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APPLICATIONS OF VARIATIONAL METHODS TO BOUNDARY-VALUE PROBLEM FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract In this paper, we investigate the existence of positive solutions to a second-order Sturm–Liouville boundary-value problem with impulsive effects. The ideas involve differential inequalities and variational methods.

Keywords: Sturm–Liouville boundary-value problem; impulsive effects; variational methods; mountain-pass theorem

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1. Introduction

In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary-value problems, by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physics phenomena are described. For relevant and recent references on impulsive differential equations, we refer the reader to [**12**, **19–21**, **25**, **26**]. For the background and applications of the theory of impulsive differential equations to different areas, we refer the reader to [**5**, **7**, **10**, **13**, **17**, **18**, **28**, **31**, **32**, **34**, **35**].

Some classical tools have been used to study impulsive differential equations in the literature. These classical tools include fixed-point theorems in cones [**1**, **9**, **11**, **14**] and the method of lower and upper solutions with monotone iterative technique (see [**15**]).

On the other hand, in the last few years, many researchers have used variational methods to study the existence of solutions for boundary-value problems [**4**,**23**,**24**,**29**,**30**]. Variational methods have become a powerful tool. For related basic information, we refer the reader to [**16**, **22**].

However, to the best of our knowledge, few authors have studied the existence of positive solutions for impulsive boundary-value problems by using variational methods. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above facts, in this paper, we study the existence of multiple positive solutions to the Sturm–Liouville boundary-value problem for the second-order impulsive differential equations

$$\left.\begin{aligned} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) &= f(t,x(t)), \quad t \neq t_i, \text{ a.e. } t \in [a,b],\\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) &= I_i(x(t_i)), \quad i = 1,2,\dots,l,\\ \alpha x'(a) - \beta x(a) = A, &\qquad \gamma x'(b) + \sigma x(b) = B, \end{aligned}\right\} \tag{1.1}$$

where $p > 1$, $\Phi_p(x) := |x|^{p-2}x$, $\rho, s \in L^\infty[a,b]$ with $\operatorname{ess\,inf}_{[a,b]}\rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $0 < \rho(a), \rho(b) < \infty$, $A \leqslant 0, B \geqslant 0$, $\alpha, \beta, \gamma, \sigma > 0$, $a = t_0 < t_1 < \cdots < t_l < t_{l+1} = b$, $\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \rho(t_i^+)\Phi_p(x'(t_i^+)) - \rho(t_i^-)\Phi_p(x'(t_i^-))$, where $x'(t_i^+)$ and $x'(t_i^-)$ denote the right and left limits, respectively, of $x'(t)$ at $t = t_i$, $I_i \in C([0,+\infty),[0,+\infty))$, $i = 1,2,\dots,l$, $f \in C([a,b] \times [0,+\infty),[0,+\infty))$, $f(t,0) \not\equiv 0$ for $t \in [a,b]$.

Our aim is to apply critical-point theory to problem (1.1) and prove the existence of at least two positive solutions. With the impulse effects and the Sturm–Liouville boundary conditions taken into consideration, difficulties such as how to construct suitable functional φ and how to prove that the critical points of φ are just the solutions of problem (1.1) must be overcome. In addition, this paper is a generalization of [**2**, **3**, **6**, **8**, **30**], in which impulse effects are not involved. Moreover, the conditions on f and I_i, $i = 1,2,\dots,l$, are easily verified.

The following lemmas will be needed in our argument, which can be found in [**9**,**16**,**33**].

Lemma 1.1 (Zeidler [33, Theorem 38.A]). *For the functional $F : M \subseteq X \to [-\infty,+\infty]$ with $M \neq \emptyset$, $\min_{u\in M} F(u) = \alpha$ has a solution for which the following hold:*

(i) *X is a real reflexive Banach space;*

(ii) *M is bounded and weak sequentially closed;*

(iii) *F is weakly sequentially lower semi-continuous on M, i.e. by definition, for each sequence (u_n) in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leqslant \underline{\lim}_{n\to\infty} F(u_n)$ holds.*

Lemma 1.2 (Mawhin and Willem [16, Theorem 4.10]). *Let E be a Banach space and let $\varphi \in C^1(E,R)$. Assume that there exist $x_0 \in E$, $x_1 \in E$ and a bounded open neighbourhood Ω of x_0 such that $x_1 \in E \setminus \bar{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \in C([0,1],E) : h(0) = x_0,\ h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

If φ satisfies the Palais–Smale $(PS)_c$-condition, i.e. the existence of a sequence (x_k) in E such that $\varphi(x_k) \to c$ and $\varphi'(x_k) \to 0$ as $k \to \infty$ implies that c is a critical value of φ, then c is a critical value of φ and $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

Lemma 1.3 (Guo [9]). *Let E be a Banach space and let $\varphi \in C^1(E, R)$ satisfy the Palais–Smale condition, i.e. every sequence $\{x_n\}$ in E satisfying $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$ has a convergent subsequence. Assume there exist $x_0, x_1 \in E$ and a bounded open neighbourhood Ω of x_0 such that $x_1 \in E \setminus \bar{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \mid h : [0,1] \to E \text{ is continuous and } h(0) = x_0,\ h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} \varphi(h(s)).$$

Then c is a critical value of φ, that is, there exists $x^* \in E$ such that $\varphi'(x^*) = \Theta$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

Proof. By Lemma 1.2, we need only to show that the (PS)-condition implies the $(\text{PS})_c$-condition for each $c \in R$. By the (PS)-condition, every sequence $\{x_n\}$ in E satisfying $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$ has a convergent subsequence; without loss of generality, we assume $(x_{n_k}) \to x_0$ as $k \to \infty$. Since φ is a continuous functional, $\varphi(x(n_k)) \to \varphi(x_0)$. Let $c = \varphi(x_0)$. Clearly, $\varphi'(x_{n_k}) \to 0 = \varphi'(x_0)$ since $\varphi \in C^1(E, R)$. So c is a critical value of φ, and φ satisfies the $(\text{PS})_c$-condition. The proof is complete. □

In this paper, we will need the following conditions.

(C1) There exist $\mu > p$, $h \in C([a,b] \times [0,+\infty), [0,+\infty))$, $g \in C([0,+\infty), [0,+\infty))$, $r \in C([a,b], [0,+\infty)), \eta > 0$ and

$$\int_a^b r(s)\,\mathrm{d}s + \eta > 0,$$

such that

$$f(t,x) = r(t)\Phi_\mu(x) + h(t,x), \qquad I_i(x) = \eta\Phi_\mu(x) + g(x).$$

(C2) There exist $c \in L^1([a,b], [0,+\infty))$, $d \in C([a,b], [0,+\infty))$, $\xi \geqslant 0$, such that

$$h(t,x) \leqslant c(t) + d(t)\Phi_p(x), \qquad g(x) \leqslant \xi\Phi_p(x).$$

The remainder of the paper is organized as follows. In § 2, some preliminary results will be given. In § 3, we will state and prove the main results of the paper, as well as some applications to (1.1).

