

APPLICATIONS OF THE SPACES OF HOMOGENEOUS POLYNOMIALS TO SOME PROBLEMS ON THE BALL ALGEBRA

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ABSTRACT. Denote by B_2 the unit ball in \mathbb{C}^2 . The existence is shown of a uniformly bounded orthonormal basis in $H^2(B_2)$, by constructing such systems in the spaces of homogeneous polynomials. In the second part of the paper, those spaces of homogeneous polynomials are exploited to disprove the existence of generalized analytic projections, the so-called $(i_p-\pi_p)$ property, for the ball algebra.

Summary. In the first part of the paper we use a construction in the spaces of homogeneous polynomials on the 2-dimensional complex ball B_2 to generate an orthonormal basis for the space $H^2(B_2)$ which is uniformly bounded. The existence of such a system answers a question raised by W. Rudin.¹ The second part is devoted to the failure of the $(i_p-\pi_p)$ theorem, known for the disc algebra $A(D)$, in case of the ball algebras $A(B_m)$, $m > 1$. It is proved that the ideals of p -summing and p -integral operators ($p \neq 2$) in $A(B_2)$ are distinct. This fact solves negatively a problem considered in [2].

1. Introduction and terminology. Denote by $\langle \cdot, \cdot \rangle$ the inner product on \mathbb{C}^2 , by B_2 the closed unit ball of \mathbb{C}^2 , and by $S_2 = \partial B_2 = \{\zeta \in \mathbb{C}^2; \|\zeta\| = \langle \zeta, \zeta \rangle^{1/2} = 1\}$ the unit sphere. Under the parametrization $\zeta = (z, w)$, $z = \sqrt{\rho} e^{i\theta}$, and $w = \sqrt{1-\rho} e^{i\psi}$, the normalized Haar measure σ on S_2 is given by $d\sigma = (1/4\pi^2) d\rho d\theta d\psi$ ($0 \leq \rho \leq 1$, $0 \leq \theta, \psi \leq 2\pi$). For $1 \leq p \leq \infty$, the spaces $L^p(S_2) = L^p(S_2, \sigma)$ are defined in the usual way. The ball algebra $A(B_2)$ consists of continuous functions in B_2 which are analytic in the interior of B_2 and identifies with a subspace of the space $C(S_2)$ of continuous functions on S_2 by restriction to the sphere. The spaces $H^p(B_2)$ ($1 \leq p < \infty$) are obtained as the closure of $A(B_2)$ in $L^p(S_2)$. They are generated by the polynomial spaces $\mathcal{P}_N = \text{span}\{z^j w^{N-j}; j = 0, 1, \dots, N\}$, $N = 0, 1, \dots$, where \mathcal{P}_N consists of the homogeneous polynomials of degree N . A remarkable property is the fact that the orthogonal projection onto \mathcal{P}_N ,

$$P_N f = Pf = c_N^{-1} \int \langle \eta, \zeta \rangle^N f(\zeta) d\sigma(\zeta), \quad c_N = \int |z|^{2N} d\sigma,$$

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is bounded under the L^p -norm ($1 \leq p \leq \infty$) independently of N . One has, indeed,

$$\|P\|_{L^\infty(S_2) \rightarrow L^\infty(S_2)} = c_N^{-1} \int |z|^N d\sigma \leq c \quad (= \text{numerical constant}).$$

For $\zeta, \eta \in S_2$, let $d(\zeta, \eta) = 1 - |\langle \zeta, \eta \rangle|$, which is the natural metric for the projective space $\mathbf{P} = \mathbf{P}^1$. Notice that the modulus of a homogeneous polynomial on S_2 can be seen as a function on \mathbf{P} . We mention [3] as a standard reference for function theory on complex balls.

If X, Y are Banach spaces, $\Pi_p(X, Y)$ and $I_p(X, Y)$ ($1 \leq p < \infty$) denote the ideals of p -summing, respectively strictly p -integral, operators from X to Y . The respective norms are given by

$$\pi_p(u) = \sup \left\{ \left(\sum \|u(x_i)\|^p \right)^{1/p} \mid \{x_i\} \subset X, \sum |\langle x_i, x^* \rangle|^p \leq 1 \right. \\ \left. \text{for } x^* \in X^*, \|x^*\| \leq 1 \right\}$$

and

$$i_p(u) = \inf \|A\| \|B\|,$$

where this infimum is taken over all factorizations $u = B \circ I \circ A$, $I: L^\infty(\Omega, \mu) \rightarrow L^p(\Omega, \mu)$ is the identity map, $A: X \rightarrow L^\infty(\mu)$, and $B: L^p(\mu) \rightarrow Y$.

