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APPLYING A SPLITTING TECHNIQUE TO ESTIMATE ELECTRICAL GRID RELIABILITY

Wander Wadman
Daan Crommelin
Jason Frank

CWI Amsterdam
Science Park 123
Amsterdam, 1098 XG, NETHERLANDS

ABSTRACT

As intermittent renewable energy penetrates electrical power grids more and more, assessing grid reliability is of increasing concern for grid operators. Monte Carlo simulation is a robust and popular technique to estimate indices for grid reliability, but the involved computational intensity may be too high for typical reliability analyses. We show that various reliability indices can be expressed as expectations depending on the rare event probability of a so-called power curtailment, and explain how to extend a Crude Monte Carlo grid reliability analysis with an existing rare event splitting technique. The squared relative error of index estimators can be controlled, whereas orders of magnitude less workload is required than when using an equivalent Crude Monte Carlo method. We show further that a bad choice for the time step size or for the importance function may endanger this squared relative error.

1 INTRODUCTION

Contemporary societies have grown accustomed to a very reliable supply by electrical power grids. However, substantial implementation of intermittent renewable generation, such as photovoltaic power or wind power, may threaten grid reliability. Power imbalances caused by generation intermittency may force grid operators to curtail power to preserve grid stability. To assess long-term grid investments or decide on short-term operational strategies, the responsible grid operator should be able to estimate grid reliability.

For this purpose, various grid reliability indices exist (Li and Billinton 1994), and many depend on the probability $\mathbb{P}(C)$, where

$$C = \{\text{A power curtailment occurs during } [0,T]\} \tag{1}$$

denotes the event of a power curtailment during the time interval $[0,T]$ of interest. We model the uncertain energy sources as stochastic processes, discretized in time. At each time step, the mapping from the state of these processes to the outcome of C (true or false) requires solving a nonlinear algebraic system. As this mapping is defined implicitly, we can not derive $\mathbb{P}(C)$ directly, and we estimate it by a Monte Carlo (MC) simulation.

However, as power curtailments are undesirable, we may expect C to be rare. In case of a time interval T equal to one week, values for $\mathbb{P}(C) \approx 10^{-4}$ or even much smaller are not uncommon (Carden and Wintermantel 2013, CEER 2011). Crude Monte Carlo (CMC) estimators for rare event probabilities require a large number of samples to achieve a fixed accuracy (Rubino and Tuffin 2009). Since one CMC sample already involves solving a large number of high dimensional nonlinear systems, CMC estimation is computationally too intensive for grid reliability analyses in general.

Rare event simulation techniques have been developed to accurately estimate very small probabilities, of which importance sampling and (importance) splitting are two well-known variants. In importance

sampling, one samples from an alternative distribution, whereafter the estimator is multiplied by an appropriate likelihood ratio to correct for the induced bias (Rubino and Tuffin 2009). Crucial for variance reduction is to find a distribution that increases rare event occurrences. Adaptive importance sampling techniques (Juneja and Shahabuddin 2006) have been developed to recursively learn this distribution. For example, the Cross-Entropy Method (CEM) iteratively changes the distribution parameters of random variables responsible for approaching the rare event in a pilot run (Rubinstein and Kroese 2004). However, in general power grids various typical paths may lead to rare event C, especially when considering a large number of stochastic sources and a large time domain. In this case CEM changes the distribution of all corresponding random variables, wherefore the resulting alternative distribution may (counterproductively) *increase* the variance by pushing too much in the direction of C.

Splitting techniques do not change the distribution, but resample trajectories as soon as they are presumed substantially closer to the rare event (Rubino and Tuffin 2009, Garvels 2000, L'Ecuyer et al. 2006). In this way, variance reduction is achieved using an increased occurrence of rare events, without the need to understand the most likely occurrences a priori. In the literature splitting techniques have rarely been applied to power systems. Wang et al. (2011) estimated small probabilities of instantaneous, unforeseen failures of grid components. Our work though considers the rare event of power curtailments over a certain time domain due to (and given) the uncertain nature of generation. Further, Schlapfer and Mancarella (2011) estimated the probability of a transmission line temperature exceeding a critical value, using Markov processes with a discrete state space. Our model considers Markov processes with a continuous state space, and allows for the assessment of more general reliability indices.

In this paper, we speed up an MC method for grid reliability estimation with an existing splitting technique called Fixed Number of Successes (FNS). In Section 2, we introduce common reliability indices for transmission power grids. In Section 3, we describe a CMC method that estimates these indices. We specify a stochastic model for the intermittent energy sources, define the mapping from these sources to the outcome of a power curtailment and explain the computational intensity of this brute force approach. In Section 4 we extend the CMC method with the FNS technique. We investigate the computational performance of the FNS technique in Section 5 on an example power grid, and demonstrate how to choose the time step size and the importance function such that the estimate accuracy can be controlled.

