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Applications of the Multivariable Popov Criterion†

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ABSTRACT

Two classes of systems are considered for the application of the multivariable Popov criterion. The first is obtained from a linear, finite-dimensional system with a state feedback law derived from a quadratic loss function minimization problem. It is shown that a non-critical part of the system is the set of transducers producing the inputs to the system, in the sense that stability is retained even when the transducers are far from ideal.

The second class of systems is derived from linear, finite-dimensional systems which are stable. It is shown that it is always possible to tolerate in general a small amount of non-linearity at virtually any point in the system without impairment of stability.

§ 1. INTRODUCTION

MANY so-called linear systems are in fact not so, and it is the aim of this paper to indicate situations where non-linearity may be tolerated in a system, with this system remaining stable if the linear one from which it is derived is also stable. The technique used for establishing stability of the non-linear system is the Popov theorem (Popov 1961) in multi-dimensional format (Anderson 1966 a, b, Tokumaru and Saito 1965).

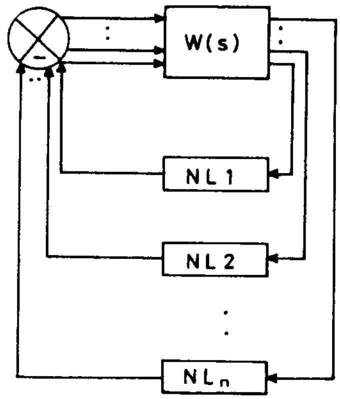
Two classes of systems are considered. The first is obtained from a linear, finite-dimensional system with a state feedback law derived from a quadratic loss function minimization problem. Non-linearity is then permitted in the transducers producing the input to the linear system. As is shown below, and as is known for the single input case (Kalman, private communication) a substantial amount of non-linearity can be tolerated with stability being retained. In practical terms, this means that a non-critical part of a regulator system is the transducer driving the system.

The second class of systems is derived from linear, finite-dimensional systems which are stable. It is shown that it is always possible to tolerate in general a small amount of non-linearity at virtually any point in the system. Moreover, it is possible to apply a sufficiency test to determine whether a non-linearity lying within prescribed bounds will not affect the stability of the system.

† Communicated by the Authors.

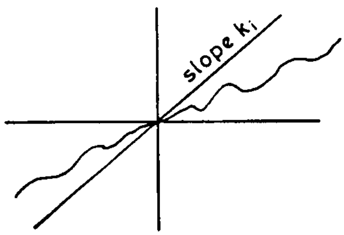
may be made strictly non-negative definite for all ω. The other requirements to guarantee positive realness (Newcomb 1966), can be shown to be satisfied in a straightforward manner.

Fig. 2



System to which Popov theorem applies.

Fig. 3



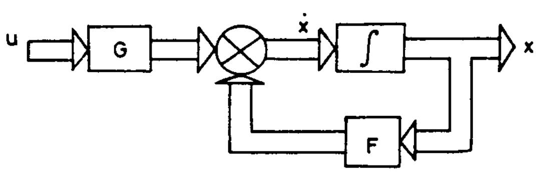
Restrictions on memoryless non linearities.

§ 3. Non-Linearities in otherwise Optimal Systems

We shall consider the system of fig. 4 described by the equation:

$$\dot{x} = Fx + Gu. \tag{2}$$

Fig. 4

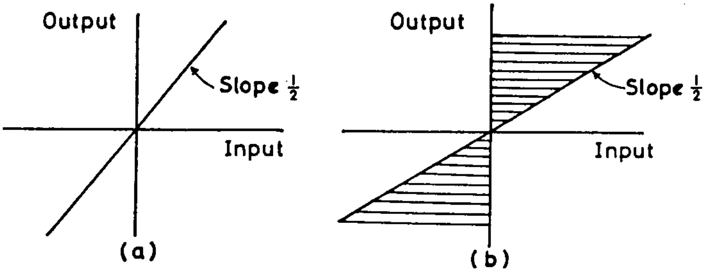


Linear finite-dimensional system.

blocks labelled transducers. In a practical situation, the realization of the block M' may well be done with electronics, while the generation of system inputs may require devices of large power-handling capability, e.g. motors, which inherently tend to contain non-linearities. Thus the transducers of fig. 6 are a generic representation of such devices.