It is well known that $\pi_p(u) \leq i_p(u)$ and $i_p(u) \leq c p^2 \pi_p(u) / (p - 1)$ ($1 < p < \infty$, $c = \text{numerical}$) whenever u is a p -summing operator on the disc algebra $A(D)$ (the reader will find details on this subject in [2]). The problem of whether or not this so-called $(i_p - \pi_p)$ property holds for ball and polydisc algebras was considered in the last section of [2]. The answer to this question turns out to be negative in both cases. The polydisc case was settled in [1] and we present here the argument for the ball algebra $A(B_2)$, relying on the spaces \mathcal{P}_N defined above. The argument in the general case $A(B_m)$ ($m \geq 2$) is completely similar.

2. Construction of a uniformly bounded orthonormal basis. Since the spaces \mathcal{P}_N are mutually orthogonal in $H^2(B_2)$, it will suffice to construct the basis in each of the spaces \mathcal{P}_N with a uniform bound on the L^∞ -norm of the systems. Our approach is completely explicit. It is clear that the polynomials

$$(1) \quad \zeta_k = (N + 1)^{-1/2} \sum_{j=0}^N \sigma_j e^{2\pi i j k / N + 1} \frac{z^j w^{N-j}}{\|z^j w^{N-j}\|_2} \quad (k = 0, 1, \dots, N)$$

form an orthonormal basis for \mathcal{P}_N , whatever the choice of the signs $\sigma_j = \pm 1$. We claim that if $\{\sigma_j\}_{j=0}^N$ is a transform of Rudin-Shapiro type (that is, coefficients of a Rudin-Shapiro polynomial [4, p. 33 for definition]) or any $N + 1$ consecutive coefficients in some Rudin-Shapiro transform, then $\|\zeta_k\|_\infty \leq c$ ($0 \leq k \leq N$), where the constant c is independent of the degree N . Replacing z by $z \cdot e^{2\pi i k / N + 1}$ and using the parametrization $z = \sqrt{\rho} e^{i\theta}$, $w = \sqrt{1 - \rho} e^{i\psi}$, the problem clearly reduces to an estimation of

$$(2) \quad \sup_{\substack{0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi}} (N + 1)^{1/2} \left| \sum_{j=0}^N \sigma_j \|z^j w^{N-j}\|_2^{-1} \rho^{j/2} (1 - \rho)^{(N-j)/2} e^{ij\theta} \right|.$$

In the sequel c denotes various numerical constants. We make use of the following fact.

LEMMA 1. For $\{\sigma_j\}_{j=0}^N$ an interval of length $N + 1$ in a Rudin-Shapiro transform, $0 \leq a < b \leq N$, and $(\lambda_j)_{a \leq j \leq b}$ a monotone sequence,

$$\left\| \sum_{j=a}^b \lambda_j \sigma_j e^{ij\theta} \right\|_{L^\infty(T)} \leq c \sup |\lambda_j| \cdot (b - a)^{1/2}.$$

PROOF. By an argument of partial summation, we may, of course, take $\lambda_j = 1$ and verify the majorization of $\|\sum_{j=a}^b \alpha_j e^{ij\theta}\|_{L^\infty(T)}$. If $[a, b]$ is a dyadic interval in a Rudin-Shapiro transform, $\{\sigma_j\}_{j=a}^b$ is in fact again such a transform, as a consequence of their construction. Hence, $\|\sum_{j=a}^b \sigma_j e^{ij\theta}\|_{L^\infty(T)} \leq 2(b - a)^{1/2}$ in this case. For a general interval we divide $[a, b]$ into dyadic intervals of decreasing length, and apply the previous estimate and the triangle inequality to get the result. Details are elementary and left to the reader.

