \tau)\} = 0$. Hence there must exist functions $\lambda_1(y, \tau)$ and $\lambda_2(y, \tau)$ such that

$$\mathfrak{X}_3 \equiv \lambda_1(y, \tau)\mathfrak{X}_1 + \lambda_2(y, \tau)\mathfrak{X}_2 .$$

It turns out that

$$\lambda_1(y, \tau) = \tfrac{1}{2}\tau, \quad \lambda_2(y, \tau) = -\tfrac{1}{2}y.$$

5. Derivation of the fundamental solution.

Corresponding to $\mathfrak{X}_1$, $\mathfrak{X}_2$

$$\zeta_1 = \tau$$

$$G_1(y, \tau) = \exp\left(-\frac{y^2}{4\tau}\right), \qquad \zeta_2 = y/\sqrt{\tau},$$

$$G_2(y, \tau) = 1/\sqrt{\tau}, \qquad G_2/G_1 = (1/\sqrt{\zeta_1}) \exp\left(\frac{\zeta_2^2}{4}\right)$$

$$\Rightarrow A(\zeta_1) = 1/\sqrt{\zeta_1} = 1/\sqrt{\tau}$$

$$\Rightarrow u = \frac{c}{\sqrt{\tau}} \exp\left(-\frac{y^2}{4\tau}\right).$$

The source condition implies that $c = 1/\sqrt{4\pi}$.

Note that a standard way to derive this solution is to consider a similarity form corresponding to invariance with respect to $\mathfrak{X}_2$ (stretching invariance or dimensional-analysis argument) and then to solve the resulting second-order ordinary differential equation.

6. Group properties of an inverse Stefan problem.

As another application of group methods to a boundary value problem we consider transient heat conduction in a melting slab [4], [5]. A finite slab originally extending from $x = 0$ to $x = a$ is melted in such a way that the face $x = 0$ is insulated and the other face is melted with heat flowing into the melting face at a rate $h(t)$. It is assumed that all of the molten material is removed immediately upon formation. At time t the melting face is located at $x = X(t)$ with $X(0) = a$.

Let Θ_m be the melting temperature and $\Theta_0(x)$ be the initial temperature distribution in the bar. Then the governing partial differential equation is (1) for $0 < x < X(t)$, $t > 0$ and the appropriate boundary conditions during melting are:

$$\begin{aligned} \Theta(X(t), t) &= \Theta_m, \qquad t > 0, \\ (\partial\Theta/\partial x)(0, t) &= 0, \qquad t > 0, \\ \Theta(x, 0) &= \Theta_0(x), \qquad 0 < x < a, \\ h(t) &= k(\partial\Theta/\partial x)(X(t), t) - \rho L(dX/dt), \qquad t > 0, \end{aligned} \tag{24}$$

where L = latent heat of fusion. Note that there are two unknowns in this nonlinear Stefan problem: the temperature distribution $\Theta(x, t)$ and the moving (free) boundary $X(t)$.

We non-dimensionalize (1), (24) by letting

$$\begin{aligned} y &= x/a, \qquad \tau = kt/c\rho a^2, \qquad s(\tau) = X(t)/a, \\ u(y, \tau) &= (\Theta(x, t) - \Theta_m)/\Theta_m, \qquad H(\tau) = (ca/Lk)h(t), \\ K &= (c/L)\Theta_m, \qquad \Phi(y) = (\Theta_0(x) - \Theta_m)/\Theta_m. \end{aligned} \tag{25}$$

Then (1), (24) become

$$\partial u/\partial\tau = \partial^2 u/\partial y^2, \qquad 0 < y < s(\tau), \quad \tau > 0 \tag{26}$$

$$u(s(\tau), \tau) = 0, \qquad \tau > 0, \tag{27}$$

$$(\partial u/\partial y)(0, \tau) = 0, \qquad \tau > 0, \tag{28}$$

$$u(y, 0) = \Phi(y), \qquad 0 < y < 1, \tag{29}$$

$$s(0) = 1, \tag{30}$$

$$H(\tau) = K(\partial u/\partial y)(s(\tau), \tau) - ds/d\tau, \qquad \tau > 0. \tag{31}$$

The unknowns are now $u(y, \tau)$ and $s(\tau)$.

