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Applications of the q-Derivative Operator to New Families of Bi-Univalent Functions Related to the Legendre Polynomials

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Abstract: By using the q-derivative operator and the Legendre polynomials, some new subclasses of q-starlike functions and bi-univalent functions are introduced. Several coefficient estimates and Fekete–Szegö-type inequalities for functions in each of these subclasses are obtained. The results derived in this article are shown to extend and generalize those in some earlier works.

Keywords: analytic functions; bi-univalent functions; q-derivative operator; subordination between analytic functions; Legendre polynomials; Fekete–Szegö inequality

MSC: 30C45; 30C80; 33C45

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1. Introduction

In the development of the Geometric Function Theory of Complex Analysis, the q-derivative is an important research tool. The application of q-calculus was first considered by Jackson (see [1–4]). Recently, many scholars have defined new subclasses of analytic functions by combining the q-derivative operator with the principle of differential subordination and studied their geometric properties (see [5–15]). In this article, we investigate two new subclasses $I^q_{\mathcal{A}}[A, B, \lambda, \beta]$ and $I^q_{\Sigma}[\phi, \lambda, \beta]$ of the class of q-starlike functions and bi-univalent functions associated with the q-derivative operator and the Legendre polynomials. For each of these subclasses, we obtain certain coefficient estimates and Fekete–Szegö-type inequalities. The results obtained in this article are also shown to extend and generalize those in some earlier works. For motivation and incentive for further researches, the reader's attention is drawn toward some of the related recent developments in [12,16–19] dealing with the coefficient inequalities and coefficient estimates of various subclasses of analytic, univalent, and bi-univalent functions involving the second, third, and fourth Hankel determinants and the Fekete–Szegö functional.

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

which have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, let $\mathcal{S} \subset \mathcal{A}$ be the class of functions that are univalent in $\mathbb{U}$. Obviously, each function $f \in \mathcal{S}$ has an inverse f^{-1}, so that

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}\big(f(w)\big) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote this class using Σ. We remark that the study of the normalized class Σ of analytic and bi-univalent functions in $\mathbb{U}$ was revived in recent years by a pioneering article on the subject by Srivastava et al. [20], which has flooded the literature on the Geometric Function Theory of Complex Analysis with a large number of sequels to [20].

For a function $f \in \mathcal{A}$, given by (1), and a function $g \in \mathcal{A}$, given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

Let $\mathcal{P}$ be the class of Carathéodory functions h that are analytic in $\mathbb{U}$ and that satisfy

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$\Re\big(h(z)\big) > 0 \quad (z \in \mathbb{U}).$$

For two analytic functions f and g, we say that f is subordinate to g and it is written as $f \prec g$ or $f(z) \prec g(z)$, if there is a Schwarz function w such that $f(z) = g\big(w(z)\big)$. Further, if g is univalent in $\mathbb{U}$, then

$$f \prec g \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ as follows:

$$[\lambda]_q := \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] 1 + \sum\limits_{j=1}^{n-1} q^j & (\lambda = n \in \mathbb{N}). \end{cases}$$

Especially, we note that $[0]_q = 0$.

Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! := \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Let $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Define $[r]_{q,n}$ by

$$[r]_{q,n} := \begin{cases} 1 & (n = 0) \\ \prod_{k=r}^{r+n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Now, we recall here the q-derivative (or the q-difference) operator D_q $(0 < q < 1)$ of a function $f \in \mathcal{A}$ as follows:

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

where $f'(0)$ exists. Also, we write

$$(D_q^{(2)} f)(z) = \big(D_q(D_q f)\big)(z).$$

The Legendre polynomials $P_n(x)$ are the particular solutions to the Legendre differential equation:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (n \in \mathbb{N}_0).$$

The Legendre polynomials $P_n(x)$ are defined by

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad (n \in \mathbb{N}_0) \tag{3}$$

for arbitrary real or complex values of the variable x. The Legendre polynomials $P_n(x)$ are generated by (see, for details, [21])

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where $(1 - 2xt + t^2)^{-\frac{1}{2}}$ is taken to be 1 when $t \to 0$. The first few Legendre polynomials are given by

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1) \ \text{ and } \ P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

The function $\phi(z)$ given by

$$\phi(z) = \frac{1 - z}{\sqrt{1 - 2z\cos\alpha + z^2}} \tag{4}$$

belongs to the class $\mathcal{P}$ for every real number α (see [22]). By using (3), it is easy to see that

$$\phi(z) = 1 + \sum_{n=1}^{\infty} [P_n(\cos\alpha) - P_{n-1}(\cos\alpha)] z^n = 1 + \sum_{n=1}^{\infty} l_n\, z^n,$$

where

$$l_n = P_n(\cos\alpha) - P_{n-1}(\cos\alpha).$$

We also note that

$$l_1 = \cos\alpha - 1 \quad \text{and} \quad l_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha).$$

For more details, one can see the earlier work [23].