To obtain a bound on (2) is a straightforward computation. Using the evaluation

$$\|z^j w^{N-j}\|_2^2 = \int_0^1 \rho^j (1 - \rho)^{N-j} d\rho = \frac{j!(N - j)!}{(N + 1)!}$$

and Stirling's formula, we see that for fixed ρ , $0 \leq \rho \leq 1$,

$$(3) \quad \alpha_j \stackrel{\text{def}}{=} (N + 1)^{-1/2} \|z^j w^{N-j}\|_2^{-1} \rho^{j/2} (1 - \rho)^{N-j/2} \\ = c(N + 1)^{-1/2} \left\{ \int_0^1 \left(\frac{u}{\rho}\right)^j \left(\frac{1-u}{1-\rho}\right)^{N-j} du \right\}^{-1/2}$$

is dominated by an expression of the form

$$(4) \quad C \exp\left[-c(j - \rho N)^2 / N\rho(1 - \rho)\right] \min(j, N - j)^{-1/4}.$$

Moreover, taking a derivative with respect to j in the right member of (3), we verify that $\{\alpha_j\}_{j=0}^N$ increases until $j = j_0 \sim \rho N$; then it decreases. We suppose $0 \leq \rho \leq 1/2$, the other case being symmetric.

Rewrite (2) as

$$(5) \quad \left\| \sum_{j=0}^N \sigma_j \alpha_j e^{ij\theta} \right\|_{L^\infty(T)},$$

which we estimate by applying Lemma 1 and (4). Write

$$\left\| \sum_{j=0}^N \sigma_j \alpha_j e^{ij\theta} \right\|_{L^\infty(T)} \leq \left\| \sum_{j=0}^{j_0} \sigma_j \alpha_j e^{ij\theta} \right\|_\infty + \left\| \sum_{j=j_0+1}^N \sigma_j \alpha_j e^{ij\theta} \right\|_\infty,$$

where thus $\{\alpha_j\}_{j=0}^{j_0}$ is increasing and $\{\alpha_j\}_{j=j_0+1}^N$ decreasing. We handle the first sum and leave the details (essentially similar) for the second sum to the reader. Before using Lemma 1, we need a further splitting of $\sum_{j=0}^{j_0}$. Define inductively

$$k_0 = 0, \quad k_1 = 1, \quad k_{r+1} = k_r + [k_r^{1/2}] \quad ([] = \text{integer part}),$$

and let s be the largest integer for which $k_s \leq j_0$. By Lemma 1

$$\begin{aligned} \left\| \sum_{j=0}^{j_0} \sigma_j \alpha_j e^{ij\theta} \right\|_{\infty} &\leq \sum_{r=0}^{s-1} \left\| \sum_{j=k_r}^{k_{r+1}-1} \sigma_j \alpha_j e^{ij\theta} \right\|_{\infty} + \left\| \sum_{j=k_s}^{j_0} \sigma_j \alpha_j e^{ij\theta} \right\|_{\infty} \\ &\leq c \sum_{r=0}^{s-1} \alpha_{k_{r+1}} (k_{r+1} - k_r)^{1/2} + c \alpha_{j_0} (j_0 - k_s)^{1/2} \\ &\leq c \sum_{r=0}^{s-1} \alpha_{k_{r+1}} k_r^{1/4} + c \alpha_{j_0} j_0^{1/4}, \end{aligned}$$

where, by (4),

$$\alpha_{k_r} \leq c k_r^{-1/4} \exp[-c'(j_0 - k_r)^2/j_0] \quad \text{and} \quad \alpha_{j_0} \leq c.$$

From the construction of the sequence $\{k_r\}_{r=0}^s$, it is clear that

$$j_0 - k_r \geq \frac{1}{4} j_0^{1/2} \min(s - r, j_0^{1/2}).$$

Hence, one ends up with the series $\sum_{r=0}^{s-1} [e^{-c(s-r)^2} + e^{-cj_0}]$, obviously converging.

REMARK. The construction of the ζ_K -polynomials provide an explicit example of N homogeneous polynomials p for which $\|p\|_2 \sim \|p\|_{\infty}$. They can be used to obtain singular positive RP-measures μ on S_2 and, hence, inner functions on the ball B_2 (see [3] for details). Denote by K the Cauchy kernel of B_2 and define the operator T acting on $C^\infty(S_2)$ into $C^\infty(S_2) \cap \text{Re } A(B_2)$ by