Instead of solving the direct problem we consider the inverse Stefan problem where $s(\tau)$ is specified and the solution is found which satisfies all of the boundary conditions except (31). The generated solution fixes the value of $H(\tau)$. In effect this is a "control" type of problem. In a future paper it will be shown how to solve numerically the direct problem by piecing together appropriate inverse solutions.

Using similarity we seek the most general expression for $s(\tau)$ leading to analytical solutions and find the corresponding temperature distribution $u(y, \tau)$ and heat flux $H(\tau)$.

Before proceeding to construct such analytical solutions we show that the solution to the inverse Stefan problem (26)–(31) is unique. Say $u = u_1$ and $u = u_2$ are solutions corresponding to a particular fixed $s(\tau)$. Let $v = u_2 - u_1$. Then v satisfies

$$\partial v/\partial\tau = \partial^2 v/\partial y^2, \qquad 0 < y < s(\tau), \quad \tau > 0, \tag{32}$$

$$v(s(\tau), \tau) = 0, \qquad \tau > 0, \tag{33}$$

$$(\partial v/\partial y)(0, \tau) > 0, \qquad \tau > 0 \tag{34}$$

$$v(y, 0) = 0, \qquad 0 < y < 1. \tag{35}$$

Let R (Fig. 1) be the region bounded by the curves $y = s(\tau)$, $\tau = 0$, $y = 0$ and $\tau = \tau'$

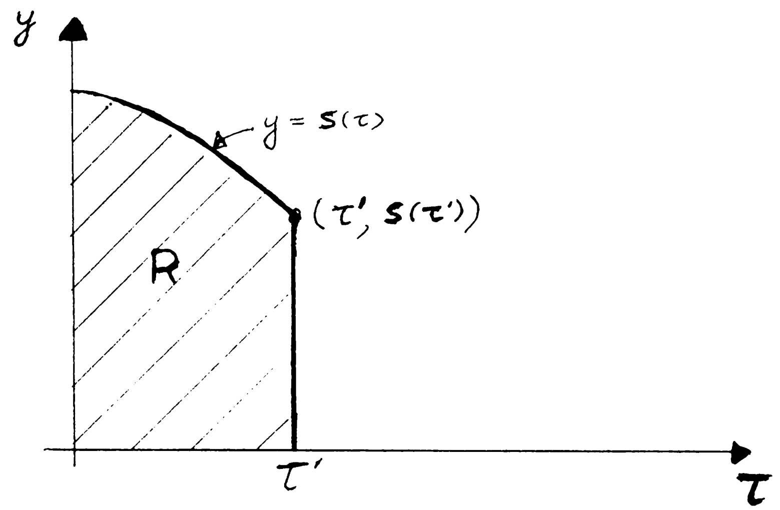

FIG. 1.

$[0 \leq s(\tau') < 1]$. Since v satisfies (32) in R, v also satisfies the identity

$$\frac{\partial}{\partial y}\left(v \frac{\partial v}{\partial y}\right) - \frac{1}{2}\frac{\partial v^2}{\partial \tau} = \left(\frac{\partial v}{\partial y}\right)^2 \tag{36}$$

in R. Let ∂R be the boundary of R. Then by Green's theorem

$$\oint_{\partial R} \frac{1}{2} v^2 \, dy + v \frac{dv}{\partial y} \, d\tau = -\int_R \int \left(\frac{\partial v}{\partial y}\right)^2 d\tau \, dy. \tag{37}$$

From the boundary conditions along ∂R, (37) reduces to

$$\int_R \int \left(\frac{\partial v}{\partial y}\right)^2 d\tau \, dy + \frac{1}{2}\int_0^{s(\tau')} v^2(y, \tau') \, dy = 0.$$

Hence $\partial v/\partial y \equiv 0$ in R. Thus $v = V(\tau)$ for some function of τ in R. But since v satisfies (32) in R and the boundary condition (35), we see that $V(\tau) = \text{const.} = V_0 = 0$. Hence the solution $u(y, \tau)$ is unique if $s(\tau)$ is specified.