2 GRID RELIABILITY INDICES

A reliability assessment of a power grid during a time interval $[0,T]$ of interest (e.g. day/week/year) involves estimating to what extent electrical constraints are violated. The power grid topology may be regarded as an undirected graph, with M edges representing connections (lines or cables), and with N nodes representing buses where power is possibly injected or extracted. Two important types of constraints (Wangdee 2005) are absolute voltage constraints at all grid nodes

$$\mathbf{V}_{\min} < |\mathbf{V}(t)| < \mathbf{V}_{\max}, \quad \text{for all } t \in [0,T], \tag{2}$$

and absolute current constraints at all grid connections

$$|\mathbf{I}(t)| < \mathbf{I}_{\max}, \quad \text{for all } t \in [0,T]. \tag{3}$$

Here $\mathbf{V}(t)$ and $\mathbf{I}(t)$ are complex-valued vector functions of nodal voltages and connection currents, respectively, at time t, and $\mathbf{V}_{\min}, \mathbf{V}_{\max}, \mathbf{I}_{\max}$ real-valued vectors of the allowed extrema. In practice, when constraints are violated, grid operators perform corrective actions to restore stability. These actions include rescheduling of generation and curtailing power. For simplicity, we assume that a violation of (2) at some node or a violation of (3) at some connection is a sufficient (and necessary) condition for a power curtailment. That is, such constraint violation immediately affects power delivery somewhere in the grid.

Various indices exist (IEEE PES T&D Committee et al. 2004, Li and Billinton 1994) to indicate the extent of load curtailments, where load refers to power consumption. These can broadly be divided in probability,

duration, frequency and severity of curtailments. The indices of interest may differ per consumer, depending on the behavior of their connected devices during and shortly after a power outage. Most conventional index definitions measure the extent of *load* curtailments only. However, in contemporary privatized energy markets grid operators must ensure power *supply* by electricity producers as well. Therefore, load-based definitions are easily generalized to power curtailments (where power denotes both load and supply) in general. We describe four of these indices below.

1. The Probability of Power Curtailments during $[0,T]$
$$\mathrm{PPC}(T) := \mathbb{P}(C), \tag{4}$$
with C as defined in (1).
2. The Expected Duration of Power Curtailments
$$\mathrm{EDPC}(T) := \mathbb{E}\left[D(T)\right], \tag{5}$$
with $D(T)$ the total duration of curtailments during $[0,T]$.
3. The Expected Number of Power Curtailments
$$\mathrm{ENPC}(T) := \mathbb{E}\left[N_C(T)\right], \tag{6}$$
with $N_C(T)$ the number of distinct power curtailments during $[0,T]$.
4. The Expected Energy Not Supplied accounts for the severity of the curtailment
$$\mathrm{EENS}(T) := \mathbb{E}\left[\int_0^T S_C(t)dt\right], \tag{7}$$
with $S_C(t)$ the size of power curtailment (in MW) at time t.

Other well-known indices account for duration, frequency and unsupplied energy *per curtailment* or *per customer*. These include SAIDI, CAIFI, SAIFI, CAIDI, denoting the System Average Interruption Duration Index, the Customer Average Interruption Frequency Index, and so on (IEEE PES T&D Committee et al. 2004), and they can easily be derived from the above indices.

3 A CONVENTIONAL RELIABILITY ESTIMATION METHOD

To assess the risk of curtailment caused by intermittent generation, one should properly model the uncertain power generation. Typically, the meteorological source (e.g. wind speed, solar radiation) is modeled instead of the amount of power generation itself, as data of the latter are often scarce, especially when considering investments in infrastructure by estimating reliability of a future grid. A source can be modeled as a Markov process, of which the distribution should be realistic and exhibit temporal periodicities due to daily and seasonal cycles. Further, one should take into account spatial correlations between sources at different locations, as they may affect reliability indices significantly (Wangdee and Billinton 2006). In case of wind power, Lojowska et al. (2010) and Lujano-Rojas et al. (2011) proposed wind speed ARMA models, whereas Wadman et al. (2012) and Wangdee and Billinton (2006) extended such models to the multivariate case, imposing spatial dependency.

Suppose we are interested in the value of one of the indices (4)-(7), and write it as $\mathbb{E}[I]$ for an appropriately defined random variable I (note that $\mathbb{P}(C) = \mathbb{E}[\mathbf{1}_C]$, with $\mathbf{1}$ the indicator function). Then at each time step, the function $f : S \mapsto I$ from the stochastic sources S to I requires solving the *power flow equations*

$$P_i = \sum_j^N |V_i||Y_{ij}||V_j|\cos(\theta_{ij} + \delta_j - \delta_i), \tag{8}$$

$$Q_i = -\sum_j^N |V_i||Y_{ij}||V_j|\sin(\theta_{ij} + \delta_j - \delta_i), \tag{9}$$

for each grid node $i = 1, \ldots, N$. Here, $P_i, Q_i \in \mathbb{R}$ denote the active and reactive power, respectively, injected at node i. $P_i, Q_i > 0$ indicates generation, whereas $P_i, Q_i < 0$ indicates extraction at node i. $|V_i|, \delta_i \in \mathbb{R}$ denote the voltage magnitude and voltage angle, respectively, at grid node i. $|Y|, \theta \in \mathbb{R}^{N \times N}$ denote the absolute value and angle, respectively, of the complex admittance matrix, containing the grid topology and electrical admittances of all grid connections. We desire the solution of this nonlinear algebraic system for the state vectors $|\mathbf{V}|, \delta$, given vectors P, Q. We can not solve system (8)-(9) directly, so we approximate the solution numerically using a Newton-Raphson method (Grainger and Stevenson 1994). Using Ohm's law we can then immediately derive all other electrical quantities required to check curtailment constraints (2) and (3).