$$Tf = 2 \text{Re } K[f] - \text{Re } \int f.$$

Clearly $T^2 = T$. Define inductively

$$f_0 = 1, \quad f_{j+1} = T[f_j(1 + \text{Re } p_{j+1})] + \epsilon_{j+1},$$

where f_{j+1} is a homogeneous polynomial of degree N_{j+1} chosen large enough and $\|p_{j+1}\|_{\infty} \leq 1, \|p_{j+1}\|_2 > \delta = \text{constant}$. For ϵ_{j+1} we take the smallest positive number to insure the positivity of f_{j+1} . It is not hard to see that the N_j 's and p_j 's can be chosen inductively such that $\sum \epsilon_j < \infty$ and each w^* -limit point of the sequence $\{f_j d\sigma\}$ in $M(S)$ yields a singular measure. Let us give some further details. By taking $N_{j+1} = \text{degree}(p_{j+1})$ large enough, we can ensure that f_{j+1} is an arbitrarily small perturbation of $f_j(1 + \text{Re } p_{j+1})$ in the uniform norm. The point is that $f_j \cdot p_{j+1}$ is "almost" in $A(B)$. To ensure that w^* -limit points of $\{f_j d\sigma\}$ are singular measures, we are led to force the Riesz product $\prod(1 + \text{Re } p_{j+1})$ to determine a singular measure. Assume p_1, \dots, p_j is obtained. After choice of $N = N_{j+1}$, consider the bounded orthonormal basis ξ in \mathcal{P}_N , which we introduced above. Since ξ is orthonormal, an invariance argument with respect to the unitary group acting on S shows that

$$(1) \quad \sum_{p \in \xi \cup (i\xi)} (\text{Re } p(\zeta))^2 = \sum_{p \in \xi} |p(\zeta)|^2 = |\xi| = N + 1$$

pointwise for $\zeta \in S$. Since, for $|x| \leq 1$,

$$(1 + x)^{1/2} \leq 1 + \frac{1}{2}x - cx^2 \quad (c > 0),$$

we find by (1) (for $\delta = \frac{1}{2}\|p\|_\infty^{-1}$),

$$\begin{aligned} & \frac{1}{2(N+1)} \sum_{p \in \xi_0} \int f_j^{1/2} (1 + \delta \operatorname{Re} p)^{1/2} \\ & \leq \int f_j^{1/2} + \frac{\delta}{4(N+1)} \sum_{p \in \xi_0} \int f_j^{1/2} \operatorname{Re} p - c' \int f_j^{1/2}, \end{aligned}$$

where $\xi_0 = \xi \cup (i\xi)$. For N large enough, the second term on the right side will essentially vanish. So, for some $p \in \xi \cup (i\xi)$ and $p_{j+1} = \delta p$, it follows that

$$\int f_{j+1}^{1/2} d\sigma \leq (1 - c) \int f_j^{1/2} d\sigma.$$

3. Absolutely summing operators. If μ is a positive Radon measure on S_2 and $1 \leq p < \infty$, define $H^p(\mu)$ as the closure of the space $A(B_2)$ in $L^p(\mu)$. The identity map $A(B) \rightarrow H^p(\mu)$ is a trivial example of a p -summing operator on the ball algebra. It turns out to be possible to find μ , $d\mu/d\sigma = \Delta \in L^1(S)$, such that the latter operator is not p -integral whenever $p \neq 2$ and does not extend to $C(S)$ for $p > 2$. The approach is virtually the same as the method described in [1] to disprove the $(i_p\text{-}\pi_p)$ property for the bidisc algebra $A(D^2)$. The following general lemma (see [1, Lemma 6.1]) will be used again.

LEMMA 2. *Assume X is a linear subspace of a $C(K)$ -space (K compact) satisfying the $(i_p\text{-}\pi_p)$ property for a fixed p , $1 \leq p < \infty$, i.e. the ratio $k_p(x) = \sup\{i_p(u) \mid u \text{ is a } p\text{-summing operator on } X \text{ with } \pi_p(u) \leq 1\}$ is assumed finite. Further, let $\epsilon > 0$ and let $\{\phi_j\}_{j=1}^n$ be a finite sequence in the unit ball of X such that the sets $[|\phi_j| > \epsilon] = \{t \in K \mid |\phi_j(t)| > \epsilon\}$ are mutually disjoint. Then there exists a decomposition in X , $\phi_j = \phi'_j + \phi''_j$ with $\phi'_j, \phi''_j \in X$, fulfilling the conditions*

(i)
$$\|\phi'_j\|_\infty \leq 3k_p(X) n^{\max(2,p)^{-1}} \epsilon \quad (1 \leq j \leq n),$$

(ii)
$$\left\| \sum_{j=1}^n |\phi''_j| \right\|_\infty \leq 3k_p(X) n^{\max(2,p')^{-1}}.$$

Of course, here $K = S_2$, and $X = A(B_2)$, considered as a subspace of $C(S_2)$. Notice that if $\phi_j \in \mathcal{P}_{N_j}$ for some integer N_j ($1 \leq j \leq n$) the existence of a decomposition for $\{\phi_j\}_{j=1}^n$ in $A(B)$ implies a decomposition with $\phi'_j, \phi''_j \in \mathcal{P}_{N_j}$ again for each j . This fact is clear upon using the projection operators

$$R_{Nj}f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}z, e^{i\theta}w) e^{-iN\theta} d\theta, \quad \zeta = (z, w),$$

from $A(B)$ onto \mathcal{P}_{N_j} .

