Applying Covariational Reasoning While Modeling Dynamic Events:

A Framework and a Study

Marilyn Carlson, Sally Jacobs, Ted Coe, and Sean Larsen, *Arizona State University*

Eric Hsu, *San Francisco State University*

ABSTRACT

The article develops the notion of covariational reasoning and proposes a framework for describing the mental actions involved in applying covariational reasoning when interpreting and representing dynamic function events. It also reports on an investigation of high-performing second semester calculus students' ability to reason about covarying quantities in dynamic situations. The study revealed that these students were able to construct images of a function's dependent variable changing in tandem with the imagined change of the independent variable, and in some situations, were able to construct images of rate of change for contiguous intervals of a function's domain. However, students appeared to have difficulty forming images of continuously changing rate and were unable to accurately represent and interpret increasing and decreasing rate for dynamic function situations.

INTRODUCTION

Dating to the late19th century, there have been repeated calls to increase the emphasis on functions in school curriculum (Klein, 1883, as cited in Hamley, 1934; MAA, 1921; NCTM, 1934, 1989, 2000; CEEB, 1959). More recently, the literature on early function instruction supports the promotion of conceptual function thinking that includes investigations of patterns of change (NCTM 1989, 2000; Sfard, 1989; Thorpe, 1989; Vinner & Dreyfus, 1989; Monk, 1992; Kaput, 1994). Both in 1989 and in 2000, the authors of the *National Council of Teachers of Mathematics Standards* documents called for students to be able to analyze patterns of change in various contexts. They recommended that students learn to interpret statements such as “the rate of inflation is decreasing”, and in general, promoted that students develop a “deeper understanding of the ways in which changes in quantities can be represented mathematically” (NCTM, 2000, p. 305). Additionally, the authors of the *National Science Education Standards* (1996) have called for the use of mathematical functions to identify patterns and anomalies in data (p. 174).

It is not clear to what extent mathematics curriculum has responded to these calls (Cooney & Wilson, 1993). Research suggests that undergraduate students are entering the university with weak understandings of functions, and entry level university courses do little to address this deficiency (Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994a; Carlson, 1998). Recent investigations of college students’ understandings of functions have documented that even academically talented undergraduate students have difficulty modeling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependency relationship with another variable (Monk & Nemirovsky, 1994; Thompson, 1994a; Carlson, 1998). Research has also shown that this ability is essential for interpreting models of dynamic events (Kaput, 1994; Rasmussen, 2000) and is foundational for understanding major concepts of calculus (Kaput, 1994; Thompson, 1994a; Cottrill, 1996; Zandieh, 2000) and differential equations (Rasmussen, 2000).

In studying the process of acquiring an understanding of dynamic functional relationships, Thompson (1994b) has suggested that the concept of rate is foundational. A mature image of rate, says Thompson, involves: the construction of an image of change in some quantity, the coordination of images of two quantities, and the formation of an image of the simultaneous covariation of two quantities. These phases run parallel to Piaget's (1977) three-stage theory about children's mental operations involved in functional thinking about variation. Also contributing to our understanding of the notion of covariation is the work of Saldhana and Thompson (1998) who describe covariation as "someone holding in mind a sustained image of two quantities values (magnitudes) simultaneously" (p. 298). This involves the coordination of the two quantities; then tracking either quantities value with the realization that the other quantity also has a value at every moment in time. In this theory, images of covariation are viewed as developmental, with the development evolving from the coordination of two quantities, to images of the continuous coordination of both quantities for some duration of time. According to Saldhana and Thompson (1998), "In early development, one coordinates two quantities' values -- think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities values persist" (p. 298).