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2. Related lemmas

To begin with, we introduce some notation. Henceforth, we assume that $[a,b]$ is a compact real interval. Define the space $X = W^{1,p}([a,b])$ equipped with the norm

$$\|x\|_X = \left(\int_a^b \rho(t)|x'(t)|^p + s(t)|x(t)|^p \,\mathrm{d}t\right)^{1/p},$$

the norm in $W^{1,p}([a,b])$ is equivalent to the usual norm. Hence, X is reflexive. F is the real function

$$F(t,\xi) = \int_0^\xi f(t,x)\,\mathrm{d}x.$$

We define the norm in $C([a,b])$ as $\|x\|_\infty = \max_{x\in[a,b]} |x(t)|$.

Definition 2.1. A function

$$x \in Z = \{x \in X : \rho\Phi_p(x')(\cdot) \in W^{1,\infty}([a,b] \setminus \{t_1,t_2,\dots,t_l\})\}$$

is said to be a classical solution of problem (1.1) if x satisfies the equation in (1.1) for a.e. $t \in [a,b] \setminus \{t_1,t_2,\dots,t_l\}$ and the impulsive condition and boundary condition of (1.1) hold. Moreover, x is said to be a positive classical solution of problem (1.1) if $x(t) \geqslant 0, x(t) \not\equiv 0, t \in [a,b]$.

Lemma 2.2. *For $x \in X$, let $x^\pm = \max\{\pm x, 0\}$. Then the following six properties hold:*

(i) $x \in X \Rightarrow x^+, x^- \in X$;

(ii) $x = x^+ - x^-$;

(iii) $\|x^+\|_X \leqslant \|x\|_X$;

(iv) *if (x_n) uniformly converges to x in $C([a,b])$, then (x_n^+) uniformly converges to x^+ in $C([a,b])$;*

(v) $x^+(t)x^-(t) = 0$, $(x^+)'(t)(x^-)'(t) = 0$ *for a.e.* $t \in [a,b]$;

(vi) $\Phi_p(x)x^+ = |x^+|^p$, $\Phi_p(x)x^- = -|x^-|^p$.

Proof. It is easy to show that properties (i)–(iv) and (vi) hold.

Now we will show that (v) holds. Since $x \in W^{1,p}([a,b])$, there exists a subset $S \subset [a,b]$ with meas $S = 0$ (i.e. the measure of S is equal to 0), such that $x'(t)$ exists on $[a,b] \setminus S$.

Let $K = [a,b] \setminus S$,

$$K_+ = \{t \in K : x(t) \geqslant 0\}, \qquad K_- = \{t \in K : x(t) < 0\},$$
$$K_1 = \{t \in K : x(t) = 0,\ x'(t) = 0\}, \qquad K_2 = \{t \in K : x(t) = 0,\ x'(t) \neq 0\}.$$

Clearly, $(x^+)'(t)(x^-)'(t) = 0$ if $t \in K_+ \cup K_- \cup K_1$. Now we prove meas $K_2 = 0$. Otherwise, there is a closed interval $J \subseteq K_2$ such that $x(t) = 0$, $x'(t) \neq 0$ for each $t \in J$ with meas $J > 0$. Then we have

$$x'(t) = 0 \quad \text{for } t \in \operatorname{Int} J,$$

which is a contradiction. So (v) holds. □

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Lemma 2.3. *If $x \in C([a,b])$ is a classical solution of problem*

$$\left.\begin{aligned} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x^+(t)), \quad & t \neq t_i,\ \text{a.e. } t \in [a,b], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = I_i(x^+(t_i)), \quad & i = 1,2,\dots,l, \\ \alpha x'(a) - \beta x(a) = A, \qquad \gamma x'(b) + \sigma x(b) = B, & \end{aligned}\right\} \tag{2.1}$$

then $x(t) \geqslant 0$, $x(t) \not\equiv 0$, $t \in [a,b]$, and hence it is a positive classical solution of problem (1.1).

Proof. If $x \in C([a,b])$ is a classical solution of problem (2.1), by Lemma 2.2 we have

$$\begin{aligned} 0 &= \int_a^b [(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t, x^+(t))] \times x^-(t)\,\mathrm{d}t \\ &= \sum_{i=0}^{l} \rho(t)\Phi_p(x'(t))x^-(t)\Big|_{t=t_i^+}^{t_{i+1}} \\ &\quad - \int_a^b [\rho(t)\Phi_p(x'(t))(x^-)'(t) + s(t)\Phi_p(x(t))x^-(t)]\,\mathrm{d}t + \int_a^b f(t,x^+(t))x^-(t)\,\mathrm{d}t \\ &= -\sum_{i=1}^{l} \Delta(\rho(t_i)\Phi_p(x'(t_i)))x^-(t_i) \\ &\quad - \rho(a)\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)x^-(a) + \rho(b)\Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)x^-(b) \\ &\quad + \int_a^b \rho(t)|(x^-)'(t)|^p + s(t)|x^-(t)|^p\,\mathrm{d}t + \int_a^b f(t,x^+(t))x^-(t)\,\mathrm{d}t \\ &\geqslant \sum_{i=1}^{l} I_i(x^+(t_i))x^-(t_i) + \rho(b)\left|\frac{B-\sigma x(b)}{\gamma}\right|^{p-2}\frac{Bx^-(b)+\sigma(x^-(b))^2}{\gamma} \\ &\quad + \rho(a)\left|\frac{A+\beta x(a)}{\alpha}\right|^{p-2}\frac{-Ax^-(a)+\beta(x^-(a))^2}{\alpha} + \|x^-\|_X^p \\ &\geqslant \|x^-\|_X^p, \end{aligned} \tag{2.2}$$

so $x^-(t) = 0$ for $t \in [a,b]$, that is $x(t) \geqslant 0$ for $t \in [a,b]$. If $x(t) \equiv 0$ for $t \in [a,b]$, the fact that $f(t,0) \not\equiv 0$ for $t \in [a,b]$ gives a contradiction. □

Remark 2.4. By Lemma 2.3, in order to find the positive classical solutions of problem (1.1) it suffices to obtain classical solutions of (2.1).

For each $x \in X$, set

$$\begin{aligned} \varphi(x) := \frac{\|x\|_X^p}{p} + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma x(b)}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta x(a)}{\alpha}\right|^p \\ - \int_a^b [F(t,x^+(t)) - f(t,0)x^-(t)]\,\mathrm{d}t - \sum_{i=1}^{l}\left[\int_0^{x^+(t_i)} I_i(s)\,\mathrm{d}s - I_i(0)x^-(t_i)\right]. \end{aligned} \tag{2.3}$$

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Clearly, φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\varphi'(x) \in X^*$ given by

$$\begin{aligned}\langle \varphi'(x), v\rangle = \int_a^b [\rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t)]\,\mathrm{d}t \\ - \rho(b)\Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)v(a) \\ - \int_a^b f(t, x^+(t))v(t)\,\mathrm{d}t - \sum_{i=1}^{l} I_i(x^+(t_i))v(t_i) \end{aligned} \tag{2.4}$$

for every $v \in X$. Obviously, $\varphi' : X \to X^*$ is continuous.