Ideally, the transducer transfer characteristics should lead to identical input and output (fig. 7 (*a*)). But in practice, some non-linearity in each transducer will always exist, and we shall demonstrate that *any non-linearity lying in the shaded sector of fig. 7 (b) will lead to a stable system.*

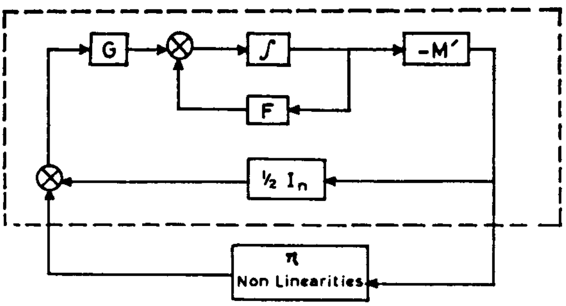
Fig. 7



(*a*) Ideal transducer characteristic. (*b*) Region of permissible transducer non-linearity retaining stability.

In order to prove this fact, it is necessary to transform the non-linearities so as to be amenable to the theory of §2. This may be done by observing that the non-linearity of fig. 7 (*b*) is equivalent to the sum of a linear transmission characteristic of slope $\frac{1}{2}$, and a non-linear transmission characteristic restricted to lying in the entire first and third quadrants. Taking the number of separate inputs to the system as n, this permits re-drawing of the system as in fig. 8.

Fig. 8



Nominally optimal non-linear system.

to be specified. Note that (10) forces, for positive x_j, the real u_i to be always greater than the ideal u_i of (9). The situation where a non-linearity causes the real u_i to be smaller than the ideal one can be covered as follows.

Because the eigenvalues of $F-GM'$ are continuous functions of the entries of F, G and M, it is true that if some of the entries m_{ij} of M are replaced by $m_{ij}+\epsilon_{ij}$, where ϵ_{ij} may have to be chosen sufficiently small, the eigenvalues of the modified $F-GM'$ will still be in the left half plane. But then the ideal u_i is now:

$$u_i = -\sum_j m_{ji}x_j - \sum_j \epsilon_{ji}x_j \tag{11}$$

and the real u_i is:

$$u_i = -\sum_j m_{ji}x_j + \sum_j [\mu_{ji}(x_j) - \epsilon_{ji}x_j]. \tag{12}$$

By still restricting the μ_{ji} to be first and third quadrant functions, we can, by the artifice of introducing a linear transformation with the ϵ_{ji}, arrange to consider non-linearities that result in values of u_i which are smaller than the ideal. All that is required is that $\mu_{ji}(x_j)-\epsilon_{ji}x_j$ be negative!

Because of the above remarks, we shall henceforth restrict attention to feedback laws as in (10), with the μ_{ij} first and third quadrant functions. Figure 9 illustrates the closed loop system incorporating the non-linearities.

To discuss the stability of the system with the aid of the Popov criterion, it is necessary to construct the $pn \times pn$ matrix $W(s)$, the transfer function matrix associated with the linear part of fig. 9, which maps the outputs of the non-linearities into the corresponding inputs, just like the $W(s)$ of fig. 2.

Neglecting the minus sign associated with the $-G$ block, $W(s)$ can be calculated explicitly by straightforward means as:

$$W(s) = \begin{bmatrix} I_p \\ I_p \\ \cdot \\ \cdot \\ I_p \end{bmatrix} (sI_p - F + GM')^{-1}G \begin{bmatrix} \overbrace{1\ 1\ \ldots\ 1}^{p} & \overbrace{0\ 0\ \ldots\ 0}^{p} & 0\ 0\ \ldots \\ 0\ 0\ \ldots\ 0 & 1\ 1\ \ldots\ 1 & 0\ \ldots \\ 0\ 0\ \ldots\ 0 & 0\ 0\ \ldots\ 0 & 1\ \ldots \\ \cdot & & \cdot \\ \cdot & & \cdot \end{bmatrix} \tag{13}$$

Stability of the closed loop non-linear system will follow if for some set of slopes $k_1, k_2 \ldots k_{pn}$ restricting the μ_{ij}, these are non-negative constants α and β for which:

$$Z(s) = \alpha K + (\alpha + \beta s)W(s), \tag{14}$$

§ 5. Conclusions

The results presented here, though constituting two useful applications of the multidimensional Popov theorem, do apply to differing practical situations. The range of tolerable non-linearities are quite different, and it is perhaps fortunate that the situation where a large amount of non-linearity can be tolerated, the nominally optimal feedback system, is also the situation wherein large amounts of non-linearity may be experienced.

The result on nominally linear stable systems is reassuring, in that it shows such systems are structurally stable. But it does go further than this, for, at the expense of the difficult calculations necessary to check positive realness, it gives a sufficiency test to see whether components of prescribed specifications can be satisfactorily incorporated in a system.

Both results illustrate the often experienced fact that saturation-type non-linearities can lead to trouble. Such non-linearities of course fall well outside a sector non-linearity in general.

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