Note that P_i, Q_i are random variables if an intermittent generator is connected at node i. By nonlinearity of (8)-(9), the function f is only implicitly defined. Therefore, we can not derive $\mathbb{E}[I]$ directly, and instead we estimate it using an MC simulation. That is, we sample a realization of the discrete time Markov processes corresponding to all intermittent generators. Then at each time step, the power flow equations are solved and curtailment constraints are checked. Repeating this for all time steps yields one realization of I. The average over many such realizations constitutes a CMC estimate for the index.

To estimate power grid reliability accurately, the time interval of the MC simulation should cover the significant temporal periodicities of generation and consumption. On the other hand, the step size should be sufficiently small to address sudden changes of consumption and generation. For example, Frunt (2011) states that less than 10 minutes is required in case of wind power. Hence, one MC sample may already require thousands of solutions of (8)-(9), each of which will take on the order of milliseconds (see Table 1).

Table 1: *Average CPU time requires on the order of milliseconds to solve the power flow equations at one time step. Typical power flow test cases (University of Washington 2006) are considered, and simulations are performed using MATLAB R2011a on an Intel Core 2.80GHz.*

Number of grid nodes	14	30	57	118	300
Average CPU time (ms)	0.72	0.78	1.5	2.2	11.6

However, as power curtailments are undesirable, we may expect their occurrence to be rare. For the unbiased CMC estimator for $\mathbb{P}(C)$

$$\hat{P}_n := \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}\{C \text{ in sample } j\},$$

the *squared relative error*

$$\mathrm{SRE}\left(\hat{P}_n\right) := \frac{\mathrm{Var}\left(\hat{P}_n\right)}{\mathbb{P}(C)^2} = \frac{1 - \mathbb{P}(C)}{n\mathbb{P}(C)} \tag{10}$$

goes to infinity as $\mathbb{P}(C)$ goes to 0. If accuracy is considered sufficient when the squared relative error is less than, say, $1/2$, we require $n > 160000$ CMC samples for probabilities smaller than 10^{-4}. Estimating all other indices requires a similar sample size: $n = 1/\mathbb{P}(C)$ CMC samples will on average yield one nonzero realization for $\mathbb{E}[I]$, which obviously is the bare minimum for a magnitude indication. Recalling the high computational intensity of one MC sample, we conclude that CMC estimation is not feasible in practice for general grid reliability analyses.

4 SPLITTING THE RELIABILITY ESTIMATION METHOD

To reduce the computational burden of the conventional reliability estimation method, we write all indices (4)-(7) as a function of a rare event probability. That is, the law of total expectation gives

$$\mathbb{E}[I] = \mathbb{P}(C)\mathbb{E}[I|C] + \mathbb{P}(C^{\mathsf{c}})\mathbb{E}[I|C^{\mathsf{c}}] = \mathbb{P}(C)\mathbb{E}[I|C].$$

Here C^{c} denotes the complement of C, i.e. no curtailment has occurred. The last equality holds since all the mentioned indices are zero given that no curtailment has occurred. Hence, $\mathbb{E}[I]$ can be written as the product of the curtailment probability and the *conditional index*. This representation suggests estimation of $\mathbb{E}[I]$ by $\hat{I} := \hat{P}\hat{I}^C$, with $\hat{P}$ and $\hat{I}^C$ independent unbiased estimators for $\mathbb{P}(C)$ and $\mathbb{E}[I|C]$, respectively. As $\mathbb{P}(C)$ and $\mathbb{E}[I|C]$ are independent, $\hat{I}$ is obviously an unbiased estimator for $\mathbb{E}[I]$, and its variance is

$$\mathrm{Var}\left(\hat{I}\right) = \mathrm{Var}\left(\hat{P}\hat{I}^C\right) = \mathbb{E}^2[I|C]\,\mathrm{Var}\left(\hat{P}\right) + \mathbb{P}(C)^2\,\mathrm{Var}\left(\hat{I}^C\right) + \mathrm{Var}\left(\hat{P}\right)\mathrm{Var}\left(\hat{I}^C\right).$$

Dividing by $\mathbb{P}(C)^2\mathbb{E}^2[I|C]$ results in a decomposition of the squared relative error of $\hat{I}$

$$\mathrm{SRE}(\hat{I}) = \mathrm{SRE}\left(\hat{P}\right) + \mathrm{SRE}\left(\hat{I}^C\right) + \mathrm{SRE}\left(\hat{P}\right)\mathrm{SRE}\left(\hat{I}^C\right). \tag{11}$$

Expression (11) basically states that to control the precision of the index estimator, one should control the precision of both the probability estimator $\hat{P}$ and the conditional estimator $\hat{I}^C$.