Lemma 2.5. *If the function $x \in X$ is a critical point of the functional φ, then x is a solution of problem (2.1).*

Proof. Let $x \in X$ be a critical point of the functional φ. Then $\langle \varphi'(x), v\rangle = 0$. By integrating (2.4), one has

$$\begin{aligned}
&\int_a^b [\rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t)]\,\mathrm{d}t - \rho(b)\Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)v(b) \\
&\quad + \rho(a)\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)v(a) - \int_a^b f(t, x^+(t))v(t)\,\mathrm{d}t - \sum_{i=1}^{l} I_i(x^+(t_i))v(t_i) \\
&= \sum_{i=0}^{l} \rho(t)\Phi_p(x'(t))v(t)\Big|_{t=t_i^+}^{t_{i+1}} - \int_a^b [(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t, x^+(t))]v(t)\,\mathrm{d}t \\
&\quad - \rho(b)\Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)v(a) - \sum_{i=1}^{l} I_i(x^+(t_i))v(t_i) \\
&= -\sum_{i=1}^{l}[\Delta(\rho(t_i)\Phi_p(x'(t_i))) + I_i(x^+(t_i))]v(t_i) + \rho(b)\Phi_p(x'(b))v(b) - \rho(a)\Phi_p(x'(a))v(a) \\
&\quad - \int_a^b [(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t, x^+(t))]v(t)\,\mathrm{d}t \\
&\quad - \rho(b)\Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)v(a) \\
&= -\sum_{i=1}^{l}[\Delta(\rho(t_i)\Phi_p(0x'(t_i))) + I_i(1x^+(t_i))]v(t_i) \\
&\quad + \rho(b)\left[\Phi_p(2x'(b)) - \Phi_p\left(3\frac{B-\sigma x(b)}{\gamma}\right)\right]v(b) \\
&\quad + \rho(a)\left[-\Phi_p(4x'(a)) + \Phi_p\left(5\frac{A+\beta x(a)}{\alpha}\right)\right]v(a) \\
&\quad - \int_a^b [(6\rho(t)\Phi_p(7x'(t)))' - s(t)\Phi_p(8x(t)) + f(9t, x^+(t))]v(t)\,\mathrm{d}t.
\end{aligned} \tag{2.5}$$

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Thus,

$$
\begin{aligned}
-\sum_{i=1}^{l}[\Delta(\rho(t_i)\Phi_p(x'(t_i))) + I_i(x^+(t_i))]v(t_i) \\
+ \rho(b)\biggl[\Phi_p(x'(b)) - \Phi_p\biggl(\frac{B-\sigma x(b)}{\gamma}\biggr)\biggr]v(b) \\
+ \rho(a)\biggl[-\Phi_p(x'(a)) + \Phi_p\biggl(\frac{A+\beta x(a)}{\alpha}\biggr)\biggr]v(a) \\
- \int_a^b[(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t,x^+(t))]v(t)\,\mathrm{d}t = 0 \qquad (2.6)
\end{aligned}
$$

holds for all $v \in X$. Without loss of generality, we assume that $v \in C_0^\infty(t_i, t_{i+1}), v(t) \equiv 0$, $t \in [a, t_i] \cup [t_{i+1}, b]$. Then, substituting it into (2.6), we get

$$
(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t,x^+(t)) = 0 \quad \text{a.e. } t \in (t_i, t_{i+1}).
$$

Thus, x satisfies the equation in (2.1). So, by (2.6),

$$
\begin{aligned}
\sum_{i=1}^{l}[\Delta(\rho(t_i)\Phi_p(x'(t_i))) + I_i(x^+(t_i))]v(t_i) \\
+ \rho(b)\biggl[\Phi_p(x'(b)) - \Phi_p\biggl(\frac{B-\sigma x(b)}{\gamma}\biggr)\biggr]v(b) \\
+ \rho(a)\biggl[-\Phi_p(x'(a)) + \Phi_p\biggl(\frac{A+\beta x(a)}{\alpha}\biggr)\biggr]v(a) = 0, \qquad (2.7)
\end{aligned}
$$

holds for all $v \in X$. Next we shall show that x satisfies the impulsive condition in (2.1). If not, without loss of generality, we assume that there exists $i \in \{1, 2, \dots, l\}$ such that

$$
\Delta(\rho(t_i)\Phi_p(x'(t_i))) + I_i(x^+(t_i)) \neq 0. \qquad (2.8)
$$

Let

$$
v(t) = \prod_{j=0,\, j\neq i}^{l+1}(t - t_j).
$$

Then

$$
\begin{aligned}
-\sum_{k=1}^{l}[\Delta(\rho(t_k)\Phi_p(x'(t_k))) + I_k(x^+(t_k))]v(t_k) \\
+ \rho(b)\biggl[\Phi_p(x'(b)) - \Phi_p\biggl(\frac{B-\sigma x(b)}{\gamma}\biggr)\biggr]v(b) \\
+ \rho(a)\biggl[-\Phi_p(x'(a)) + \Phi_p\biggl(\frac{A+\beta x(a)}{\alpha}\biggr)\biggr]v(a)
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{k=1}^{l}[\Delta(\rho(t_k)\Phi_p(x'(t_k))) + I_k(x^+(t_k))] \prod_{j=0,\, j\neq i}^{l+1}(t_k - t_j) \\
&\qquad + \rho(b)\biggl[\Phi_p(x'(b)) - \Phi_p\biggl(\frac{B-\sigma x(b)}{\gamma}\biggr)\biggr] \prod_{j=0,\, j\neq i}^{l+1}(t_{l+1} - t_j) \\
&\qquad + \rho(a)\biggl[-\Phi_p(x'(a)) + \Phi_p\biggl(\frac{A+\beta x(a)}{\alpha}\biggr)\biggr] \prod_{j=0,\, j\neq i}^{l+1}(t_0 - t_j) \\
&= -[\Delta(\rho(t_i)\Phi_p(x'(t_i))) + I_i(x^+(t_i))] \prod_{j=0,\, j\neq i}^{l+1}(t_i - t_j) \neq 0, \qquad (2.9)
\end{aligned}
$$

which contradicts (2.7). So x satisfies the impulsive condition in (2.1). Similarly, x satisfies the boundary condition. Therefore, x is a solution of problem (2.1). □

Lemma 2.6. *For $x \in X$, we then have $\|x\|_\infty \leqslant \bar{\gamma}\|x\|_X$, where*

$$
\bar{\gamma} = 2^{1/q} \times \max\biggl\{\frac{1}{(b-a)^{1/p}(\operatorname{ess\,inf}_{[a,b]} s)^{1/p}}, \frac{(b-a)^{1/q}}{(\operatorname{ess\,inf}_{[a,b]} \rho)^{1/p}}\biggr\}, \quad \frac{1}{p} + \frac{1}{q} = 1.
$$