For $f \in \mathcal{A}$, the q-Ruscheweyh operator $R_{q,\lambda}$ is defined as follows (see [24]):

$$R_{q,\lambda} f(z) = f(z) * F_{q,\lambda+1}(z) \quad (z \in \mathbb{U};\ \lambda > -1),$$

where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!} z^n.$$

Let $f \in \mathcal{A}$. The q-integral operator $R_{q,\lambda}^{-1}$ is defined by (see [5,25])

$$R_{q,\lambda}^{-1}(z) * R_{q,\lambda}(z) = z\big(D_q f(z)\big).$$

Further, we have

$$R_{q,\lambda}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n,$$

where

$$\psi_{n-1} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \quad (n \geqq 2). \tag{5}$$

When $q \to 1{-}1$, the q-integral operator $R_{q,\lambda}^{-1}$ reduces to an integral operator studied by Noor [26].

For $f \in \mathcal{A}$, the q-integral operator I_q^λ is defined by (see [5])

$$I_q^\lambda f(z) = f(z) * R_{q,\lambda}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n, \tag{6}$$

where ψ_{n-1} is given by (5). Clearly, one can see that

$$I_q^0 f(z) = z\big(D_q f(z)\big) \quad \text{and} \quad I_q^1 f(z) = f(z).$$

Next, we will define the analytic function class $I_{\mathcal{A}}^q[A, B, \lambda, \beta]$ and the bi-univalent function class $I_\Sigma^q[\phi, \lambda, \beta]$.

Definition 1. *Let $\lambda > -1$, $-1 \leqq B < A \leqq 1$, and $0 \leqq \beta \leqq 1$. A function $f \in \mathcal{A}$ is said to be in the class $I_{\mathcal{A}}^q[A, B, \lambda, \beta]$ if*

$$\frac{z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta) I_q^\lambda f(z) + \beta z (D_q I_q^\lambda f)(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{7}$$

or equivalently,

$$\left| \frac{\frac{z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta) I_q^\lambda f(z) + \beta z (D_q I_q^\lambda f)(z)} - 1}{A - B \frac{z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta) I_q^\lambda f(z) + \beta z (D_q I_q^\lambda f)(z)}} \right| < 1. \tag{8}$$

Remark 1. *(i) For $\lambda = 1$ and $\beta = 0$, we have*

$$I_{\mathcal{A}}^q[A, B, 1, 0] = \mathcal{S}_q^*[A, B],$$

where the class $\mathcal{S}_q^*[A, B]$ was introduced by Srivastava et al. [27].

(ii) For $\lambda = 1$, $\beta = 0$, and $q \to 1^-$, we get

$$\lim_{q \to 1^-} I_{\mathcal{A}}^q[A, B, 1, 0] = \mathcal{S}^*[A, B],$$

where the class $\mathcal{S}^*[A, B]$ was considered by Janowski [28].

(iii) *For $A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ and $B = -1$, the class $\mathcal{S}^*[A, B]$ reduces to the class $\mathcal{S}^*(\alpha)$, which was studied by Silverman [29].*

Definition 2. *Let $\lambda > -1$ and $0 \leqq \beta \leqq 1$. A function $f \in \Sigma$ is said to be in the class $I_\Sigma^q[\phi, \lambda, \beta]$ if*

$$\begin{cases} \frac{z(D_q I_q^\lambda f)(z)+\beta z^2(D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z)+\beta z(D_q I_q^\lambda f)(z)} \prec \phi(z) \\ \\ \frac{w(D_q I_q^\lambda g)(w)+\beta w^2(D_q^{(2)} I_q^\lambda g)(w)}{(1-\beta)I_q^\lambda g(w)+\beta w(D_q I_q^\lambda g)(w)} \prec \phi(w), \end{cases} \tag{9}$$

where the functions g and ϕ are given by (2) and (4), respectively.

To derive our main results, we need the following lemmas.