Confrey and Smith (1995) see a covariation approach to creating and conceptualizing functions as involving the formation of linkages between values in a function's domain and range. In the case of tables, it involves the coordination of the variation in two or more columns as one moves up and down the table (Confrey & Smith, 1994). For both Confrey and Smith (1995), and Thompson (1994a) *coordinating* is described as foundational for reasoning about dynamic function relationships. Even though the covariation of two quantities does not always require the notion of time, it is through the metaphor of "exact time" for the imagined location of a "moving point" that Confrey and Smith, and many others discuss covarying quantities (e.g., Monk's (1992) "across-time" function view and Nemirovsky's (1996) variational versus pointwise approach).

In this paper, we propose a framework for the study of covariational reasoning and illustrate how this framework can be used to analyze students' understanding about dynamic situations involving two simultaneously changing quantities. We also present problems that evoke and require the use of covariational reasoning, and in doing so, illustrate features of curriculum that emphasize a covariational approach to learning functions. We describe our research findings about high-performing second-semester calculus students' covariational reasoning abilities and discuss implications of these results.

DEFINITIONS

Informed by these studies and our research from the last several years (Carlson, 1998; Carlson & Larsen, 2001; Carlson, Jacobs, & Larsen, 2001) we define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other. We concur with Saldanha's and Thompson's (1998) view that images of covariation are developmental. We use the term developmental in the Piagetian sense (Piaget, 1970) to mean that the images of covariation can be defined by level and the levels emerge in an order of succession. Throughout this paper, our use of the word *image* is consistent with the description provided by Thompson (1994b). The construct of *image* is portrayed as, "dynamic, originating in bodily actions and movements of attention, and as the source and carrier of mental operations" (p. 231). The notion of *image* is not inconsistent with that of *concept image* as defined by Vinner and Dreyfus (1989) (i.e., the mental pictures, visual representations, experiences, properties and impressions associated with a concept name by an individual in a given context); however, its focus is on the dynamics of mental operations (Thompson, 1994b).

The term *pseudo-analytical thought processes* and *behaviors* are characterized as processes and behaviors that lack understanding, and *pseudo-analytic behaviors* are produced by *pseudo-analytical thought processes* (Vinner, 1997). "*Pseudo-analytic behaviors* describe a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior" (Vinner, 1997, p. 100). According to Vinner these behaviors and thought processes are not necessarily negative and may be the result of "spontaneous, natural, but uncontrolled associations" (p. 125). *Pseudo-analytical behaviors* differ from *pseudo-conceptual* in that the focus is on the analytic process rather than the concept; however, these two ideas should not be seen as mutually exclusive as there are some contexts in which both the analytical and conceptual modes are involved.

We use the word rate to mean the average rate of change in the case of imagining a subinterval, or instantaneous rate of change in the case of imagining a function over its entire domain.

BACKGROUND

In recent years our understanding of ways in which college students interpret and represent dynamic function situations has been informed by considerable research (Monk, 1992; Sierpinska, 1992; Kaput, 1994; Nemirovsky, 1996; Thompson, 1994b; Carlson, 1998). In examining the thinking of calculus students when attempting to interpret the changing nature of "rate of change" for intervals of a function's domain, several studies (Monk, 1992; Monk &

Nemirovsky, 1994; Thompson, 1994a; Nemirovsky, 1996; Carlson, 1998) have revealed that this ability is slow to develop, with specific problems reported in students' ability to interpret graphical function information. Studies by Monk (1992) and Kaput (1992) have noted that calculus students show a strong tendency to become distracted by the changing shape of the graph, and in general, do not appear to view a graph of a function as a means of defining a covarying relationship between two variables. Other studies have found that calculus students have difficulty interpreting and representing concavity and inflection points on a graph (Monk, 1992; Carlson, 1998). Even when directly probed to describe their meaning in the context of a dynamic real-world situation, students made statements such as, "second derivative positive, concave up; and second derivative equal to zero, inflection point" (Carlson, 1998). Further probing revealed that these students appeared to have no understanding of why this procedure worked and in general did not appear to engage in behaviors that were suggestive of their coordinating images of two variables changing concurrently. Tall (1992) also found that, although college students' concept images of function included a correspondence notion, the idea of operation, an equation, a formula and a graph, it did *not* include the conception of two variables changing in tandem with each other.