Proof. For $x \in X$, it follows from the mean-value theorem that

$$
x(\tau) = \frac{1}{b-a}\int_a^b x(\theta)\,\mathrm{d}\theta
$$

for some $\tau \in [a,b]$. Hence, for $t \in [a,b]$, using Hölder's inequality,

$$
\begin{aligned}
|x(t)| &= \biggl|x(\tau) + \int_\tau^t x'(\theta)\,\mathrm{d}\theta\biggr| \\
&\leqslant \frac{1}{b-a}\int_a^b |x(\theta)|\,\mathrm{d}\theta + \int_a^b |x'(\theta)|\,\mathrm{d}\theta \\
&\leqslant (b-a)^{-1/p}\biggl(\int_a^b |x(\theta)|^p\,\mathrm{d}\theta\biggr)^{1/p} + (b-a)^{1/q}\biggl(\int_a^b |x'(\theta)|^p\,\mathrm{d}\theta\biggr)^{1/p} \\
&\leqslant \frac{1}{(b-a)^{1/p}(\operatorname{ess\,inf}_{[a,b]} s)^{1/p}}\biggl(\int_a^b s(\theta)|x(\theta)|^p\,\mathrm{d}\theta\biggr)^{1/p} \\
&\qquad + \frac{(b-a)^{1/q}}{(\operatorname{ess\,inf}_{[a,b]} \rho)^{1/p}}\biggl(\int_a^b \rho(\theta)|x'(\theta)|^p\,\mathrm{d}\theta\biggr)^{1/p} \\
&\leqslant 2^{1/q} \times \max\biggl\{\frac{1}{(b-a)^{1/p}(\operatorname{ess\,inf}_{[a,b]} s)^{1/p}}, \frac{(b-a)^{1/q}}{(\operatorname{ess\,inf}_{[a,b]} \rho)^{1/p}}\biggr\}\|x\|_X,
\end{aligned}
$$

which completes the proof. □

Lemma 2.7. *Suppose that (C1) and (C2) hold. Furthermore, we assume the following.*

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(C3) *We have*

$$\mu - p > \frac{\mu\|d\|_\infty}{\text{ess inf}_{[a,b]} s} + \mu b l \bar{\gamma}^p,$$

where $\bar{\gamma}$ is defined in Lemma 2.6.

Then the functional φ satisfies the Palais–Smale condition.

Proof. First we prove that (x_n) is a bounded sequence in X. By Lemma 2.2 (vi) and (2.4) we have

$$\begin{aligned}
\langle \varphi'(x_n), x_n^- \rangle &= \int_a^b [\rho(t)\Phi_p(x_n'(t))(x_n^-)'(t) + s(t)\Phi_p(x_n(t))x_n^-(t) - f(t, x_n^+(t))x_n^-(t)]\,\mathrm{d}t \\
&\quad - \rho(b)\Phi_p\biggl(\frac{B - \sigma x_n(b)}{\gamma}\biggr)x_n^-(b) + \rho(a)\Phi_p\biggl(\frac{A + \beta x_n(a)}{\alpha}\biggr)x_n^-(a) \\
&\quad - \sum_{i=1}^{l} I_i(x_n^+(t_i))x_n^-(t_i) \\
&= \int_a^b [-\rho(t)|(x_n^-)'(t)|^p - s(t)|x_n^-(t)|^p - f(t, x_n^+(t))x_n^-(t)]\,\mathrm{d}t \\
&\quad - \rho(b)\Phi_p\biggl(\frac{B - \sigma x_n(b)}{\gamma}\biggr)x_n^-(b) + \rho(a)\Phi_p\biggl(\frac{A + \beta x_n(a)}{\alpha}\biggr)x_n^-(a) \\
&\quad - \sum_{i=1}^{l} I_i(x_n^+(t_i))x_n^-(t_i) \\
&= -\|x_n^-\|_X^p - \int_a^b f(t, x_n^+(t))x_n^-(t)\,\mathrm{d}t \\
&\quad - \rho(b)\biggl|\frac{B - \sigma x_n(b)}{\gamma}\biggr|^{p-2}\biggl(\frac{Bx_n^-(b) + \sigma(x_n^-(b))^2}{\gamma}\biggr) \\
&\quad + \rho(a)\biggl|\frac{A + \beta x_n(a)}{\alpha}\biggr|^{p-2}\biggl(\frac{Ax_n^-(a) - \beta(x_n^-(a))^2}{\alpha}\biggr) \\
&\quad - \sum_{i=1}^{l} I_i(x_n^+(t_i))x_n^-(t_i) \\
&\leqslant -\|x_n^-\|_X^p. \qquad (2.10)
\end{aligned}$$

Set $w_n^- = x_n^-/\|x_n^-\|_X$. Dividing by $\|x_n^-\|_X$ on both sides of the above inequality, we have

$$\|x_n^-\|_X^{p-1} \leqslant -\langle \varphi'(x_n), w_n^- \rangle \to 0 \quad \text{as } n \to \infty.$$

So $x_n^- \to 0$ in X. Now we shall show that (x_n^+) is bounded.

Let

$$\begin{aligned}
J(x_n) &= \mu\frac{\gamma\rho(b)}{\sigma p}\biggl|\frac{B - \sigma x_n(b)}{\gamma}\biggr|^p + \mu\frac{\alpha\rho(a)}{\beta p}\biggl|\frac{A + \beta x_n(a)}{\alpha}\biggr|^p \\
&\qquad + \rho(b)\Phi_p\biggl(\frac{B - \sigma x_n(b)}{\gamma}\biggr)x_n^+(b) - \rho(a)\Phi_p\biggl(\frac{A + \beta x_n(a)}{\alpha}\biggr)x_n^+(a).
\end{aligned}$$

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By (2.3) and (2.4) we have

$$\begin{aligned}\frac{\mu}{p}\|x_n\|_X^p - \|x_n^+\|_X^p &= \mu\varphi(x_n) - \langle\varphi'(x_n), x_n^+\rangle - J(x_n)\\ &\quad + \mu\int_a^b [F(t, x_n^+(t)) - f(t,0)x_n^-(t)]\,\mathrm{d}t - \int_a^b f(t, x_n^+(t))x_n^+(t)\,\mathrm{d}t\\ &\quad + \mu\sum_{i=1}^{l}\biggl[\int_0^{x^+(t_i)} I_i(s)\,\mathrm{d}s - I_i(0)x_n^-(t_i)\biggr] - \sum_{i=1}^{l} I_i(x_n^+(t_i))x_n^+(t_i).\end{aligned} \tag{2.11}$$

By (C1), (C2) and Lemma 2.6, one obtains

$$\begin{aligned}&\mu\int_a^b [F(t, x_n^+(t)) - f(t,0)x_n^-(t)]\,\mathrm{d}t - \int_a^b f(t, x_n^+(t))x_n^+(t)\,\mathrm{d}t\\ &\qquad\leqslant \mu\int_a^b H(t, x_n^+(t))\,\mathrm{d}t\\ &\qquad\leqslant \mu\int_a^b \biggl[c(t)x_n^+(t) + \frac{d(t)}{p}|x_n^+(t)|^p\biggr]\,\mathrm{d}t\\ &\qquad\leqslant \mu\|c\|_{L^1}\bar{\gamma}\|x_n^+\|_X + \frac{\mu\|d\|_\infty}{p(\operatorname{ess\,inf}_{[a,b]} s)}\|x_n^+\|_X^p, \quad \text{where } H(t,x) = \int_0^x h(t,\tau)\,\mathrm{d}\tau,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\mu\sum_{i=1}^{l}\biggl[\int_0^{x_n^+(t_i)} I_i(s)\,\mathrm{d}s - I_i(0)x_n^-(t_i)\biggr] - \sum_{i=1}^{l} I_i(x_n^+(t_i))x_n^+(t_i) &\leqslant \frac{\mu\xi}{p}\sum_{i=1}^{l}|x_n^+(t_i)|^p\\ &\leqslant \frac{\mu\xi l}{p}\bar{\gamma}^p\|x_n^+\|_X^p.\end{aligned} \tag{2.13}$$