If the group (5) leaves invariant $y = s(\tau)$ and $y = 0$ $[s(0) = 1]$ then $s(\tau)$ satisfies

$$Y(0, \tau) = 0, \quad Y(s(\tau), \tau) = \tau s'(\tau), \quad s(0) = 1 \tag{38}$$

$\Rightarrow$

$$\kappa = \delta = 0, \quad \alpha = 1$$

Thus

$$s(\tau) = (1 - 2\beta\tau - \gamma\tau^2)^{1/2}, \qquad \beta > 0 \tag{39†}$$

Case I: $\beta^2 \neq -\gamma$. Integrating out the characteristic equations (6), we obtain the similarity variable

$$\zeta = y/(1 - 2\beta\tau - \gamma\tau^2)^{1/2}. \tag{40}$$

† $\gamma \geq -\beta^2$ if the slab is to melt completely. If $\gamma < -\beta^2$ then $\tau \leq (-\beta/\gamma)$ and $s_{\min} = (1 + \beta^2/\gamma)^{1/2}$.

The similarity curves fill the region $0 \le \zeta \le 1$, $\zeta = 0 \leftrightarrow y = 0$, $\zeta = 1 \leftrightarrow y = s(\tau)$.

The corresponding similarity solutions are:

$I(a): \gamma > -\beta^2$:

$$u(y, \tau) = \frac{f(z)}{(1 - 2\beta\tau - \gamma\tau^2)^{1/4}} \left|\frac{c + \beta + \gamma\tau}{c - \beta - \gamma\tau}\right|^{\rho} \cdot \exp\left[\tfrac{1}{4}\zeta^2(\beta + \gamma\tau)\right], \tag{41}$$

where $f(z)$ satisfies a differential equation of confluent hypergeometric type, namely

$$d^2f/dz^2 + [\tfrac{1}{2} + \nu - \tfrac{1}{4}z^2]f = 0. \tag{42}$$

$I(b): \gamma < -\beta^2$:

$$u(y, \tau) = \frac{f(z)}{(1 - 2\beta\tau - \gamma\tau^2)^{1/4}} \cdot \exp\left[2\rho \tan^{-1}\frac{\gamma t + \beta}{C} + \frac{1}{4}\zeta^2(\beta + \gamma\tau)\right] \tag{43}$$

where $f(z)$ satisfies the differential equation:

$$d^2f/dz^2 + [\tfrac{1}{2} + \nu + \tfrac{1}{4}z^2]f = 0. \tag{44}$$

In both subcases $I(a)$ and $I(b)$, $C = (|\beta^2 + \gamma|)^{1/2}$, $z = \zeta\sqrt{C}$, ν is an arbitrary constant and $\rho = \frac{1}{2}\nu + \frac{1}{4}$.

The boundary conditions (27), (28) imply that $f(z)$ satisfies a Sturm–Liouville problem with boundary conditions $f'(0) = f(\sqrt{C}) = 0$.

Consider $I(a)$

$$f'(0) = 0$$

$$\Rightarrow \quad f(z) = \exp(-\tfrac{1}{4}z^2)M(-\tfrac{1}{2}\nu, \tfrac{1}{2}, \tfrac{1}{2}z^2)$$

where $M(a, b, z)$ is Kummer's function. $f(\sqrt{C}) = 0$ leads to the eigenvalue equation

$$M(-\tfrac{1}{2}\nu, \tfrac{1}{2}, \tfrac{1}{2}C) = 0.$$

Let $\{\nu_n\}$ and $\{f_n\}$, $n = 1, 2, \cdots$, be the corresponding eigenvalues and eigenfunctions:

$$f_n(z) = \exp(-\tfrac{1}{4}z^2)M(-\tfrac{1}{2}\nu_n, \tfrac{1}{2}, \tfrac{1}{2}z^2)$$