LEMMA 3. *There exists a sequence $\{p_j\}_{j=1}^n$ of homogeneous polynomials on B_2 , $d(p_j) = \text{degree}(p_j) = N_j$, for which the sets $[|p_j| > \epsilon]$ are disjoint and $\|\sum_{j=1}^n |q_j|\|_\infty > c n \epsilon$ whenever $\{q_j\}_{j=1}^n$ are homogeneous polynomials, $d(q_j) \leq N_j$ and $\|p_j - q_j\|_\infty < c$ ($1 \leq j \leq n$) (where $c > 0$ is numerical).*

To see that this gives the desired result, take $\varepsilon = (c/4)k_p(X)^{-1}n^{-\max(2,p)^{-1}}$ in Lemma 2. Assuming $k_p(X) < \infty$ for $X = A(B_2)$ gives, by Lemma 3 ($\phi_j'' = q_j$),

$$cn\varepsilon < 3k_p(X)n^{\max(2,p')^{-1}}, \quad ck_p(X)^2 > n^{1/2-\max(p,p')^{-1}},$$

which leads to a contradiction by letting $n \rightarrow \infty$ if $p \neq 2$.

LEMMA 4. *Let $1 = (1, 0)$ and $V_\delta = \{\zeta \in S_2 \mid |\langle \zeta, 1 \rangle| > 1 - \delta\}$. Then the inequality*

$$\sup\{|f(\zeta)| \mid \zeta \in S \setminus V_\delta\} \geq c(1 - \delta)^N |f(1)|$$

holds whenever $f \in \mathcal{P}_N$.

PROOF. For $f \in L^\infty(S)$ the inequality

$$\|f\|_{L^\infty(S \setminus V_\delta)} \geq cN(1 - \delta)^{-N} \left| \int_S f(\eta) \langle l, \eta \rangle^N \chi_{\{|\eta_1| \leq 1 - \delta\}} d\eta \right|$$

clearly holds. On the other hand, for $f \in \mathcal{P}_N, f = P_N f$,

$$f(\eta) = c_N^{-1} \int f(\zeta) \langle \eta, \zeta \rangle^N d\zeta, \quad c_N \sim \frac{1}{N}.$$

Substitution yields

$$\|f\|_{L^\infty(S \setminus V_\delta)} \geq cN^2(1 - \delta)^{-N} \left| \iint f(\zeta) \langle \eta, \zeta \rangle^N \langle 1, \eta \rangle^N \chi_{\{|\eta_1| \leq 1 - \delta\}} d\eta d\zeta \right|.$$

Performing the integration in η gives

$$\int \langle \eta, \zeta \rangle^N \langle 1, \eta \rangle^N \chi_{\{|\eta_1| \leq 1 - \delta\}} d\eta = \alpha(1 - \delta)^{2N} \zeta_1^N \quad \text{with } \alpha \sim \frac{1}{N}.$$

So

$$\|f\|_{L^\infty(S \setminus V_\delta)} \geq cN(1 - \delta)^N \left| \int f(\zeta) \zeta_1^N \right| \geq c'(1 - \delta)^N |f(1)|.$$

The proof of Lemma 3 is now very much in the spirit of Lemma 6.3 in [1]. For $a \in S_2$ and $\alpha > 0$, let $B(a, \alpha) = \{\zeta \in S_2 \mid d(a, \zeta) < \alpha\}$, where $d(\zeta, \eta) = 1 - |\langle \zeta, \eta \rangle|$ is the distance in projective space. Since \mathcal{P}_N and d are invariant under the unitary group acting on S_2 , Lemma 4 also implies

$$\|f\|_{L^\infty(S \setminus B(a, \alpha))} \geq c(1 - \alpha)^N |f(a)| \quad \text{if } f \in \mathcal{P}_N.$$

Moreover, this inequality is, up to a constant, best possible, considering polynomial $f(\zeta) = \langle \zeta, a \rangle^N$.