We compute

$$\begin{aligned}-J(x_n) &= -\rho(b)\biggl|\frac{B - \sigma x_n(b)}{\gamma}\biggr|^{p-2}\frac{Bx_n^+(b) - \sigma x_n(b)x_n^+(b)}{\gamma}\\ &\quad + \rho(a)\biggl|\frac{A + \beta x_n(a)}{\alpha}\biggr|^{p-2}\frac{Ax_n^+(a) + \beta x_n(a)x_n^+(a)}{\alpha}\\ &\quad - \frac{\mu\gamma\rho(b)}{\sigma p}\biggl|\frac{B - \sigma x_n(b)}{\gamma}\biggr|^p - \frac{\mu\alpha\rho(a)}{\beta p}\biggl|\frac{A + \beta x_n(a)}{\alpha}\biggr|^p\\ &\leqslant \frac{\rho(b)B(2\mu - p)}{\gamma p}\biggl|\frac{B - \sigma x_n(b)}{\gamma}\biggr|^{p-2} x_n^+(b)\\ &\quad + \frac{\rho(a)A(p - 2\mu)}{\alpha p}\biggl|\frac{A + \beta x_n(a)}{\alpha}\biggr|^{p-2} x_n^+(a).\end{aligned} \tag{2.14}$$

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Substituting (2.12)–(2.14) into (2.11), in view of Lemma 2.2 (ii) one has

$$\begin{aligned}\left(\frac{\mu}{p}-1\right)\|x_n^+\|_X^p \leqslant \mu\varphi(x_n) - \langle\varphi'(x_n), x_n^+\rangle + \frac{\rho(b)B(2\mu-p)}{\gamma p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2} x_n^+(b) \\ + \frac{\rho(a)A(p-2\mu)}{\alpha p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2} x_n^+(a) \\ + \mu\|c\|_{L^1}\bar{\gamma}\|x_n^+\|_X + \frac{\mu\|d\|_\infty}{p(\operatorname{ess\,inf}_{[a,b]} s)}\|x_n^+\|_X^p + \frac{\mu\xi l}{p}\bar{\gamma}^p\|x_n^+\|_X^p. \qquad (2.15)\end{aligned}$$

Suppose that (x_n^+) is unbounded. Passing to a subsequence, we may assume, if necessary, that $\|x_n^+\|_X \to \infty$ as $n \to \infty$. Dividing both sides of (2.15) by $\|x_n^+\|_X^p$, with $w_n^+ = x_n^+/\|x_n^+\|_X$, we have

$$\begin{aligned}\frac{\mu}{p} - 1 \leqslant \frac{\mu\varphi(x_n)}{\|x_n^+\|_X^p} - \frac{\langle\varphi'(x_n), w_n^+\rangle}{\|x_n^+\|_X^{p-1}} \\ + \frac{\rho(b)B(2\mu-p)}{\gamma p\|x_n^+\|_X^p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2} x_n^+(b) \\ + \frac{\rho(a)A(p-2\mu)}{\alpha p\|x_n^+\|_X^p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2} x_n^+(a) \\ + \frac{\mu\bar{\gamma}\|c\|_{L^1}}{\|x_n^+\|_X^{p-1}} + \frac{\mu\|d\|_\infty}{p(\operatorname{ess\,inf}_{[a,b]} s)} + \frac{\mu\xi l}{p}\bar{\gamma}^p. \qquad (2.16)\end{aligned}$$

Since $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$, $x_n^- \to 0$ in X, let $n \to \infty$ in the above inequality. We have

$$\frac{\mu}{p} - 1 \leqslant \frac{\mu\|d\|_\infty}{p(\operatorname{ess\,inf}_{[a,b]} s)} + \frac{\mu\xi l}{p}\bar{\gamma}^p,$$

which contradicts (C3). Therefore, (x_n) is bounded in X.

From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call (x_n), $x_n \rightharpoonup x$. In the following we will show that (x_n) strongly converges to x. By (2.4) we have

$$\begin{aligned}&\langle\varphi'(x_n) - \varphi'(x), x_n - x\rangle \\ &\quad = \int_a^b \{\rho(t)[\Phi_p(x_n'(t)) - \Phi_p(x'(t))] \times (x_n'(t) - x'(t)) \\ &\qquad\qquad + s(t)[\Phi_p(x_n(t)) - \Phi_p(x(t))] \times (x_n(t) - x(t))\}\,\mathrm{d}t \\ &\qquad - \int_a^b [f(t, x_n^+(t)) - f(t, x^+(t))](x_n(t) - x(t))\,\mathrm{d}t \\ &\qquad - \sum_{i=1}^{l} [I_i(x_n^+(t_i)) - I_i(x^+(t_i))](x_n(t_i) - x(t_i))\end{aligned}$$

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$$- \rho(b)\left[\Phi_p\left(\frac{B-\sigma x_n(b)}{\gamma}\right) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)\right] \times (x_n(b)-x(b))$$
$$+ \rho(a)\left[\Phi_p\left(\frac{A+\beta x_n(a)}{\alpha}\right) - \Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right)\right] \times (x_n(a)-x(a)). \quad (2.17)$$

By $x_n \rightharpoonup x$ in X, we see that (x_n) uniformly converges to x in $C([a,b])$. So

$$\left.\begin{aligned} &\int_a^b [f(t,x_n^+(t)) - f(t,x^+(t))](x_n(t)-x(t))\,\mathrm{d}t \to 0,\\ &\sum_{i=1}^{l}[I_i(x_n^+(t_i)) - I_i(x^+(t_i))](x_n(t_i)-x(t_i)) \to 0,\\ &x_n(b)\to x(b), \quad x_n(a)\to x(a) \quad \text{as } n\to\infty. \end{aligned}\right\} \quad (2.18)$$

By $\varphi'(x_n) \to 0$ and $x_n \rightharpoonup x$, we have

$$\langle \varphi'(x_n) - \varphi'(x), x_n - x\rangle \to 0 \quad \text{as } n \to \infty. \quad (2.19)$$

By [**27**, Equation (2.2)], there exist $c_p, d_p > 0$ such that

$$\begin{aligned} &\int_a^b \{\rho(t)[\Phi_p(u'(t)) - \Phi_p(v'(t))] \times (u'(t)-v'(t))\\ &\qquad + s(t)[\Phi_p(u(t)) - \Phi_p(v(t))] \times (u(t)-v(t))\}\,\mathrm{d}t\\ &\qquad\qquad \geqslant \begin{cases} c_p \displaystyle\int_a^b [\rho(t)|u'(t)-v'(t)|^p + s(t)|u(t)-v(t)|^p]\,\mathrm{d}t & \text{if } p \geqslant 2,\\ d_p \displaystyle\int_a^b \left[\frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}} + \frac{s(t)|u(t)-v(t)|^2}{(|u(t)|+|v(t)|)^{2-p}}\right]\mathrm{d}t & \text{if } 1<p<2. \end{cases} \end{aligned} \quad (2.20)$$

If $p \geqslant 2$, then (2.17)–(2.20) yield that $\|x_n - x\|_X \to 0$ in X.