$$\begin{aligned}\int_0^{\sqrt{C}} f_m(z)f_n(z)\,dz &= 0 \quad \text{if} \quad n \neq m \\ &= N_n \quad \text{if} \quad n = m,\end{aligned} \tag{45}$$

where N_n is the normalizing factor. Hence a formal solution satisfying the boundary conditions (27), (28) is:

$$u(y, \tau) = \frac{\exp\left[\dfrac{1}{4}\zeta^2(\beta + \gamma\tau)\right]}{[1 + 2\beta\tau - \gamma\tau^2]^{1/4}} \sum_{n=1}^{\infty} A_n f_n(z) \left|\frac{C + \beta + \gamma\tau}{C - \beta - \gamma\tau}\right|^{1/2\nu_n + 1/4}. \tag{46}$$

The constants $\{A_n\}$ are determined from the initial condition (29) and the orthogonality relations (45):

$$A_n = \frac{\sqrt{C}}{N_n}\left|\frac{C - \beta}{C + \beta}\right|^{1/2\nu_n + 1/4} \int_0^1 \exp(-\tfrac{1}{4}\beta\zeta^2)f_n(\zeta\sqrt{C})\Phi(\zeta)\,d\zeta, \qquad n = 1, 2, \cdots. \tag{47}$$

Correspondingly we generate a two-parameter family of heating fluxes

$$H(\tau) = \frac{K\sqrt{C}\cdot \exp\left[\frac{1}{4}(\beta + \gamma\tau)\right]}{(1 - 2\beta\tau - \gamma\tau^2)^{3/4}} \cdot \sum_{n=1}^{\infty} A_n f_n'(\sqrt{C}) \left|\frac{C + \beta + \gamma\tau}{C - \beta - \gamma\tau}\right|^{1/2\nu_n + 1/4} + \frac{(\beta + \gamma\tau)}{(1 - 2\beta\tau - \gamma\tau^2)^{3/2}}. \tag{48}$$

The subcase $\gamma = 0$ was considered by Sanders [5] following the work of Landau [3]. The moving boundary is eliminated by a clever change of variables. Then separation of variables is applied with ν playing the role of an eigenvalue in superposition of solutions of the form (41) where $\gamma = 0$. Langford [4] also only considers the case $\gamma = 0$.

Case II: $\gamma = -\beta^2$. In this case the melting boundary $s(\tau)$ moves at a constant velocity $-\beta$:

$$s(\tau) = 1 - \beta\tau, \tag{49}$$

and the similarity variable is

$$\zeta = y/(1 - \beta\tau), \quad 0 \leq \zeta \leq 1, \tag{50}$$

where

$$\zeta = 0 \leftrightarrow y = 0, \qquad \zeta = 1 \leftrightarrow y = s(\tau) = 1 - \beta\tau.$$

The similarity curves are straight lines (Fig. 2) in the $y - \tau$ plane, filling a triangular region. When $\tau = 1/\beta$ the bar has melted completely. From the characteristic equations the similarity form for the solution is:

$$u(y, \tau) = \frac{F(\zeta)}{(1 - \beta\tau)^{1/2}} \exp\left[\frac{\zeta^2\beta}{4}(1 - \beta\tau) - \frac{\nu^2}{\beta(1 - \beta\tau)}\right] \tag{51}$$