4.1 Controlling the Probability Estimator Precision

As $\hat{P}$ is the estimator of a rare event probability, a splitting technique may control $\mathrm{SRE}(\hat{P})$ using significantly less workload compared to a CMC method. The basic idea of a splitting technique is to decompose the probability into several conditional probabilities that are separately estimated using less total computational effort. This is done by splitting each sample path into multiple paths whenever the process is substantially closer to the rare event set. This subsection we adapt a splitting technique for the described MC reliability estimation method.

By defining the vector of discrete time Markov processes $\mathbf{X}(t) = \begin{pmatrix} -|\mathbf{V}(t)| & |\mathbf{V}(t)| & |\mathbf{I}(t)| \end{pmatrix}$ with state space $\mathscr{E}$, we can concatenate the three vector inequalities of curtailment constraints (2)-(3):

$$\mathbf{X}(t) \leq \mathbf{U}, \quad \text{with} \quad \mathbf{U} = \begin{pmatrix} -\mathbf{V}_{\min} & \mathbf{V}_{\max} & \mathbf{I}_{\max} \end{pmatrix}.$$

In accordance with a general splitting procedure, we should choose an appropriate importance function $h : \mathscr{E} \mapsto \mathbb{R}$, assigning importance values to the states of $\mathbf{X}(t)$. Increasing values of h should correspond to approaching the rare event. We propose an intuitive importance function that takes the maximum over all ratios between the state variables and their allowed extrema:

$$h(\mathbf{X}(t)) = \max_i \left(\frac{\mathbf{X}_i(t) - \mathbf{L}_i}{\mathbf{U}_i - \mathbf{L}_i} \right), \quad \text{with} \quad \mathbf{L} = \begin{pmatrix} -\frac{\mathbf{V}_{\max}+\mathbf{V}_{\min}}{2} & \frac{\mathbf{V}_{\max}+\mathbf{V}_{\min}}{2} & \mathbf{0} \end{pmatrix}. \tag{12}$$

Here the subscript $i = 1, \ldots, 2N+M$ denotes the index of the vector. For $h(\mathbf{X}(t)) < 1$, h indicates how relatively close we are to a constraint violation at time t. If for some constraint $\mathbf{X}_i(t) > \mathbf{U}_i$, then $h(\mathbf{X}(t)) > 1$, signifying that the rare event is hit, i.e. a curtailment occurs at time t. The linear transformation of each $\mathbf{X}_i$ in (12) ensures that the ratio of each constraint type has the same codomain $[0,1]$ as long as the rare event is not hit, and $h \nearrow 1$ corresponds to approaching the rare event. In this sense, each constraint is assumed equally important when evaluating $h(\mathbf{X}(t))$.

Using importance function (12), a general splitting technique is set up. The codomain $[0,1]$ is partitioned into m subintervals with boundaries $0 = l_0 < l_1 < \cdots < l_m = 1$. We define $T_k = \inf\{t > 0 : h(\mathbf{X}(t)) \geq l_k\}$ as the time of hitting the k-th level and $D_k = \{T_k < T\}$ as the event that the k-th level is hit during $[0,T]$. Obviously, $\mathbb{P}(D_0) = 1$, and $\mathbb{P}(D_m) = \mathbb{P}(C)$ is the value that we are interested in. As $D_m \subset D_{m-1} \subset \cdots \subset D_0$, we decompose the probability of interest

$$\mathbb{P}(C) = \prod_{k=1}^{m} \mathbb{P}(D_k|D_{k-1})$$

in m conditional probabilities $p_k := \mathbb{P}(D_k|D_{k-1})$, which we will estimate separately. Generating independent sample paths from G_{k-1}, the distribution of the entrance state $(T_{k-1}, \mathbf{X}(T_{k-1}))$ conditional on D_{k-1} would

give us an estimate for p_k. However, as we do not know G_{k-1} for $k > 1$, we use the empirical distribution $\hat{G}_{k-1}$ for G_{k-1}, obtained from samples of the previous stage. In this way, we proceed recursively, and we estimate p_k at each stage k by the proportion of level hits

$$\hat{p}_k = R_k / N_{k-1},$$

with R_k the number of sample paths where D_k occurs and N_k the total number of sample paths at stage k.