If $1 < p < 2$, by Hölder's inequality, for $u, v \in X$, we obtain

$$\begin{aligned} &\int_a^b \rho(t)|u'(t)-v'(t)|^p\,\mathrm{d}t\\ &\leqslant \left(\int_a^b \frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}}\,\mathrm{d}t\right)^{p/2}\left(\int_a^b \rho(t)(|u'(t)|+|v'(t)|)^p\,\mathrm{d}t\right)^{(2-p)/2}\\ &\leqslant \left(\int_a^b \frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}}\,\mathrm{d}t\right)^{p/2} 2^{(p-1)(2-p)/2}\left(\int_a^b \rho(t)[|u'(t)|^p+|v'(t)|^p]\,\mathrm{d}t\right)^{(2-p)/2}\\ &\leqslant 2^{(p-1)(2-p)/2}\left(\int_a^b \frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}}\,\mathrm{d}t\right)^{p/2}(\|u\|_X + \|v\|_X)^{(2-p)p/2}. \end{aligned} \quad (2.21)$$

Similarly,

$$\int_a^b s(t)|u(t)-v(t)|^p \,\mathrm{d}t \leqslant 2^{(p-1)(2-p)/2}\biggl(\int_a^b \frac{s(t)|u(t)-v(t)|^2}{(|u(t)|+|v(t)|)^{2-p}}\,\mathrm{d}t\biggr)^{p/2}(\|u\|_X+\|v\|_X)^{(2-p)p/2}. \tag{2.22}$$

So (2.20)–(2.22) yield

$$\begin{aligned}\int_a^b &\rho(t)[\Phi_p(x_n'(t))-\Phi_p(x'(t))](x_n'(t)-x'(t)) \\ &+ s(t)[\Phi_p(x_n(t))-\Phi_p(x(t))](x_n(t)-x(t))\,\mathrm{d}t \\ &\geqslant d_p\int_a^b\biggl[\rho(t)\frac{|x_n'(t)-x'(t)|^2}{(|x_n'(t)|+|x'(t)|)^{2-p}}+s(t)\frac{|x_n(t)-x(t)|^2}{(|x_n(t)|+|x(t)|)^{2-p}}\biggr]\mathrm{d}t \\ &\geqslant \frac{d_p}{2^{(p-1)(2-p)/p}(\|x_n\|_X+\|x\|_X)^{2-p}} \\ &\quad\times\biggl\{\biggl(\int_a^b\rho(t)|x_n'(t)-x'(t)|^p\,\mathrm{d}t\biggr)^{2/p}+\biggl(\int_a^b s(t)|x_n(t)-x(t)|^p\,\mathrm{d}t\biggr)^{2/p}\biggr\} \\ &\geqslant \frac{d_p}{2^{(p-1)(2-p)/p}\max\{2^{(2/p)-1},1\}}\frac{\|x_n-x\|_X^2}{(\|x_n\|_X+\|x\|_X)^{2-p}}.\end{aligned} \tag{2.23}$$

Then (2.17)–(2.19) and (2.23) yield that $\|x_n-x\|_X\to 0$ in X, i.e. (x_n) strongly converges to x in X. □

3. Main results

Theorem 3.1. *Suppose that (C1)–(C3) hold. Furthermore, we assume the following.*

(C4) *There exists an $M_0>0$ such that*

$$\frac{1}{p}\biggl[1-\frac{\|d\|_\infty}{\operatorname{ess\,inf}_{[a,b]} s}-\xi l\bar{\gamma}^p\biggr]M_0^p > \frac{[\|r\|_\infty(b-a)+\eta l]\bar{\gamma}^\mu M_0^\mu}{\mu}+\|c\|_{L^1}\bar{\gamma}M_0+\frac{\rho(b)|B|^p}{\sigma p\gamma^{p-1}}+\frac{\rho(a)|A|^p}{\beta p\alpha^{p-1}}.$$

Then problem (1.1) has at least two positive classical solutions, x_0, x^*, with $\|x_0\|_X<M_0$.

Proof. We complete the proof in three steps.

Step 1. By Lemma 2.7, the functional φ satisfies the Palais–Smale condition.

Step 2. We shall show that there exists $M>0$ such that the functional φ has a local minimum $x_0\in B_M:=\{x\in X:\|x\|_X<M\}$.

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Let $M>0$, which will be determined later. First we claim that $\bar{B}_M$ is bounded and weak sequentially closed. In fact, let $(u_n)\subseteq \bar{B}_M$ and $(u_n)\rightharpoonup u$ as $n\to\infty$. By the Mazur theorem [**16**], there exists a sequence of convex combinations

$$v_n=\sum_{j=1}^{n}\alpha_{n_j}u_j,\quad \sum_{j=1}^{n}\alpha_{n_j}=1,\quad \alpha_{n_j}\geqslant 0,\ j\in N,$$

such that $v_n\to u$ in X. Since $\bar{B}_M$ is a closed convex set, $(v_n)\subset \bar{B}_M$ and $u\in\bar{B}_M$. Now we claim that φ has a minimum $x_0\in\bar{B}_M$. We will show that φ is weak sequentially lower semi-continuous on $\bar{B}_M$. For this, let

$$\varphi^1(x)=\frac{1}{p}\int_a^b[\rho(t)|x'(t)|^p+s(t)|x(t)|^p]\,\mathrm{d}t$$

and

$$\begin{aligned}\varphi^2(x)=-\int_a^b&[F(t,x^+(t))-(f(t,0),x^-(t))]\,\mathrm{d}t\\&-\sum_{i=1}^{l}\biggl[\int_0^{x^+(t_i)}I_i(s)\,\mathrm{d}s-I_i(0)x^-(t_i)\biggr]\\&\qquad+\frac{\gamma\rho(b)}{\sigma p}\biggl|\frac{B-\sigma x(b)}{\gamma}\biggr|^p+\frac{\alpha\rho(a)}{\beta p}\biggl|\frac{A+\beta x(a)}{\alpha}\biggr|^p.\end{aligned}$$

Then $\varphi(x)=\varphi^1(x)+\varphi^2(x)$. By $x_n\rightharpoonup x$ on X we see that (x_n) uniformly converges to x in $C([a,b])$. So φ^2 is weak sequentially continuous. Clearly, φ^1 is continuous, which, together with the convexity of φ^1, implies that φ^1 is weak sequentially lower semi-continuous. Therefore, φ is weak sequentially lower semi-continuous on $\bar{B}_M$. Besides, X is a reflexive Banach space and $\bar{B}_M$ is a bounded and weak sequentially closed set, so our claim follows from Lemma 1.1. Without loss of generality, we assume that $\varphi(x_0)=\min_{x\in\bar{B}_M}\varphi(x)$. Now we will show that

$$\varphi(x_0)<\inf_{x\in\partial B_M}\varphi(x). \tag{3.1}$$

If this is true, the result of Step 2 holds.