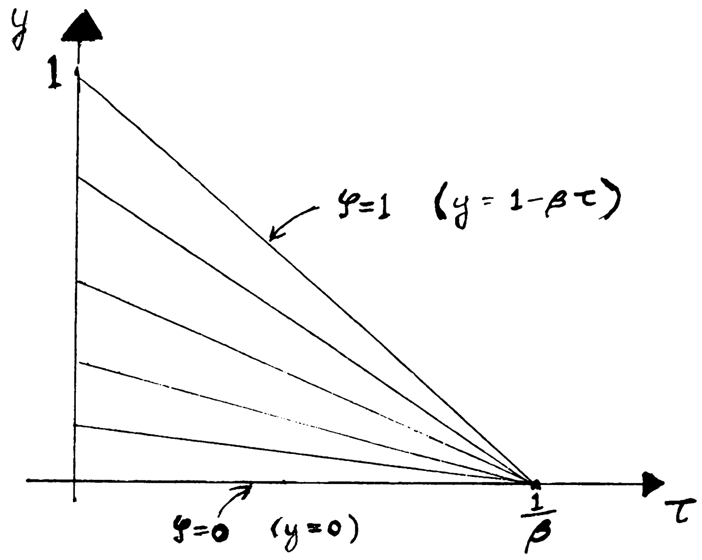

FIG. 2. Similarity curves for Case II: $\gamma = -\beta^2$.

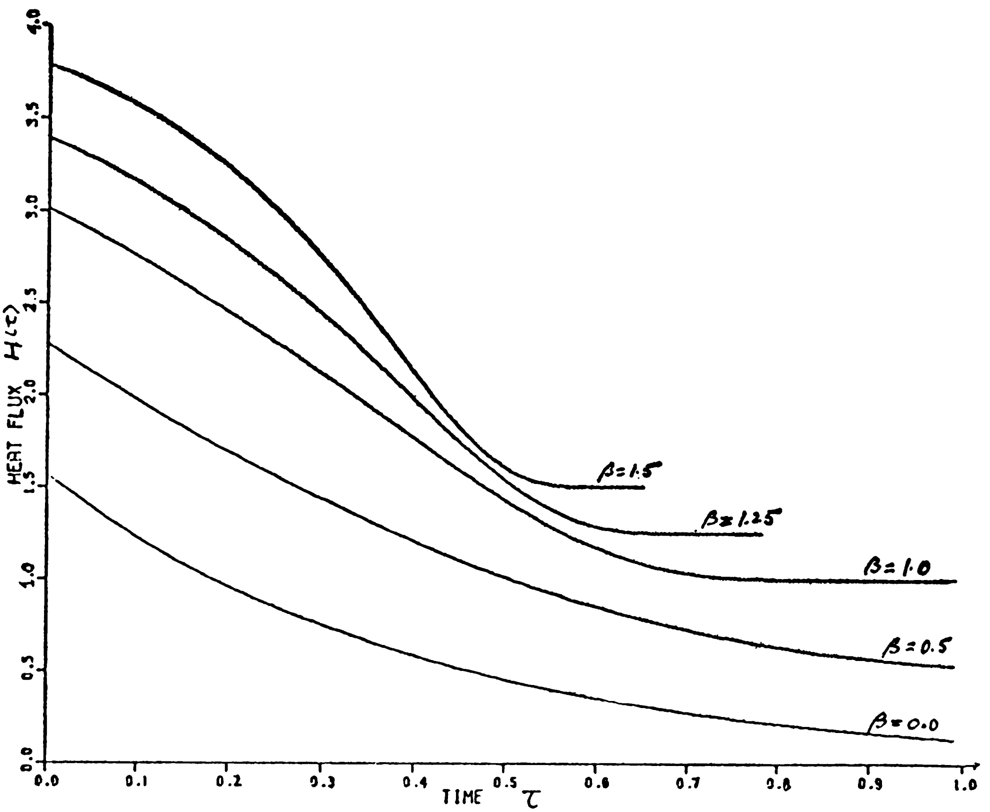

FIG. 3. Heat flux vs. time for initial temperature distribution $\Phi(y) = -\exp(+\beta y^2/4)\cos(\pi/2)y$.

where $F(\zeta)$ satisfies the elementary differential equation

$$d^2F/d\zeta^2 + \nu^2 F = 0. \tag{52}$$

The boundary conditions (27), (28) imply that $F(1) = F'(0) = 0$. The corresponding Sturm–Liouville problem for $F(\zeta)$ has as eigenvalues $\{\nu_n\}$ and eigenfunctions $\{F_n(\zeta)\}$:

$$\nu_n = (n - \tfrac{1}{2})\pi, \quad F_n(\zeta) = \cos \nu_n\zeta, \quad n = 1, 2, \cdots . \tag{53}$$