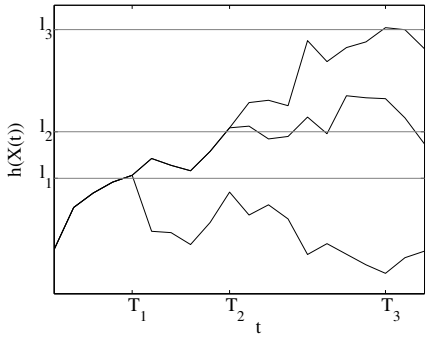

Figure 1: *Example of an importance function realization splitting two times with* $N_1 = 1$, $N_2 = N_3 = 2$ *and* $R_1 = R_2 = R_3 = 1$. *The hit* $h(\mathbf{X}(t)) > l_3$ *would correspond to a constraint violation somewhere in the grid.*

Amrein and Künsch (2011) prove that for several splitting techniques,

$$\hat{P} := \prod_{k=1}^{m} \hat{p}_k$$

is an unbiased estimate for $\mathbb{P}(D_m)$. One of these techniques, Fixed Splitting (FS), determines in advance the number of splittings per level hit. A disadvantage of this technique is that, especially when the number of stages and the level heights are far from optimally chosen, paths may hit the next level many times or never. The former situation results in a high computational burden, the latter situation in a useless estimate $\hat{P} = 0$. The splitting technique Fixed Effort (FE) attempts to overcome these issues by fixing N_k, and thereby fixing the expected workload per stage. Unfortunately, as the realized number of hits r_k may differ from the expected number of hits $\mathbb{E}[R_k]$, the same issues, although to a lesser extent, may persist. Amrein and Künsch (2011) proposes the Fixed Number of Successes (FNS) splitting technique, where the authors fix the number of hits r_k, repeating the simulation at one stage until r_k hits are observed. This technique avoids the issues of path explosion and extinction by adjusting the computational effort. We choose to extend the CMC method with this splitting technique.

The optimal level heights and number of stages are not known beforehand, and the authors recommend a pilot run to determine these parameters. This pilot run uses a large number of equidistant levels and $r := r_k$ of moderate size, say $r = 20$, yielding first estimates $\hat{p}_k$. The optimal value for p_k in terms of variance reduction is $p_{\text{opt}} \approx 0.2032$ (Amrein and Künsch 2011). A pilot run estimate for $\hat{p}_k$ close to one suggests to merge stage k with a neighboring stage, whereas an estimate close to zero suggests to divide the stage into multiple stages. More precisely, one finds an improved stage partition for the final run by interpolating the pilot stage partition on the log scale.

An unbiased estimator for the variance of the FNS (and FE) estimate is not known. However, Amrein and Künsch (2011) showed that under the assumption that the conditional hitting probability does not depend on the entrance state of the previous stage,

$$\mathbb{P}(D_k | D_{k-1}, (T_{k-1}, \mathbf{X}(T_{k-1}))) = \mathbb{P}(D_k | D_{k-1}) \quad \text{for all} \quad (T_{k-1}, \mathbf{X}(T_{k-1})), \quad \text{for all} \quad k, \tag{13}$$

one can bound the squared relative error of $\hat{P}$ by choosing r_k appropriately:

$$\mathrm{SRE}(\hat{P}) \leq q := -1 + \prod_{k=1}^{m} \left(\frac{1}{r_k - 2} + 1 \right). \tag{14}$$

In this way, we are able to control the precision of $\hat{P}$.

4.2 Controlling the Conditional Estimator Precision

We can reuse the sample paths that hit the rare event set to estimate $\mathbb{E}[I|C]$, by performing one additional splitting stage. We randomly choose one of the r_m realizations of the last stage entrance state $(T_m, \mathbf{X}(T_m))$ and continue to generate the path from this point up till time T, obtaining an index realization $\hat{I}_i^C$. Repeating this n times yields the estimator

$$\hat{I}^C := \frac{1}{n} \sum_{i=1}^{n} \hat{I}_i^C \tag{15}$$

for $\mathbb{E}[I|C]$. One might question whether I_i^C is unbiased as it depends on a randomly chosen realization of $(T_m, \mathbf{X}(T_m))$, instead of directly on $(T_m, \mathbf{X}(T_m))$. However, randomly choosing this realization is equivalent to sampling it from the empirical distribution $\hat{G}_m$ of the distribution G_m of $(T_m, \mathbf{X}(T_m))$. Therefore, as $\hat{G}_m$ is an unbiased estimator for G_m, we conclude that I_i^C (and thus I^C) is indeed an unbiased estimator for $\mathbb{E}[I|C]$. Appendix A contains a more rigorous proof.

Estimators $\hat{I}^C$ and $\hat{P}$ are indeed independent, a fact that we used at the start of Section 4, before we explicitly defined these estimators. Further, the construction of FNS (in contrast to FS and FE) ensures the existence of realizations $(T_m, \mathbf{X}(T_m))$ and thus $\hat{I}^C$ as long as the rare event probability $\mathbb{P}(C)$ is not zero. We can estimate all desired indices using only one simulation run, just as in a similar CMC run. Furthermore, we attain an arbitrarily small $\mathrm{SRE}(\hat{I}^C)$ by choosing n sufficiently large, thereby controlling the conditional estimate precision. As we do not estimate a rare event probability here, we expect the computational intensity to be negligible compared to the estimation of $\mathbb{P}(C)$.

5 PERFORMANCE RESULTS ON AN EXAMPLE GRID

We will investigate the computational intensity of the proposed FNS technique for grid reliability estimation on a very simple transmission grid. As shown in Figure 2, the grid exists of one wind farm node, one consumption node and one so-called slack node, where the total surplus or shortage of power is absorbed or emitted, respectively.