In fact, for any $x\in\partial B_M$, by (2.3), (C1) and Lemma 2.6, we have

$$\begin{aligned}\varphi(x)&\geqslant\frac{M^p}{p}-\int_a^b F(t,x^+(t))\,\mathrm{d}t-\sum_{i=1}^{l}\int_0^{x^+(t_i)}I_i(s)\,\mathrm{d}s\\&\geqslant\frac{M^p}{p}-\int_a^b\biggl[\frac{r(t)|x^+(t)|^\mu}{\mu}+c(t)x^+(t)+\frac{d(t)|x^+(t)|^p}{p}\biggr]\mathrm{d}t\\&\quad-\sum_{i=1}^{l}\biggl[\frac{\eta}{\mu}|x^+(t_i)|^\mu+\frac{\xi}{p}|x^+(t_i)|^p\biggr]\\&\geqslant\frac{M^p}{p}-\frac{\|r\|_\infty(b-a)}{\mu}\|x\|_\infty^\mu-\|c\|_{L^1}\|x\|_\infty\\&\quad-\frac{\|d\|_\infty}{p(\operatorname{ess\,inf}_{[a,b]}s)}\|x\|_X^p-\frac{\eta l\bar{\gamma}^\mu}{\mu}\|x^+\|_X^\mu-\frac{\xi l\bar{\gamma}^p}{p}\|x^+\|_X^p\end{aligned}$$

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$$\begin{aligned}&\geqslant \frac{M^p}{p} - \frac{\|r\|_\infty(b-a)}{\mu}\bar\gamma^\mu\|x\|_X^\mu - \|c\|_{L^1}\bar\gamma\|x\|_X \\ &\quad - \frac{\|d\|_\infty}{p(\text{ess inf}_{[a,b]} s)}\|x\|_X^p - \frac{\eta l\bar\gamma^\mu}{\mu}\|x^+\|_X^\mu - \frac{\xi l\bar\gamma^p}{p}\|x^+\|_X^p \\ &= \frac{M^p}{p} - \frac{\|r\|_\infty(b-a)}{\mu}\bar\gamma^\mu M^\mu - \|c\|_{L^1}\bar\gamma M \\ &\quad - \frac{\|d\|_\infty}{p(\text{ess inf}_{[a,b]} s)}M^p - \frac{\eta l\bar\gamma^\mu}{\mu}M^\mu - \frac{\xi l\bar\gamma^p}{p}M^p. \end{aligned} \tag{3.2}$$

So

$$\inf_{x\in\partial B_M}\varphi(x) \geqslant \frac{M^p}{p} - \frac{\|r\|_\infty(b-a)}{\mu}\bar\gamma^\mu M^\mu - \|c\|_{L^1}\bar\gamma M - \frac{\|d\|_\infty}{p(\text{ess inf}_{[a,b]} s)}M^p - \frac{\eta l}{\mu}\bar\gamma^\mu M^\mu - \frac{\xi l\bar\gamma^p}{p}M^p.$$

Noting that

$$\varphi(x_0) \leqslant \varphi(0) = \frac{\rho(b)|B|^p}{\sigma p\gamma^{p-1}} + \frac{\rho(a)|A|^p}{\beta p\alpha^{p-1}},$$

by (C4) there exists $M_0 > 0$ such that $\varphi(x) > \varphi(0) \geqslant \varphi(x_0)$ for any $x \in \partial B_{M_0}$. So (3.1) holds and $x_0 \in B_{M_0}$.

Step 3. We shall show that there exists x_1 with $\|x_1\|_X > M_0$ such that $\varphi(x_1) < \inf_{x\in\partial B_{M_0}}\varphi(x)$.

Let $\tilde e(t) = 1 \in X$, $\bar\lambda > 0$. Then

$$\begin{aligned}\varphi(\bar\lambda\tilde e) &= \frac{\bar\lambda^p}{p}\int_a^b s(t)\,\mathrm{d}t - \int_a^b [F(t,\bar\lambda) - f(t,0)\bar\lambda]\,\mathrm{d}t \\ &\quad - \sum_{i=1}^{l}\bigg[\int_0^{\bar\lambda} I_i(s)\,\mathrm{d}s - I_i(0)\bar\lambda\bigg] + \frac{\gamma\rho(b)}{\sigma p}\bigg|\frac{B-\sigma\bar\lambda}{\gamma}\bigg|^p + \frac{\alpha\rho(a)}{\beta p}\bigg|\frac{A+\beta\bar\lambda}{\alpha}\bigg|^p \\ &\leqslant \frac{\bar\lambda^p}{p}\int_a^b s(t)\,\mathrm{d}t - \int_a^b\bigg[\frac{r(t)\bar\lambda^\mu}{\mu} + H(t,\bar\lambda)\bigg]\mathrm{d}t + \bar\lambda\int_a^b f(t,0)\,\mathrm{d}t \\ &\quad - \frac{l\eta\bar\lambda^\mu}{\mu} + \frac{\gamma\rho(b)}{\sigma p}\bigg|\frac{B-\sigma\bar\lambda}{\gamma}\bigg|^p + \frac{\alpha\rho(a)}{\beta p}\bigg|\frac{A+\beta\bar\lambda}{\alpha}\bigg|^p \\ &\leqslant \frac{\bar\lambda^p}{p}\int_a^b s(t)\,\mathrm{d}t - \frac{\bar\lambda^\mu}{\mu}\int_a^b r(t)\,\mathrm{d}t + \bar\lambda\int_a^b f(t,0)\,\mathrm{d}t \\ &\quad - \frac{l\eta\bar\lambda^\mu}{\mu} + \frac{\gamma\rho(b)}{\sigma p}\bigg|\frac{B-\sigma\bar\lambda}{\gamma}\bigg|^p + \frac{\alpha\rho(a)}{\beta p}\bigg|\frac{A+\beta\bar\lambda}{\alpha}\bigg|^p. \end{aligned} \tag{3.3}$$

Since $\mu > p$ and (C1), we have $\lim_{\bar\lambda\to+\infty}\varphi(\bar\lambda\tilde e) = -\infty$. Therefore, there exists a sufficiently large $\lambda_0 > 0$ with $\|\bar\lambda_0\tilde e\| > M_0$ such that $\varphi(\bar\lambda_0\tilde e) < \inf_{x\in\partial B_{M_0}}\varphi(x)$. Therefore, let $x_1 = \bar\lambda_0\tilde e$ and $\varphi(x_1) < \inf_{x\in\partial B_{M_0}}\varphi(x)$.

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Lemma 1.3 now gives the critical value

$$c = \inf_{h\in\gamma} \max_{t\in[0,1]} \varphi(h(t)),$$

where

$$\gamma = \{h \mid h : [0,1] \to E \text{ is continuous and } h(0) = x_0,\ h(1) = x_1\},$$

that is, there exists $x^* \in X$ such that $\varphi'(x^*) = 0$. Therefore, x_0 and x^* are two critical points of φ, $\|x_0\|_X < M_0$, and hence they are classical solutions of (2.1). Lemma 2.3 means that x_0 and x^* are positive classical solutions of problem (1.1). □

Example 3.2. Consider the following problem:

$$\left.\begin{aligned} -\left(\frac{1}{1+t}\Phi_3(x'(t))\right)' + \frac{1}{1+t}\Phi_3(x(t)) = f(t,x), \qquad t \neq t_i,\ t \in [0,1], \\ -\Delta\left(\frac{1}{1+t_i}\Phi_3(x'(t_i))\right) = I_i(x(t_i)), \quad i = 1,2, \\ x'(0) - 2x(0) = -\tfrac{1}{8}, \qquad x'(1) + 3x(1) = \tfrac{1}{4}, \end{aligned}\right\} \tag{3.4}$$

where

$$f(t,x) = \frac{t}{96}x^{13} + \frac{t^{11}}{48} + \frac{1}{4+2t}x^2,$$

and $I_i(x) = \frac{1}{192}x^{13} + \frac{1}{64}x^2$, $i = 1,2$.

