APPLICATIONS OF THE SPACE DIFFERENTAL GEOMETRY AT THE STUDY OF PRODUCTION FUNCTIONS

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Abstract:

This paper is a new onset about production functions. Because all papers on this subject use the projections of production functions on a plan, the analysis becomes heavy and less general in conclusions, and for this reason we made a treatment from the point of view of differential geometry in space.

On the other hand, we generalise the Cobb-Douglas, CES and Sato production functions to a unique form and we made the analysis on this.

The conclusions of the paper allude to the principal directions of the surface (represented by the graph of the production function) i.e. the directions in which the function varies the best. Also the concept of the total curvature of a surface is applied here and we obtain that it is null in every point, that is all points are parabolic.

We compute also the surface element which is useful to finding all production (by means the integral) when both labour and capital are variable.

Keywords: production function, differential geometry, curvature, principal direction, Cobb-Douglas

JEL Classificiation: C65 - Miscellaneous Mathematical Tools

1. INTRODUCTION

Let a production function Q=Q(K,L) where:

- Q=product;
- K=capital;
- L=labour

The function $Q:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ must satisfiy the conditions:

1. Q(0,0)=0;
2. Q is differentiable of order 2 in any interior point of the production set;
3. $\frac{\partial Q}{\partial K}\geq 0, \frac{\partial Q}{\partial L}\geq 0$;
4. $\frac{\partial^2 Q}{\partial K^2}\leq 0, \frac{\partial^2 Q}{\partial L^2}\leq 0$
5. Q is a homogenous function of degree 1, that is Q(tK,tL)=tQ(K,L) $\forall t\in\mathbf{R}$

The meaning of the first condition is that at a vanishing of one factor the product is null.

The second condition is useful just for mathematical calculus.

The third means that at an increase of one factor (labour or capital) the product also grow.

The fourth, because the second derivative is the speed of variation of the first, means that the product has a slower speed when one factor becomes constant and the other varies.

The graph representation of a production function is a surface.

Let:

$$p=\frac{\partial Q}{\partial L},\ q=\frac{\partial Q}{\partial K},\ r=\frac{\partial^2 Q}{\partial L^2},\ s=\frac{\partial^2 Q}{\partial L \partial K},\ t=\frac{\partial^2 Q}{\partial K^2}.$$

For a constant value of one parameter we obtain a curve on the surface. For exemple: $Q=Q(K,L_0)$ or $Q=Q(K_0,L)$ are both curves on the production surface. They are obtained from the intersection of the plane $L=L_0$ or $K=K_0$ with the surface $Q=Q(K,L)$.

The curvature of a curve is from an elementary point of view the degree of deviation of the curve relative to a straight line.

In the study of the surfaces, two quadratic forms are very useful.

The first fundamental quadratic form of the surface is:

$$g=EdL^2+2FdLdK+GdK^2$$

where:

- $E=1+p^2$;
- $F=pq$;
- $G=1+q^2$.

The area element is $d\sigma=\sqrt{EG-F^2}\,dKdL$ and the surface area A when $(K,L)\in R$ (a region in the plane K-O-L) is $A=\iint_R d\sigma dKdL$.

The second fundamental form of the surface is:

$$h=\lambda dL^2+2\mu dLdK+\nu dK^2$$

where:

- $\lambda=\dfrac{r}{\sqrt{1+p^2+q^2}}$;
- $\mu=\dfrac{s}{\sqrt{1+p^2+q^2}}$;

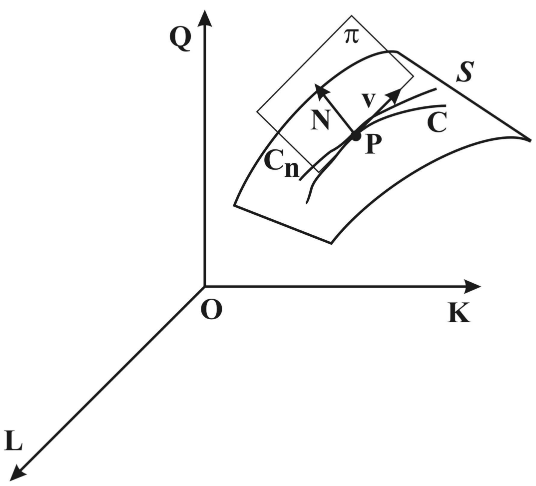
- $\nu = \frac{t}{\sqrt{1+p^2+q^2}}$.