Hence a general formal solution satisfying (26), (27) and (28) is

$$u(y, \tau) = \frac{\exp\left(\dfrac{\zeta^2}{4}\beta(1 - \beta\tau)\right)}{(1 - \beta\tau)^{1/2}} \sum_{n=1}^{\infty} A_n \cos \nu_n \zeta \exp\left(\frac{-\nu_n^{\,2}}{\beta(1 - \beta\tau)}\right). \tag{54†}$$

From the initial condition (29) we see that

† If $\beta = 0$, i.e. the boundary $y = 1$ is fixed, then the solution (54), (55) corresponds to the well-known Fourier series solution satisfying the boundary conditions (27), (28) and (29) which is obtained by separation of variables.

$$A_n = 2 \exp\left(+\frac{(n - \frac{1}{2})^2\pi^2}{\beta}\right) \int_0^1 \Phi(y)F_n(y) \exp\left(-\frac{\beta y^2}{4}\right) dy, \qquad n = 1, 2, \cdots . \tag{55}$$

Correspondingly the heating flux is

$$H(\tau) = -\frac{K \exp\left(\frac{1}{4}\beta(1 - \beta\tau)\right)}{(1 - \beta\tau)^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1}\nu_n A_n \exp\left(-\frac{\nu_n^2}{\beta(1 - \beta\tau)}\right) + \beta. \tag{56}$$

As an example, consider

$$\Phi(y) = -B \exp\left(\frac{\beta y^2}{4}\right) \cos\frac{\pi}{2}\, y, \qquad 0 < y < 1 \tag{57}$$

where $\beta < \pi^2/2$, $\beta > 0$. For this initial temperature distribution

$$A_1 = -B \exp\left(\frac{\pi^2}{4\beta}\right), \qquad A_n = 0, \quad n > 1.$$

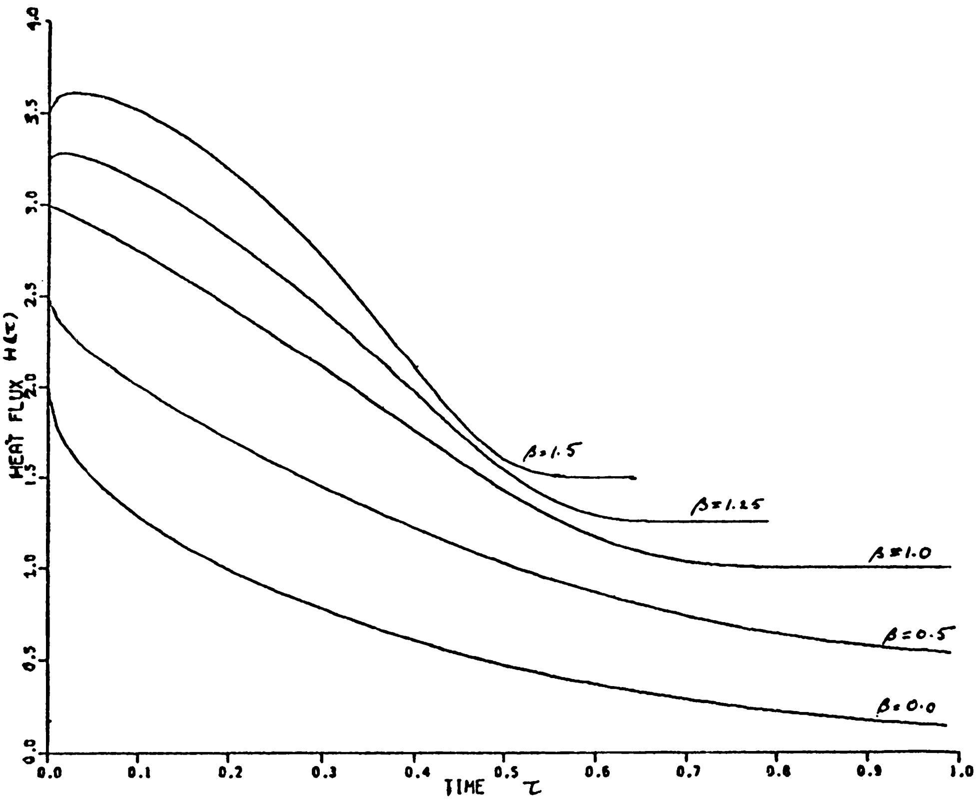

FIG. 4. Heat flux vs. time for fixed initial temperature distribution $\Phi(y) = y^2 - 1$: dependence on melting rate β.