Lemma 1 (see [30]). *Let $\varphi(z) = 1 + \omega_1 z + \omega_2 z^2 + \cdots \in \mathcal{P}$. Then,*

$$|\omega_2 - \nu\omega_1^2| \leqq \begin{cases} -4\nu + 2 & (\nu < 0) \\ \\ 2 & (0 \leqq \nu \leqq 1) \\ \\ 4\nu - 2 & (\nu > 1). \end{cases} \tag{10}$$

Lemma 2 (see [13]). *Let*

$$M(z) = 1 + \sum_{n=1}^{\infty} C_n z^n \prec H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

If $H(z)$ is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then

$$|C_n| \leqq |d_1| \quad (n \in \mathbb{N}).$$

Lemma 3 (see [31]). *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then*

$$|c_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

2. Main Results

In this section, we derive certain coefficient estimates and the Fekete–Szegö-type inequalities for functions in the classes $I_\mathcal{A}^q[A, B, \lambda, \beta]$ and $I_\Sigma^q[\phi, \lambda, \beta]$, which are defined above (see Definitions 1 and 2). Many special cases and consequences of our main findings are pointed out.

Theorem 1. *Let a function $f \in I_\mathcal{A}^q[A, B, \lambda, \beta]$ be of the form given by (1). Then,*

$$\begin{aligned} |a_n| \leqq & \frac{A - B}{(q(1-\beta) + \beta[n]_q)\psi_{n-1}} \\ & \cdot \prod_{j=1}^{n-1} \frac{\{q(1-\beta) + \beta[j]_q\}[j-1]_q + \{(1-\beta) + \beta[j]_q\}(A - B)}{\{q(1-\beta) + \beta[j]_q\}[j]_q} \quad (n \geqq 2), \end{aligned} \tag{11}$$

where ψ_{n-1} is given by (5).

Proof. For $f \in I_\mathcal{A}^q[A, B, \lambda, \beta]$, we have

$$\nu(z) := \frac{z(D_q I_q^\lambda f)(z) + \beta z^2(D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z) + \beta z(D_q I_q^\lambda f)(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{12}$$

where

$$\frac{1+Az}{1+Bz} = 1 + \sum_{n=0}^{\infty}(A-B)(-B)^n z^{n+1} = 1 + (A-B)z - B(A-B)z^2 + \cdots .$$

Since $\nu(z) = 1 + \sum_{n=1}^{\infty} \nu_n z^n$, we get from Lemma 2 that

$$|\nu_n| \leqq A - B \quad (n \in \mathbb{N}). \tag{13}$$

From (12), we have

$$z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z) = \nu(z)[(1-\beta) I_q^{\lambda} f(z) + \beta z\big(D_q I_q^{\lambda} f(z)\big)]$$

which shows that

$$\begin{aligned} &z + \sum_{n=2}^{\infty} [n]_q [1 + \beta[n-1]_q] a_n \psi_{n-1} z^n \\ &\quad = \left(1 + \sum_{n=1}^{\infty} \nu_n z^n\right)\left(z + \sum_{n=2}^{\infty} [1 + \beta([n]_q - 1)] a_n \psi_{n-1} z^n\right). \end{aligned} \tag{14}$$

Comparing the coefficients of z^n on both sides of the Equation (14), we get

$$[n-1]_q [q(1-\beta) + \beta[n]_q] a_n \psi_{n-1} = \sum_{l=1}^{n-1} [1 + \beta([l]_q - 1)] a_l \psi_{l-1} \nu_{n-l},$$

where $a_1 = 1$, $\nu_1 = 1$ and $\psi_0 = 1$. The above equation gives

$$|a_n| \leqq \frac{A-B}{[n-1]_q [q(1-\beta) + \beta[n]_q] \psi_{n-1}} \sum_{l=1}^{n-1} [1 + \beta([l]_q - 1)] |a_l| \psi_{l-1}.$$