Considering the quantity $\delta=\lambda\nu-\mu^2$ we have that:

- If $\delta>0$ in each point of the surface, we will say that it is eliptical. Such surfaces are the hyperboloid with two sheets, the eliptical paraboloid and the elypsoid.
- If $\delta<0$ in each point of the surface, we will say that it is hyperbolic. Such surfaces are the the hyperbolid with one sheet and the hyperbolic paraboloid.
- If $\delta=0$ in each point of the surface, we will say that it is parabolic. Such surfaces are the cone surfaces and the cylinder surfaces.

Considering a surface S and an arbitrary curve through a point P of the surface who has the tangent vector v in P, let the plane π determined by the vector v and the normal N in P at S. The intersection of π with S is a curve C_n named normal section of S. Its curvature is called normal curvature.

Figure-1: The normal section of a curve



If we have a direction $m=\frac{dL}{dK}$ in the tangent plane of the surface in an arbitrary point P we have that the normal curvature is given by:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}$$

The extreme values k_1 and k_2 of the function k(m) call the principal curvatures of the surface in that point. They satisfy also the equation:

$$(EG-F^2)k^2-(E\nu-2F\mu+G\lambda)k+(\lambda\nu-\mu^2)=0$$

The values of m who give the extremes call principal directions in that point.

They also satisfy the equation:

$$(E\mu-F\lambda)m^2+(E\nu-G\lambda)m+(F\nu-G\mu)=0$$

or

$(Es-Fr)m^2+(Et-Gr)m+(Ft-Gs)=0$

The curve $\frac{dL}{dK}=m$ (where m is one of the principal directions) is called line of curvature on the surface. On such a curve we have the maximum or minimum variation of the value of Q in a neighbourhood of P.

The quantity $K=k_1k_2$ is named the total curvature in the considered point and $H=\frac{k_1+k_2}{2}$ is named the mean curvature of the surface in that point.

We have therefore:

$$K=\frac{\lambda\nu-\mu^2}{EG-F^2} \text{ and } H=\frac{E\nu-2F\mu+G\lambda}{EG-F^2}$$

A surface with K=constant call surface with constant total curvature and if H=0 call minimal surface.

Considering now in the tangent plane π at the surface in a point P a direction m, if $\lambda m^2+2\mu m+\nu=0$ we will say that m is an asymptotic direction, and the equation:
$\lambda\left(\frac{dL}{dK}\right)^2+2\mu\frac{dL}{dK}+\nu=0$ gives the asymptotic curves of the surface in the point P.

2. THE GENERAL PRODUCTION FUNCTION

Let the production function:

$$Q=A\frac{K^{\alpha}L^{\beta}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega}},\ \alpha,\beta,\rho\in[0,1],\ \omega\in\mathbf{R},\ \varepsilon+\gamma\neq0$$

- For $\omega=0$, γ,ε,ρ=arbitrary, $\alpha,\beta\in[0,1]$ we have the Cobb-Douglas function: $Q=AK^{\alpha}L^{\beta}$;
- For $\alpha=0$, $\beta=0$, $\omega=-\frac{1}{\rho}$ we have the CES function: $Q=A(\gamma K^{\rho}+\varepsilon L^{\rho})^{\frac{1}{\rho}}$;
- For $\alpha=2$, $\beta=2$, $\rho=3$ and $\omega=1$ we have the SATO function: $Q=A\frac{K^2L^2}{\gamma K^3+\varepsilon L^3}$.