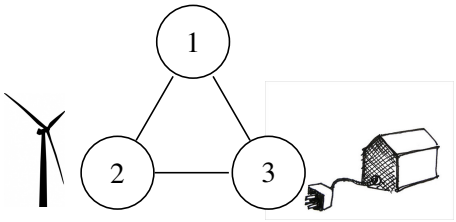

Figure 2: *A small power grid with one slack node (1), one intermittent source node (2), and one consumption node (3).*

All nodes are connected by identical transmission lines. The discrete time domain is one week. For simplicity, we omit periodicities and model the active power $P_2(t)$ of the wind farm in node 2 as an Ornstein-Uhlenbeck (OU) process

$$dP_2(t) = \theta(\mu - P_2(t))dt + \sigma dW(t).$$

Here $W(t)$ denotes a Brownian motion and the long-term mean $\mu = P_3 = 15$ base MVA (using the scalable per-unit system (Grainger and Stevenson 1994)) equals the constant active power consumption at node 3. The process initiates under normal conditions by setting $P_2(0) = \mu$, and realistic values for the mean-reverting rate $\theta = 0.13$ per hour and volatility $\sigma = 1.3$ are chosen by fitting the model to historical wind power measurements (KNMI 2013). We choose the extrema $\mathbf{V}_{\min}, \mathbf{V}_{\max}$ and $\mathbf{I}_{\max}$ such that $\mathbb{P}(C)$ is indeed small (around 10^{-4}), using a preliminary CMC simulation with 100 000 MC samples. Other parameter values can be found in Appendix B.

The results of the pilot run as described in Section 4.1 recommend to use $m = 6$ stages in this case. Then according to the squared relative error bound in (14), choosing $r_k = r = 100$ will yield a squared relative error smaller than $q = 0.063$. Assuming asymptotic normality of $\hat{P}$, we will obtain the conservative 95% confidence interval

$$[\hat{P} - 1.96\sqrt{q}\hat{P}, \hat{P} + 1.96\sqrt{q}\hat{P}] \approx [\hat{P}/2, 3\hat{P}/2].$$

Similarly, $r_k = r = 40$ will yield $q = 0.17$ and thus a conservative confidence interval of the form $[c, 10c]$ for some c, rather indicating the order of magnitude. For the final run, we choose $r = 100$, a time step size of 6 minutes and the importance function h as in (12). We further assume that (13) and thus (14) hold. The resulting index estimates are displayed in the first column of Table 2. The second column lists estimates $\widehat{\mathrm{SRE}}\left(\hat{I}^C\right)$ of $\mathrm{SRE}(\hat{I}^C)$. Finally, (11) suggests to estimate the bound for $\mathrm{SRE}(\hat{I})$ by

$$q_{\hat{I}} := q + \widehat{\mathrm{SRE}}\left(\hat{I}^C\right) + q\widehat{\mathrm{SRE}}\left(\hat{I}^C\right).$$

Values for $q_{\hat{I}}$ are displayed in the third column. Since the estimates for $\mathrm{SRE}(\hat{I}^C)$ will contain an error, the bound $\mathrm{SRE}(\hat{I}) < q_{\hat{I}}$ for the relative error of the index estimate is not guaranteed. However, as explained in Section 4.2, we can have $\mathrm{SRE}(\hat{I}^C)$ arbitrarily small by choosing sufficiently large n in (15). The estimates for $\mathrm{SRE}(\hat{I}^C)$ are indeed significantly smaller than q (using only $n = 5r = 500$ samples, with a workload comparable to that of one splitting stage). Therefore, the conservative estimate $\mathrm{SRE}(\hat{I}) \approx q_{\hat{I}}$ will be satisfactory for most practical reliability assessments.

Table 2: *Estimates $\hat{I}$ for indices (4)-(7) of the three node grid reliability over one week, the squared relative error estimate $\widehat{SRE}\left(\hat{I}^C\right)$ of the conditional index (standard error in parentheses), and the corresponding bound estimate $q_{\hat{I}}$ for $SRE(\hat{I})$. The time step size is 6 minutes, and $q = 0.063$.*

	$\hat{I}$	$\widehat{\mathrm{SRE}}\left(\hat{I}^C\right)$	$q_{\hat{I}}$
$PPC(T)$	6.35e-5	0	0.0628
$EDPC(T)$ (hour)	2.89e-5	2.4e-3 (1.5e-4)	0.0710
$ENPC(T)$	1.01e-4	6.1e-4 (3.9e-5)	0.0666
$EENS(T)$ (per-unit MWh)	2.18e-5	5.7e-3 (3.6e-4)	0.0819

In total 3075 MC samples where required using the FNS technique. To obtain a CMC estimate for $PPC(T)$ with a comparable squared relative error, equation (10) suggests to use as many as 250 000 MC samples. This workload decrease of a factor 79 illustrates the computational gain of the FNS technique compared to the CMC method. For smaller values of $PPC(T)$, the gain will be even larger.