Thus, we get

$$\begin{aligned} |a_2| &\leqq \frac{A-B}{(q+\beta)\psi_1}; \\ |a_3| &\leqq \frac{A-B}{(q(1-\beta) + \beta[3]_q)\psi_2} \cdot \frac{(q+\beta) + (1+\beta q)(A-B)}{[2]_q (q+\beta)}; \\ |a_4| &\leqq \frac{A-B}{(q(1-\beta) + \beta[4]_q)\psi_3} \cdot \left(\frac{(q+\beta) + (1+\beta q)(A-B)}{[2]_q (q+\beta)}\right) \\ &\qquad \cdot \left(\frac{[2]_q \{q(1-\beta) + \beta[3]_q\} + \big((1-\beta) + \beta[3]_q\big)(A-B)}{[3]_q}\right); \\ &\cdots\cdots \\ |a_n| &\leqq \frac{A-B}{(q(1-\beta) + \beta[n]_q)\psi_{n-1}} \\ &\qquad \cdot \prod_{j=1}^{n-1} \frac{\{q(1-\beta) + \beta[j]_q\}[j-1]_q + \{(1-\beta) + \beta[j]_q\}(A-B)}{\{q(1-\beta) + \beta[j]_q\}[j]_q}. \end{aligned}$$

This proves Theorem 1. □

For $\lambda = 1$ and $\beta = 0$ in Theorem 1, we obtain a result of the class $\mathcal{S}_q^*[A, B]$, which was considered by Srivastava et al. [27].

Corollary 1. *Let a function $f \in \mathcal{S}_q^*[A,B]$ be of the form given by (1). Then,*

$$|a_n| \leqq \frac{1}{q}\prod_{j=1}^{n-1}\frac{q[j-1]_q+(A-B)}{q[j]_q} \qquad (n \geqq 2).$$

Theorem 2. *Let a function $f \in I_\Sigma^q[\phi,\lambda,\beta]$ be given by (1). Then,*

$$|a_2| \leqq \min\left\{ \begin{array}{l} \dfrac{|\cos\alpha-1|}{(q+\beta)\psi_1}, \\ \dfrac{\sqrt{2}|\cos\alpha-1|}{\sqrt{|2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2|}} \end{array}\right. \tag{15}$$

and

$$|a_3| \leqq \min\left\{ \begin{array}{l} \dfrac{|\cos\alpha-1|^2}{(q+\beta)^2\psi_1^2}+\dfrac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}, \\ \frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}+\frac{2(\cos\alpha-1)^2}{|\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2\}|}. \end{array}\right. \tag{16}$$

Proof. From (9), we know that there are two Schwarz functions $u(z)$ and $\nu(w)$, such that

$$\frac{z(D_qI_q^\lambda f)(z)+\beta z^2(D_q^{(2)}I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z)+\beta z\big(D_qI_q^\lambda f(z)\big)} = \phi\big(u(z)\big) \tag{17}$$

and

$$\frac{w(D_qI_q^\lambda g)(w)+\beta w^2(D_q^{(2)}I_q^\lambda g)(w)}{(1-\beta)I_q^\lambda g(w)+\beta w\big(D_qI_q^\lambda g(w)\big)} = \phi\big(\nu(w)\big). \tag{18}$$

Now we define the functions $s(z)$ and $t(w)$ by

$$s(z) = \frac{1+u(z)}{1-u(z)} = 1+s_1z+s_2z^2+\cdots \in \mathcal{P}$$

and

$$t(w) = \frac{1+\nu(w)}{1-\nu(w)} = 1+t_1z+t_2z^2+\cdots \in \mathcal{P}.$$

Since

$$\phi(z) = 1+\sum_{n=1}^{\infty}[P_n(\cos\alpha)-P_{n-1}(\cos\alpha)]z^n = 1+\sum_{n=1}^{\infty}l_nz^n,$$

we get

$$\begin{aligned} \phi\big(u(z)\big) &= 1+\tfrac{1}{2}l_1s_1z+\left[\tfrac{1}{2}l_1\left(s_2-\tfrac{s_1^2}{2}\right)+\tfrac{1}{4}l_2s_1^2\right]z^2+\cdots, \\ \phi\big(\nu(w)\big) &= 1+\tfrac{1}{2}l_1t_1w+\left[\tfrac{1}{2}l_1\left(t_2-\tfrac{t_1^2}{2}\right)+\tfrac{1}{4}l_2t_1^2\right]w^2+\cdots. \end{aligned} \tag{19}$$