The resulting solution is

$$u(y, \tau) = \frac{-B}{(1 - \beta\tau)^{1/2}} \cos \frac{\pi y}{2(1 - \beta\tau)} \exp \left[\frac{\pi^2 - \beta^2 y^2}{4\beta(\beta\tau - 1)} \right] \tag{58}$$

where

$$H(\tau) = \frac{K\pi}{2(1 - \beta\tau)^{3/2}} \exp \left[\frac{\beta(1 - \beta\tau)}{4} + \frac{\pi^2}{4} \left(1 + \frac{1}{\beta(\beta\tau - 1)} \right) \right]. \tag{59}$$

7. Numerical calculations. In Figs. 3–6 for the case of a boundary moving at a constant velocity the nondimensional heat flux $H(\tau)$ is plotted against the nondimensional time τ ($K = 1$). For values of the melting rate $\beta = 0, 0.5, 1.0, 1.25$ and 1.5 Fig. 3 shows the dependence of the heat flux for an initial temperature distribution $\Phi(y) = -\exp\,(+(\beta y^2/4)) \cos\,(\pi/2)y$. For the same values of β Fig. 4 shows the dependence of heat flux on the melting rate for a fixed initial temperature distribution $\Phi(y) =$

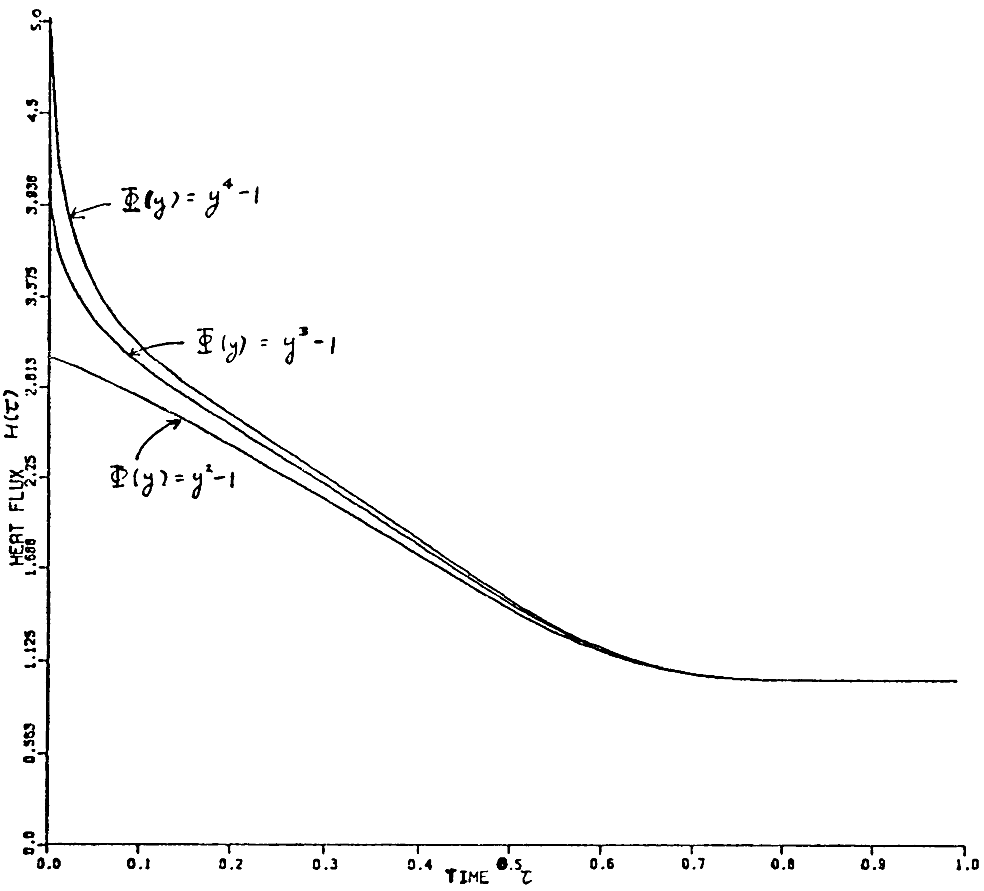