5.1 Choice for the Time Step Size

The approximation $\mathrm{SRE}(\hat{I}) \approx q_l$ relies on assumption (13), which holds as long as the level hitting probability does not depend on the entrance state $(T_{k-1}, \mathbf{X}(T_{k-1}))$ of the previous stage. We will explain that this is not necessarily the case in the proposed power grid model. Repeating the FNS index estimation would yield an unbiased estimate of the variance under general circumstances. However, this will heavily increase the computational effort, so we are interested under which circumstances (13) and thus (14) hold approximately.

Note that a sample path of $h(\mathbf{X}(t))$ may skip stage k entirely by a very large increment in a single time step. Then any sample path started from this entrance state has immediately hit level k, as shown in Figure 3. This example shows the dependence of the hitting probability on $\mathbf{X}(T_{k-1})$. In general, large increments are undesirable as they may increase the squared relative error beyond bound q.

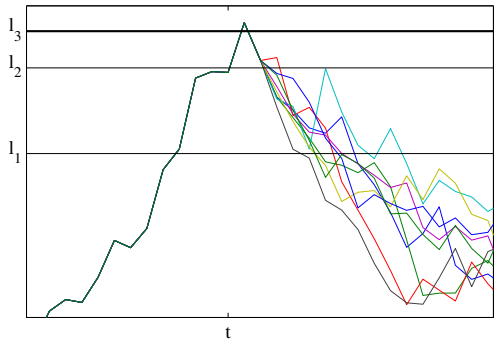

Figure 3: *The problem of too large increments in an extreme example: the only survived path skips level 3 entirely at time t, wherefore* $\hat{p}_3 = 1$ *will probably overestimate* p_3.

As we can not bound an OU increment, there is no step size in general that bounds the increments of $h(\mathbf{X}(t))$. However, decreasing the time step size Δ may reduce the variance of an OU increment sufficiently for (13) to hold approximately. Using the exact solution of the OU process (Gillespie 1996), one can derive the OU increment variance

$$\mathrm{Var}(P_2(t+\Delta) - P_2(t)) = \frac{\sigma^2}{2\theta}\left(2\left(1-e^{-\theta\Delta}\right) - e^{-2\theta t}\left(1-e^{-\theta\Delta}\right)^2\right),$$

which indeed approaches zero as $\Delta \to 0$. By computing FNS estimates for $PPC(T)$ 50 times, we estimate $\mathrm{SRE}(\hat{P})$ and investigate for which time step sizes the squared relative error bound (14) holds. Table 3 shows that a step size of one hour is too large (as $\mathrm{SRE}(\hat{P}) \not< q$), whereas runs with a step size of 6 minutes or smaller are consistent with (14). This is in agreement with the statement in Frunt (2011) that one will not fully capture the typical variability of wind power generation when using step sizes larger than 10 minutes. Although the optimal time step size is a priori unknown, such knowledge on the variability of the stochastic sources may hint a proper choice for Δ. Further, the typical increment size of h in the pilot run described in Section 5 may test as well whether the time step is sufficiently small.

Table 3: *Estimates for* $SRE(\hat{P})$*, using importance function h in (12). Estimates are consistent with bound (14) for sufficiently small time step sizes* Δ*. We used 50 estimates for* $\hat{P}$.

Δ	$q = 0.169,\ (r = 40)$	$q = 0.063,\ (r = 100)$
1 hour	0.847	0.482
6 minutes	0.168	0.051
1 minute	0.164	0.034

5.2 Choice for the Importance Function

We would like to address the relevance of choosing a suitable importance function. For example, one may have chosen the importance function

$$h_1(\mathbf{X}(t)) = \max_i \left((\mathbf{X}_i(t) - \mathbf{U}_i)/\mathbf{U}_i\right) \tag{16}$$

instead of (12), where $h_1(\mathbf{X}(t)) \nearrow 0$ corresponds to approaching the rare event. Since now the codomain of arguments $(\mathbf{X}_i(t) - \mathbf{U}_i)/\mathbf{U}_i$ differ per constraint i, some constraint ratios may be much more volatile than others, which may significantly increase the probability of large increments. This can be seen in the three node power grid if we reestimate $\mathrm{SRE}(\hat{P})$ using h_1. Table 4 shows that $\mathrm{SRE}(\hat{P})$ is larger than q, even for small step sizes. This illustrates the relevance of a choosing a suitable importance function.

Table 4: *Estimates for SRE($\hat{P}$), using importance function h_1 in (16). Estimates exceed bound q, even for small step sizes Δ. We used 100 estimates for $\hat{P}$.*

Δ	$q = 0.169,\ (r = 40)$	$q = 0.063,\ (r = 100)$
1 hour	2.18	1.26
6 minutes	1.28	0.38
1 minute	1.18	0.32

A priori known (or recursively learned) relations between certain system states and the conditional probabilities may help to improve the choice for the importance function. Finding and using these relations is part of further research.