Using the Taylor series formula, we have

$$\begin{aligned} \frac{z(D_qI_q^\lambda f)(z)+\beta z^2(D_q^{(2)}I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z)+\beta z\big(D_qI_q^\lambda f(z)\big)} &= 1+(q+\beta)\psi_1a_2z \\ &+\{[q(q+1)+(q^2+1)[2]_q\beta]\psi_2a_3-(1+\beta q)(q+\beta)\psi_1^2a_2^2\}z^2+\cdots \end{aligned}$$

and

$$\frac{w(D_q I_q^\lambda g)(w) + \beta w^2 (D_q^{(2)} I_q^\lambda g)(w)}{(1-\beta) I_q^\lambda g(w) + \beta w\big(D_q I_q^\lambda g(w)\big)} = 1 - (q+\beta)\psi_1 a_2 w$$
$$+ \{[q(q+1) + (q^2+1)[2]_q \beta]\psi_2 (2a_2^2 - a_3) - (1+\beta q)(q+\beta)\psi_1^2 a_2^2\} w^2 + \cdots.$$

Comparing the left-side and right-side coefficients of (17) and (18), we obtain

$$(q+\beta)\psi_1 a_2 = \frac{1}{2} l_1 s_1, \tag{20}$$

$$[2]_q [q + \beta(q^2+1)]\psi_2 a_3 - (1+\beta q)(q+\beta)\psi_1^2 a_2^2 = \frac{1}{2} l_1 \left(s_2 - \frac{s_1^2}{2} \right) + \frac{1}{4} l_2 s_1^2, \tag{21}$$

$$-(q+\beta)\psi_1 a_2 = \frac{1}{2} l_1 t_1 \tag{22}$$

and

$$[2]_q [q + \beta(q^2+1)]\psi_2 (2a_2^2 - a_3) - (1+\beta q)(q+\beta)\psi_1^2 a_2^2 = \frac{1}{2} l_1 \left(t_2 - \frac{t_1^2}{2} \right) + \frac{1}{4} l_2 t_1^2. \tag{23}$$

From (20) and (22), we have

$$a_2 = \frac{l_1 s_1}{2(q+\beta)\psi_1} = \frac{-l_1 t_1}{2(q+\beta)\psi_1}. \tag{24}$$

Thus, we find that

$$s_1 = -t_1 \tag{25}$$

and

$$8(q+\beta)^2 \psi_1^2 a_2^2 = l_1^2 (s_1^2 + t_1^2). \tag{26}$$

Using Lemma 3, we find from (24) that

$$|a_2| \leqq \frac{|\cos\alpha - 1|}{(q+\beta)\psi_1}. \tag{27}$$

Now from (21), (23), (24) and (25), we have

$$4\{[2]_q [q + \beta(q^2+1)] l_1^2 \psi_2 - (q+\beta)[(1+\beta q) l_1^2 + (q+\beta)(l_2 - l_1)]\psi_1^2\} a_2^2 = l_1^3 (s_2 + t_2).$$

Since

$$l_1 = \cos\alpha - 1 \quad \textit{and} \quad l_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha),$$

we obtain

$$\begin{aligned} a_2^2 &= \frac{l_1^3 (s_2 + t_2)}{4\{[2]_q [q + \beta(q^2+1)] l_1^2 \psi_2 - (q+\beta)[(1+\beta q) l_1^2 + (q+\beta)(l_2 - l_1)]\psi_1^2\}} \\ &= \frac{(\cos\alpha - 1)^2 (s_2 + t_2)}{2\{2[2]_q [q + \beta(q^2+1)](\cos\alpha - 1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha - 1) + (q+\beta)(3\cos\alpha - 1)]\psi_1^2\}}. \end{aligned} \tag{28}$$

Applying Lemma 3 to the coefficients s_2 and t_2, we have

$$|a_2| \leqq \frac{\sqrt{2}|\cos\alpha - 1|}{\sqrt{|2[2]_q [q + \beta(q^2+1)](\cos\alpha - 1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha - 1) + (q+\beta)(3\cos\alpha - 1)]\psi_1^2|}}.$$

By subtracting (23) from (21), we have

$$2[2]_q[q+\beta(q^2+1)](a_3-a_2^2)\psi_2=\frac{1}{2}l_1(s_2-t_2)+\frac{1}{4}(l_2-l_1)(s_1^2-t_1^2). \tag{29}$$

From (24), (25) and (29), we obtain

$$a_3=a_2^2+\frac{l_1(s_2-t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2}. \tag{30}$$

Now taking the modulus of (30) and using Lemma 3, we get

$$|a_3|\leqq|a_2|^2+\frac{l_1}{[2]_q[q+\beta(q^2+1)]\psi_2}. \tag{31}$$

Further, by using (27) and (31), we find

$$\begin{aligned}|a_3|&\leqq\frac{l_1^2}{(q+\beta)^2\psi_1^2}+\frac{|l_1|}{[2]_q[q+\beta(q^2+1)]\psi_2}\\&=\frac{|\cos\alpha-1|}{(q+\beta)^2\psi_1^2}+\frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}.\end{aligned}$$