PROOF OF LEMMA 3. Fix n and $\varepsilon > 0$. We introduce inductively positive numbers and positive integers $1 = \delta_1 > \varepsilon_1 > \delta_2 > \varepsilon_2 > \dots > \delta_j > \varepsilon_j > \dots > \delta_n > \varepsilon_n$ and $1 = N_1 < N_2 < \dots < N_j < \dots < N_n$. For each j , ξ_j will be a δ_j -net in the set $R_j = S \setminus \bigcup_{s=1}^{j-1} \bigcup_{a \in \xi_s} B(a, \varepsilon_s)$, and we assume $B(a, \frac{1}{10}\delta_j) \subset R_j$ for all $a \in \xi_j$. We take $\xi_1 = \{1\}$. The polynomials $\{p_j\}_{j=1}^n$ are given by

$$p_j(\zeta) = \sum_{a \in \xi_j} \langle \zeta, a \rangle^{N_j},$$

where N_j is chosen large enough to imply $\|p_j\|_\infty \approx 1$ and

$$\{|p_j| > \varepsilon\} \subset \bigcup_{a \in \xi_j} B(a, \varepsilon_j),$$

where ϵ_j is defined by $(1 - \epsilon_j)^{N_j} = \epsilon/2$. In the construction, N_j depends on δ_j . For N_j sufficiently large, ϵ_j becomes arbitrarily small, in particular, $\epsilon_j < \frac{1}{10}\delta_j$. Hence $\bigcup_{a \in \xi_j} B(a, \epsilon_j) \subset R_j$ and, consequently,

$$\left[|p_j| > \epsilon \right] \cap \left[|p_s| > \epsilon \right] = \emptyset \quad \text{for } s = 1, \dots, j - 1.$$

Consider now a sequence $\{g_j\}_{j=1}^n$ s. t. $g_j \in \mathcal{P}_{N_j}$ and $\|p_j - g_j\|_\infty < \frac{1}{10}$ for $j = 1, \dots, n$. We prove a minorization of $\|\sum |g_j|\|_\infty$ using a simple reasoning of successive perturbations based on Lemma 4.

Proceeding by induction, we define inductively points $x_j \in S$ so that $x_j \in \xi_{j+1}$ and $|q_s(x_j)| > c\epsilon$ for $s = 1, \dots, j$. Let us describe the inductive step. Since $x_j \in \xi_{j+1}$, we have $p_{j+1}(x_j) \approx 1$, hence $|q_{j+1}(x_j)| > \frac{4}{5}$. Define

$$h(\zeta) = \langle \zeta, x_j \rangle^L q_{j+1}(\zeta) \quad (L \text{ to be specified}),$$

which is of degree $L + N_{j+1}$. Since $|h(x_j)| > \frac{4}{5}$ still, Lemma 4 yields a point $x \in S$ satisfying $d(x_j, x) \geq \epsilon_{j+1}$ and $|h(x)| > c(1 - \epsilon_{j+1})^{N_{j+1}+L}$. Thus if $L = o(N_{j+1})$, we get, by definition of ϵ_{j+1} ,

$$|q_{j+1}(x)| \geq |h(x)| \geq c'\epsilon.$$

Since also $|\langle x, x_j \rangle|^L > c'\epsilon$, it will follow that $d(x, x_j) = o(\delta_{j+1})$ provided L is sufficiently large (compatible with $L = o(N_{j+1})$). Again, by construction, since $B(x_j, \frac{1}{10}\delta_{j+1}) \subset R_{j+1}$, we get $x \in R_{j+1}$. Moreover, $x \notin B(x_j, \epsilon_{j+1})$ and, since the elements of ξ_{j+1} are δ_{j+1} -separated and $d(x, x_j) = o(\delta_{j+1})$, also $x \notin \bigcup_{a \in \xi_{j+1}} B(a, \epsilon_{j+1})$. Hence, $x \in R_{j+2}$ and $x_{j+1} \in \xi_{j+2}$ can be chosen with $d(x, x_{j+1}) < 2\delta_{j+2}$. The fact that δ_{j+2} is small enough with respect to $N_{j+1} = d(q_{j+1})$ implies $|q_{j+1}(x_{j+1})| \geq c'\epsilon/2$. Further, $d(x_j, x_{j+1}) = o(\delta_{j+1})$, and the inequalities $|q_s(x_{j+1})| > c\epsilon$, $1 \leq s \leq j$, are therefore essentially preserved. This completes the argument.

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