In order to have a homogenous function of degree 1, we have that: $Q(tK,tL)=tQ(K,L)\ \forall t\in\mathbf{R}$

We have therefore:

$$Q(tK,tL)=A\,t^{\alpha+\beta-\rho\omega}\frac{K^{\alpha}L^{\beta}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega}}=t^{\alpha+\beta-\rho\omega}Q(K,L)\Rightarrow\alpha+\beta-\rho\omega=1.$$

In consequence: $\omega=\frac{\alpha+\beta-1}{\rho}$ and the general expression of Q will be:

$$Q=A\frac{K^{\alpha}L^{\beta}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\frac{\alpha+\beta-1}{\rho}}},\ \alpha,\beta,\rho\in[0,1],\ \varepsilon+\gamma\neq 0$$

We have now:

$$\frac{\partial Q}{\partial K}=AK^{\alpha-1}L^{\beta}\frac{\gamma(\alpha-\rho\omega)K^{\rho}+\alpha\varepsilon L^{\rho}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega+1}}.$$

Because $(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega}=\frac{AK^{\alpha}L^{\beta}}{Q}$ we obtain:

$$q=\frac{\partial Q}{\partial K}=Q\frac{(1-\beta)\gamma K^{\rho}+\alpha\varepsilon L^{\rho}}{K(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

Through analogy:

$$p=\frac{\partial Q}{\partial L}=Q\frac{(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}}{L(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

With the upper relations we have now:

$$t=\frac{\partial^2 Q}{\partial K^2}=-\frac{Q}{K^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$r=\frac{\partial^2 Q}{\partial L^2}=-\frac{Q}{L^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$s=\frac{\partial^2 Q}{\partial K\partial L}=\frac{Q}{KL(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

Let note now:

$P=\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}$

$U=(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}$

$V=\alpha\varepsilon L^{\rho}+(1-\beta)\gamma K^{\rho}$

from where:

$U+V=\varepsilon L^{\rho}+\gamma K^{\rho}$.

If $\alpha+\beta-1\neq 0$ we have:

$K^{\rho}=\frac{(1-\alpha)V-\alpha U}{(1-\alpha-\beta)\gamma}$ and $L^{\rho}=\frac{(1-\beta)U-\beta V}{(1-\alpha-\beta)\varepsilon}$.

We have now:

$$p=\frac{\partial Q}{\partial L}=\frac{QU}{L(U+V)};$$

$$q=\frac{\partial Q}{\partial K}=\frac{QV}{K(U+V)};$$

$$E=1+p^2=1+Q^2\frac{U^2}{L^2(U+V)^2};$$

$$F=pq=Q^2\frac{UV}{KL(U+V)^2};$$

$$G=1+q^2=1+Q^2\frac{V^2}{K^2(U+V)^2}.$$

With $\Delta=1+p2+q2=1+Q2\frac{K^2U^2+L^2V^2}{K^2L^2(U+V)^2}$ we have:

$$\lambda=\frac{r}{\sqrt{\Delta}},\ \mu=\frac{s}{\sqrt{\Delta}},\ \nu=\frac{t}{\sqrt{\Delta}}.$$

$$t=-\frac{QP}{K^2(U+V)^2},\ r=-\frac{QP}{L^2(U+V)^2},\ s=\frac{QP}{KL(U+V)^2}.$$

After an easy computing we have $EG-F^2=1+\frac{Q^2(K^2U^2+L^2V^2)}{K^2L^2(U+V)^2}$

from where:

$$d\sigma=\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

and the surface area will be compute by:

$$A=\iint_R\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

The principal directions will be given by:

$$K^2[L^2(U+V)^2+Q^2U^2+Q^2UV]m^2+KL[-L^2(U+V)^2-Q^2U^2+K^2(U+V)^2+Q^2V^2]m-L^2[K^2(U+V)^2+Q^2V^2+Q^2UV]=0$$

from where:

$$m_1=\frac{L}{K},\ m_2=-\frac{L}{K}\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}.$$

For a direction m we have:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}=\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}\cdot$$

$$\frac{-K^2m^2+2KLm-L^2}{K^2[L^2(U+V)^2+Q^2U^2]m^2+2KLQ^2UVm+L^2[K^2(U+V)^2+Q^2V^2]}$$