Also, using (26) and (31), we derive

$$\begin{aligned}|a_3|&\leqq|a_2^2|+\frac{l_1}{[2]_q[q+\beta(q^2+1)]\psi_2}\\&=\frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}\\&\quad+\frac{2(\cos\alpha-1)^2}{|\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2\}|}.\end{aligned}$$

This completes the proof of Theorem 2. □

Theorem 3. *Let a function $f\in I_{\mathcal{A}}^q[A,B,\lambda,\beta]$ be of the form given by (1). Then,*

$$|a_3-\mu a_2^2|\leqq\begin{cases}\dfrac{A-B}{(1+q)[q+(q^2+1)\beta]\psi_2}\Lambda(q) & (\mu<\sigma_1)\\[2ex]\dfrac{A-B}{(1+q)[q+(q^2+1)\beta]\psi_2} & (\sigma_1\leqq\mu\leqq\sigma_2)\\[2ex]\frac{B-A}{(1+q)[q+(q^2+1)\beta]\psi_2}\Lambda(q) & (\mu>\sigma_2),\end{cases}$$

where μ is real and

$$\Lambda(q)=\frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)B]\psi_1^2-\mu(1+q)[q+(q^2+1)\beta](A-B)\psi_2}{(q+\beta)^2\psi_1^2},$$

$$\sigma_1=\frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)(B+1)]\psi_1^2}{(1+q)[q+(q^2+1)\beta](A-B)\psi_2}$$

and

$$\sigma_2=\frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)(B-1)]\psi_1^2}{(1+q)[q+(q^2+1)\beta](A-B)\psi_2}.$$

Proof. Let $f \in I_{\mathcal{A}}^{q}[A, B, \lambda, \beta]$. Using the Taylor series formula, we have

$$\frac{z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta) I_q^\lambda f(z) + \beta z \big(D_q I_q^\lambda f(z)\big)} = 1 + (q+\beta)\psi_1 a_2 z + \{[q(q+1) + (q^2+1)[2]_q \beta]\psi_2 a_3 - (1+\beta q)(q+\beta)\psi_1^2 a_2^2\} z^2 + \cdots . \tag{32}$$

From (7), we know that there exists a Schwarz function h such that

$$\frac{z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z)}{(1-\beta) I_q^\lambda f(z) + \beta z \big(D_q I_q^\lambda f(z)\big)} = \frac{1 + Ah(z)}{1 + Bh(z)}.$$

We now define a function $w \in \mathcal{P}$ by

$$w(z) = \frac{1 + h(z)}{1 - h(z)} = 1 + w_1 z + w_2 z^2 + \cdots .$$

This implies that

$$h(z) = \frac{w(z) - 1}{w(z) + 1} = 1 + \frac{1}{2} w_1 z + \left(\frac{1}{2} w_2 - \frac{1}{4} w_1^2\right) z^2 + \cdots .$$

Also, we have

$$\frac{1 + Ah(z)}{1 + Bh(z)} = 1 + \frac{1}{2}(A-B) w_1 z + \left[\frac{1}{2}(A-B) w_2 - \frac{1}{4}(B+1)(A-B) w_1^2\right] z^2 + \cdots . \tag{33}$$

Therefore, we obtain

$$\begin{aligned} a_2 &= \left(\tfrac{A-B}{2(q+\beta)\psi_1}\right) w_1, \\ a_3 &= \left(\tfrac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2}\right)\left\{ w_2 - \tfrac{1}{2}\left[(B+1) - \left(\tfrac{1+\beta q}{q+\beta}\right)(A-B)\right] w_1^2 \right\}. \end{aligned} \tag{34}$$