6 CONCLUSION AND RECOMMENDATIONS

We demonstrated the high computational intensity of a typical Crude Monte Carlo method for reliability estimation of electrical power grids. We showed in Table 2 that a splitting technique may decrease the workload of estimating various reliability indices by orders of magnitude while controlling the squared relative error of the estimators. The reliability indices may either be the rare event probability of a power curtailment or an expectation depending on this probability. An implementation on a small transmission network shows that the proposed method requires 79 times less workload to estimate four common reliability indices than an equivalent Crude Monte Carlo simulation would require. To control the squared relative error of the estimator, an appropriate choice for the time step size and the importance function is crucial. The time step size should be sufficiently small to capture the typical variability of the stochastic power sources. Furthermore, the importance function should assign equal importance to all curtailment risks, like importance function (12) does.

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A PROOF THAT ESTIMATOR IS UNBIASED

To prove that $\hat{I}^C$ in (15) is an unbiased estimator for $\mathbb{E}[I|C]$, note that $\hat{I}_i^C = \hat{I}_i^C(Z)$ is a function of $Z \sim \hat{G}_m$, the entrance state randomly chosen from all simulated entrance states $Z_1, \ldots Z_{r_m} \sim G_m$ into the rare event. So $\hat{G}_m$ is the empirical distribution of G_m and we can write

$$\mathbb{E}_{\hat{G}_m}\left[\hat{I}_i^C(Z)\middle|\ \text{all but } Z\right] = \frac{1}{r_m}\sum_{j=1}^{r_m}\hat{I}_i^C(Z_j).$$

Here we conditioned on all random variables that I_i^C depends on except Z, that is: $Z_1, \ldots, Z_{r_m}$ and the sample path from Z on. Unbiasedness of $\hat{I}^C$ immediately follows since

$$\mathbb{E}\left[\hat{I}^C\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\hat{I}_i^C\right] = \mathbb{E}\left[\hat{I}_1^C(Z)\right] = \mathbb{E}\left[\mathbb{E}_{\hat{G}_m}\left[\hat{I}_1^C(Z)\middle|\ \text{all but } Z\right]\right] = \frac{1}{r_m}\sum_{j=1}^{r_m}\mathbb{E}\left[\hat{I}_1^C(Z_j)\right] = \mathbb{E}\left[\hat{I}_1^C(Z_1)\right] = \mathbb{E}[I|C],$$

where we used the law of total expectation in the third equality.

B PARAMETERS OF THE THREE NODE GRID

The injected reactive wind power is equal to one third of the active power: $Q_2(t) = P_2(t)/3$. Similarly, for the consumed reactive power in node 3: $Q_3 = P_3/3$. Admittance matrix Y consists of elements $Y_{ii} = -200\imath$ and $Y_{ij} = 100\imath$ for $i \neq j$, where $\imath$ is the imaginary unit.

To estimate the curtailment size $S_C(t)$ in (7) at an unfeasible time step (that is, when a constraint is violated), we assume a simple strategy: the grid operator instantaneously reschedules generation as it was during the last feasible time step. The absolute differences between all nodal powers at the feasible and unfeasible situations sum up to the power curtailment size at time t:

$$S_C(t) = \sum_{\text{nodes } i} \left| P_i^{\text{feasible}} - P_i^{\text{unfeasible}} \right|. \tag{17}$$

After obtaining $S_C(t)$ at all discrete times steps, numerical time integration yields a realization of $EENS(T)$. More realistic strategies would include a linear optimization problem where the curtailment size is minimized such that all curtailment constraints are satisfied. As we expect a curtailment to be rare, we do not expect significant increase of the total simulation workload if we replace the curtailment strategy (17) by a more realistic one.

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AUTHOR BIOGRAPHIES

WANDER WADMAN is a PhD student in the Computational Dynamics group at CWI Amsterdam. He received a master's degree in Applied Mathematics at Delft University of Technology. His PhD topic is the stochastic modeling of electric power systems, and his research interests include stochastic modeling, power flow problems and Monte Carlo techniques with a focus on rare event simulation. His email address is w.wadman@cwi.nl and his web page is http://homepages.cwi.nl/~wadman/.

DAAN CROMMELIN is a researcher in the Computational Dynamics group at CWI Amsterdam. His research focuses on stochastic methods and statistical inference for dynamical systems, with applications in atmosphere-ocean science and energy systems. He received his PhD from Utrecht University in 2003, worked as a postdoc at New York University's Courant Institute and joined CWI Amsterdam in 2006. His email address is daan.crommelin@cwi.nl and is webpage is at www.cwi.nl/~dtc.

JASON FRANK is professor at the University of Amsterdam in the Chair for Numerical Analysis and Dynamical Systems. He received his PhD in numerical analysis at Delft University of Technology and CWI Amsterdam. Since January 2000 he is employed by CWI Amsterdam as researcher and leader of the research group Computational Dynamics. His research addresses geometric numerical methods for partial differential equations, with applications in geophysical fluid dynamics and nonlinear waves. His email address is j.e.frank@cwi.nl.