FIG. 5. Heat flux vs. time for melting rate $\beta = 1$, $\Phi(0) = -1$: dependence on initial temperature distribution.

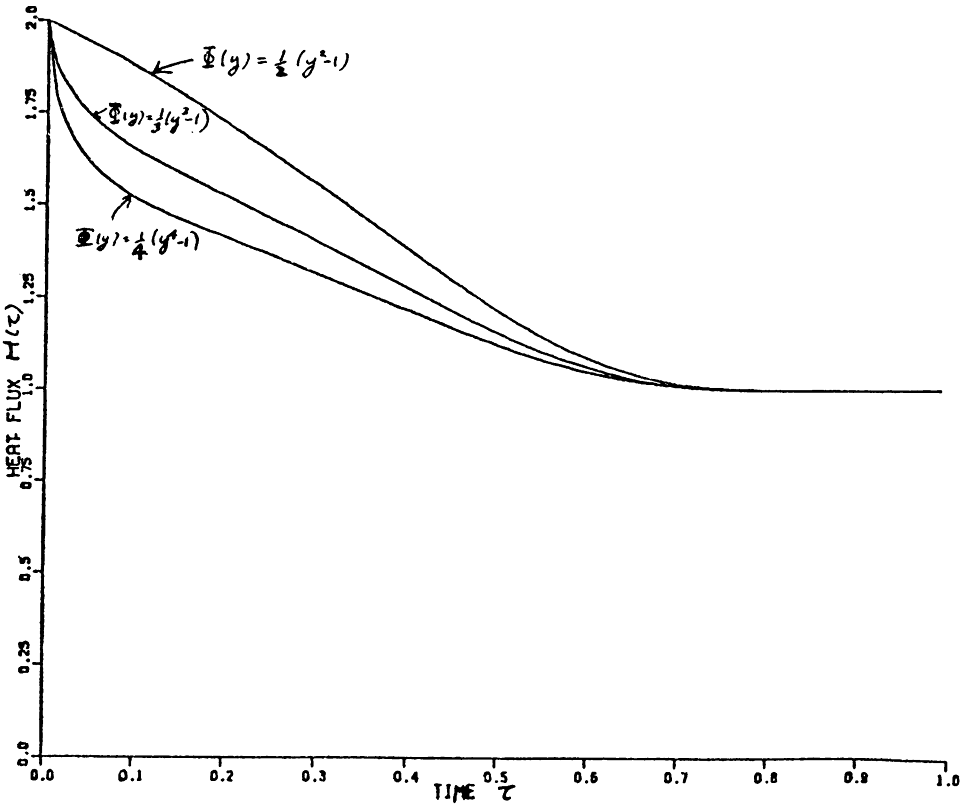

FIG. 6. Heat flux vs. time for melting rate $\beta = 1$, $\Phi'(1) = 1$: dependence on initial temperature distribution.

$y^2 - 1$. In Figs. 3 and 4 $\Phi(0) = -1$. In Figs. 5 and 6 the effects of various initial temperature distributions are shown for a melting rate $\beta = 1$. In Fig. 5 $\Phi(0) = -1$ and in Fig. 6 $\Phi'(1) = 1$.

8. Conclusions. New classes of analytical solutions for an inverse Stefan problem have been derived using group methods. In the case of a phase change boundary moving at a constant velocity the solution is of a simple form for computational purposes. In a future paper we will show how to solve a quite general direct Stefan problem by a numerical procedure based on these similarity solutions.

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