Now, we can find that

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} \left| \left\{ w_2 - \frac{1}{2}\left[(B+1) - \left(\frac{1+\beta q}{q+\beta}\right)(A-B)\right] w_1^2 \right\} \right. \\ &\quad \left. - \mu \frac{(A-B)^2}{4(q+\beta)^2 \psi_1^2} w_1^2 \right| \\ &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} \left| \left\{ w_2 - \frac{1}{2}[(B+1) \right.\right. \\ &\quad \left.\left. - \left(\frac{(1+\beta q)(q+\beta)\psi_1^2 + \mu(1+q)[q+(q^2+1)\beta]\psi_2}{(q+\beta)^2\psi_1^2}\right)(A-B)\right] w_1^2 \right\} \right| \\ &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} |\{ w_2 - k_1(q) w_1^2 \}|, \end{aligned} \tag{35}$$

where

$$k_1(q) = \frac{(q+\beta)[(q+\beta)(B+1) - (1+\beta q)(A-B)]\psi_1^2 + \mu(1+q)[q+(q^2+1)\beta](A-B)\psi_2}{2(q+\beta)^2\psi_1^2}.$$

Applying Lemma 1 in (35), we get the desired results. The proof of Theorem 3 is completed. □

For $\lambda = 1$, $\beta = 0$, and $q \to 1^-$, we get a result of the class $\mathcal{S}^*[A, B]$ that was considered by Janowski [28].

Corollary 2. *Let a function $f \in \mathcal{S}^*[A,B]$ be of the form given by (1). Then,*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \left(\frac{A-B}{2}\right)[(A-2B) - \mu(A-B)] & \left(\mu < \frac{A-2B-1}{2(A-B)}\right) \\ \frac{A-B}{2} & \left(\frac{A-2B-1}{2(A-B)} \leqq \mu \leqq \frac{A-2B+1}{2(A-B)}\right) \\ \left(\frac{B-A}{2}\right)[(A-2B) - \mu(A-B)] & \left(\mu > \frac{A-2B+1}{2(A-B)}\right). \end{cases}$$

Theorem 4. *Let a function $f \in I_{\Sigma}^{q}[\phi, \lambda, \beta]$ be of the form given by (1). Then,*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{|\cos\alpha - 1|}{[2]_q[q+\beta(q^2+1)]\psi_2} & \left(0 \leqq |h(\mu)| \leqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right) \\ 4|h(\mu)| & \left(|h(\mu)| \geqq \frac{|\cos\alpha - 1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right), \end{cases}$$

where μ is real and

$$h(\mu) = \frac{(1-\mu)(\cos\alpha-1)^2}{2\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha-1) + (q+\beta)(3\cos\alpha-1)]\psi_1^2}. \quad (36)$$

Proof. From (30), we have

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + \frac{l_1(s_2 - t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2}. \quad (37)$$

Using (28) and (37), we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(\cos\alpha-1)(s_2-t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2} \\ &+ \frac{(1-\mu)(\cos\alpha-1)^2(s_2+t_2)}{2\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha-1) + (q+\beta)(3\cos\alpha-1)]\psi_1^2\}} \\ &= \left(h(\mu) + \frac{\cos\alpha-1}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right)s_2 + \left(h(\mu) - \frac{\cos\alpha-1}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right)t_2, \end{aligned} \quad (38)$$

where $h(\mu)$ is given by (36).

Taking the modulus of each side in (38), we get

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{|\cos\alpha - 1|}{[2]_q[q+\beta(q^2+1)]\psi_2} & \left(0 \leqq |h(\mu)| \leqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right) \\ 4|h(\mu)| & \left(|h(\mu)| \geqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right). \end{cases}$$

This proves Theorem 4. □

3. Conclusions

In our present investigation, we have used the q-derivative (or the q-difference) operator D_q, as well as the Legendre polynomials $P_n(x)$ to introduce and study two new subclasses of the class of q-starlike functions and the class of analytic and bi-univalent functions. For each of these subclasses, we have derived a number of coefficient estimates and Fekete–Szegö-type inequalities. The results derived in this article are also shown to extend and generalize those in some earlier works. For motivation and incentive for further research, the reader's attention is drawn toward some of the related recent developments in [16–19] dealing with the coefficient inequalities and coefficient estimates of various subclasses of analytic, univalent, and bi-univalent functions involving the second, third, and fourth Hankel determinants, and the Fekete–Szegö functional.

In concluding this article, we choose to discourage the current trend of some amateurish-type publications in which there are falsely-claimed "generalizations" of known q-theory and known q-results by forcing-in an obviously superfluous (or redundant) parameter p. In this connection, the reader should refer to [32] (p. 340) and [33] (pp. 1511–1512) for a detailed exposition and demonstration, where it is stated clearly that the current trend of trivially and inconsequentially translating known q-results into the corresponding (p, q)-results leads to no more than a straightforward and shallow publication involving an additional forced-in parameter p that is obviously redundant (or superfluous).

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