For $m_1=\frac{L}{K}$ we have that $k_1=k(m_1)=0$ and for $m_2=-\frac{L}{K}\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}$ we have $k_2=k(m_2)=$

$$-\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}\cdot$$

$$\frac{[(U+V)K^2+VQ^2]^2+[(U+V)K^2+VQ^2][(U+V)L^2+UQ^2]+[(U+V)L^2+UQ^2]^2}{[L^2(U+V)^2+Q^2U^2][(U+V)K^2+VQ^2]^2-2Q^2UV[(U+V)K^2+VQ^2][(U+V)L^2+UQ^2]+[K^2(U+V)^2+Q^2V^2][(U+V)L^2+UQ^2]^2}$$

The total curvature of the surface is $K=k_1k_2=0$.

The mean curvature is also:

$$H=\frac{E\nu-2F\mu+G\lambda}{2(EG-F^2)}=-\frac{KLQP(U+V)\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2(U+V)^2+Q^2[K^2U^2+L^2V^2]}^3}$$

We obtain that the production surface is with null total curvature but it is not minimal in any point.

The line of curvature equation is:

$$(E\mu-F\lambda)\left(\frac{dL}{dK}\right)^2+(E\nu-G\lambda)\left(\frac{dL}{dK}\right)+(F\nu-G\mu)=0$$

Like at upper, we obtain easy that:

$\frac{dL}{dK}=\frac{L}{K}$ that is: $\frac{dL}{L}=\frac{dK}{K}\Rightarrow L=CK$ with $C\in(0,\infty)$

respectively:

$$\frac{dL}{dK}=-\frac{L}{K}\,\frac{(\varepsilon L^\rho+\gamma K^\rho)K^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+(\alpha\varepsilon L^\rho+(1-\beta)\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}{(\varepsilon L^\rho+\gamma K^\rho)L^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+((1-\alpha)\varepsilon L^\rho+\beta\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}$$

The asymptotic directions satisfy:

$\lambda m^2+2\mu m+\nu=0$

that is:

$rm^2+2sm+t=0$

from where:

$-K^2m^2+2KLm-L^2=0$ therefore $m_1=m_2=\frac{L}{K}$.

The asymptotic curves have the equation:

$\frac{dL}{dK}=m$ (with m asymptotic direction) therefore they are: $L=CK$ with $C\in(0,\infty)$.

3. APPLICATIONS FOR THE COBB-DOUGLAS FUNCTION

For the Cobb-Douglas production function, that is for $\alpha+\beta=1$, $\gamma=1$, $\varepsilon=0$, $\rho=1$ we have:

$U=\beta K$

$V=\alpha K$

$U+V=K$

$P=\alpha\beta K^2$

$$m_1=\frac{L}{K},\ m_2=-\frac{L}{K}\frac{K^3+\alpha KA^2K^{2\alpha}L^{2\beta}}{KL^2+\beta KA^2K^{2\alpha}L^{2\beta}}=-\frac{K+\alpha A^2K^{2\alpha-1}L^{2\beta}}{L+\beta A^2K^{2\alpha}L^{2\beta-1}}$$

and denoting with $g=\frac{K}{L}$ the endowment with capital we obtain:

$$m_1=\frac{1}{g},\ m_2=-g\frac{1+\alpha A^2g^{-2\beta}}{1+\beta A^2g^{2\alpha}}.$$

$k_1=0$

$$k_2=-\frac{Q\alpha\beta KL}{\sqrt{K^2L^2+Q^2(\beta^2K^2+\alpha^2L^2)}}\cdot$$

$$\frac{[K^2+\alpha Q^2]^2+[K^2+\alpha Q^2][L^2+\beta Q^2]+[L^2+\beta Q^2]^2}{[L^2+Q^2\beta^2][K^2+\alpha Q^2]^2-2Q^2\alpha\beta[K^2+\alpha Q^2][L^2+\beta Q^2]+[K^2+Q^2\alpha^2][L^2+\beta Q^2]^2}$$

The total curvature of the surface is $K=k_1k_2=0$ and the mean curvature is:

$$H=-\frac{\alpha\beta LQ\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2+Q^2[\beta^2K^2+\alpha^2L^2]}^3}.$$

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