Compared to (1.1), $\rho(t) = 1/(1+t)$, $s(t) = 1/(1+t)$, $p = 3$, $l = 2$, $a = 0$, $b = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = 1$, $\sigma = 3$, $A = -\frac{1}{8}$, $B = \frac{1}{4}$.

Let

$$\mu = 14, \quad r(t) = \frac{t}{96}, \quad h(t,x) = \frac{t^{11}}{48} + \frac{1}{4+2t}x^2, \quad c(t) = \frac{t^{11}}{48}, \quad d(t) = \frac{1}{4+2t},$$

$$\eta = \frac{1}{192}, \quad g(x) = \frac{x^2}{64}, \quad \xi = \frac{1}{64}.$$

Clearly, (C1)–(C3) are satisfied. Setting $M_0 = \frac{1}{2}$ satisfies condition (C4). Applying Theorem 3.1, the boundary-value problem (3.4) has at least two positive solutions, x_0 and x^*, with $\|x_0\|_X < \frac{1}{2}$.

Corollary 3.3. *Suppose that (C1) holds. Moreover, we assume the following.*

(C2′) *There exist $0 \leqslant \theta < p$, $c \in L^1([a,b],[0,+\infty))$, $d \in C([a,b],[0,+\infty))$ such that*

$$h(t,x) \leqslant c(t) + d(t)\Phi_\theta(x), \qquad g(x) \leqslant \xi\Phi_\theta(x).$$

(C4′) *There exists an $M_0 > 0$ such that*

$$\frac{1}{p}M_0^p > \frac{[\|r\|_\infty(b-a) + \eta l]\bar{\gamma}^\mu M_0^\mu}{\mu} + \|c\|_{L^1}\bar{\gamma}M_0 + \frac{\rho(b)|B|^p}{\sigma p\gamma^{p-1}} + \frac{\rho(a)|A|^p}{\beta p\alpha^{p-1}} + \left(\frac{\xi l\bar{\gamma}^\theta}{\theta} + \frac{\|d\|_\infty}{\theta\cdot \operatorname{ess\,inf}_{[a,b]} s}\right)M_0^\theta.$$

Then problem (1.1) has at least two positive classical solutions x_0, x^* with $\|x_0\|_X < M_0$.

Corollary 3.4. *Suppose that (C1) and (C2′) hold. Moreover, we assume the following.*

(C5) *There exists* $M_0 > 0$ *such that*

$$\frac{1}{p}M_0^p > \frac{\|r\|_\infty(b-a)+\eta l}{\mu}\bar{\gamma}^\mu M_0^\mu + \|c\|_{L^1}\bar{\gamma}M_0 + \left(\frac{\|d\|_\infty}{\theta\cdot \operatorname{ess\,inf}_{[a,b]} s} + \frac{\xi l\bar{\gamma}^\theta}{\theta}\right)M_0^\theta.$$

Then problem (1.1) with $A = B = 0$ *has at least two positive solutions* x_0, x^*.

Example 3.5. Consider the following problem:

$$\left.\begin{aligned} -x'' + x &= r(t)x^{\mu-1} + d(t)x^{\theta-1}, && t \neq t_i,\ t\in[0,1],\\ -\Delta x'(t_i) &= \eta(x(t_i))^{\mu-1} + \xi(x(t_i))^{\theta-1}, && i = 1,2,\dots,l,\\ \alpha x'(0) - \beta x(0) &= 0, \qquad \gamma x'(1) + \sigma x(1) = 0, \end{aligned}\right\} \tag{3.5}$$

where $\mu > 2 > \theta$, $r \in C([0,1],[0,+\infty))$, $\eta, \xi \geqslant 0$. On applying Corollary 3.4, problem (3.5) has at least two positive solutions provided there exists an $M_0 > 0$ such that

$$\tfrac{1}{2}M_0^2 > \frac{\|r\|_\infty + \eta l}{\mu}\bar{\gamma}^\mu M_0^\mu + \left(\frac{\|d\|_\infty}{\theta} + \frac{\xi l\bar{\gamma}^\theta}{\theta}\right)M_0^\theta, \quad \text{where } \gamma = \sqrt{2}.$$

According to the proof of Lemma 2.7 and Theorem 3.1, we have the following result.

Theorem 3.6. *Suppose that the following conditions hold:*

(D1) $f(t,x) = \circ(|x|^{p-1})$, $I_i(x) = \circ(|x|^{p-1})$ *as* $|x| \to 0$ *uniformly for* $t \in [a,b]$*;*

(D2) *there exist constants* $M > 0, \mu > p$ *such that*

$$0 < \mu F(t,x) < xf(t,x), \quad 0 < \mu\int_0^x I_i(s)\,\mathrm{d}s < xI_i(x) \quad \text{for any } x \geqslant M, t\in[a,b].$$

Then problem (1.1) with $A = B = 0$ *has at least two positive solutions.*

Proof. In the proof of (2.12), (2.13) in Lemma 2.7, we substitute conditions (D1) and (D2) for (C1) and (C2). Then it is easy to show that (x_n^+) is bounded. In the proof of (3.2) in Theorem 3.1, we apply (D1) (not (C1), (C2)). In fact, (D1) means that, for $0 < \varepsilon < 1/(p(b-a+l)\bar{\gamma}^p)$, there exists an $M > 0$ such that

$$F(t,x) \leqslant \varepsilon|x|^p, \qquad \int_0^x I_i(s)\,\mathrm{d}s \leqslant \varepsilon|x|^p$$

hold for $t\in[a,b]$, $|x| < M$. Thus,

$$\begin{aligned}\varphi(x) &\geqslant \frac{M^p}{p} - \int_a^b F(t,x^+(t))\,\mathrm{d}t - \sum_{i=1}^{l}\int_0^{x^+(t_i)} I_i(s)\,\mathrm{d}s\\ &\geqslant \frac{M^p}{p} - \varepsilon\int_a^b |x(t)|^p\,\mathrm{d}t - \sum_{i=1}^{l}\varepsilon|x(t_i)|^p\\ &= \frac{M^p}{p} - \varepsilon(b-a)\bar{\gamma}^pM^p - \varepsilon l\bar{\gamma}^pM^p\\ &= \left[\frac{1}{p} - \varepsilon(b-a)\bar{\gamma}^p - \varepsilon l\bar{\gamma}^p\right]M^p > 0 = \varphi(0).\end{aligned}$$

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Finally, we apply (D2) to (3.3). Then the result of Step 3 follows. In fact, (D2) means that there exist $a_1, a_2 \in C([a,b],(0,+\infty))$, $a_3, a_4 > 0$ such that

$$F(t,x) \geqslant a_1(t)|x|^{\mu} - a_2(t), \qquad \int_0^x I_i(s)\,\mathrm{d}s \geqslant a_3|x|^{\mu} - a_4,$$

which yields the result. □

Example 3.7. For problem (3.5), if $d(t) \equiv 0$, $t \in [0,1]$, $\xi = 0$, then problem (3.5) has at least two positive solutions by using Theorem 3.6.

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