Research into students' developing function conception has revealed that a view of function as a process that accepts input and produces output (Breidenbach, Dubinsky, Hawks & Nichols, 1992) is essential for the development of a mature image of function. This view has also been shown to be foundational for coordinating images of two variables changing in tandem with each other (Thompson, 1994a; Carlson, 1998). According to Thompson (1994a), "Once students are adept at imagining expressions being evaluated continually as they 'run rapidly' over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs" (p. 27).

The covariation view of function has also been found to be essential for understanding concepts of calculus (Kaput, 1992; Thompson, 1994b; Cottrill, 1996; Zandieh, 2000). Students' difficulties in learning the limit concept have been linked to impoverished covariational reasoning abilities. In a recent study, Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic (1996) recommended that the limit concept should begin with the informal dynamic notion of the "values of a function approaching a limiting value as the values in the domain approach some quantity" (p.6). The development of this "coordinated" process schema of limit was found to be non-trivial for students and has been sited as a major obstacle to students' developing limit conception (Cottrill et al., 1996).

In describing her framework for analyzing student understanding of derivative, Zandieh (2000) also suggested that a view of function as the covariation of the input values with the output values is essential. In her framework, she stated that "the derivative function acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at that point" (p.107). In this description, she emphasizes the notion that the derivative function results from covarying the input values of the derivative function with the "rate of change" values of the original function.

Thompson (1994b) suggested that covariational reasoning is foundational for students' understanding of the Fundamental Theorem of Calculus. "The Fundamental Theorem of Calculus—the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related is one of the intellectual hallmarks in the development of the calculus" (p. 130). When interpreting the information conveyed by a speed function, the total distance traveled relative to the amount of time passed is imagined as the coordination of accruals of distance and accruals of time (Thompson, 1994b).

Collectively, these studies suggest that covariational reasoning is foundational for understanding major concepts of calculus and that conventional curriculum has not been effective in promoting this reasoning ability in students. Building on these findings, our study investigated the complexity of constructing mental processes involving the *rate of change* as it continuously changes in a functional relationship. A framework for investigating covariational reasoning is described in the next section.

THEORETICAL FRAMEWORK

Covariational Reasoning

The covariation framework was informed by past studies (Carlson, 1998; Carlson & Larsen, 2001; Carlson, Jacobs & Larsen, 2001) and provides a lens for analyzing and reporting students' covariational reasoning abilities when responding to dynamic function tasks. The framework describes five developmental levels of images of covariation that are successively more sophisticated and complex (Table 2). The five developmental levels are described in terms of the mental actions or operations that each image supports (Table 1). Table 1 provides a description of the five mental actions and associated behaviors that have previously been identified in students (Carlson, 1998).

Table 1
Mental Actions of the Covariation Framework

Mental Action	***Description of Mental Action***	***Behaviors***
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• constructing an increasing straight line • verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• plotting points/constructing secant lines • verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• constructing contiguous secant lines for the domain • verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function	• constructing a smooth curve with clear indications of concavity changes • verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)

The notion of image used in describing the levels of the framework is consistent with Thompson's (1994a) characterization of an image as that which "focuses on the dynamics of mental operations" (p. 231). As an individual's image of covariation develops, it supports more sophisticated covariational reasoning (recall that we define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other). The framework begins with an initial image of two variables changing in tandem (Level 1). We say that one's covariational reasoning ability has reached a given level of development when it supports the mental actions associated with that level *and* the actions associated with all lower levels. A student who is using Level 5 covariational reasoning is able to coordinate the instantaneous rate of change of the function with continuous changes in the independent variable (MA5), and is able to unpack that information to apply mental actions 1-4 as desired or needed.

Table 2
Levels of the Covariation Framework

Covariational Reasoning Levels

The covariation framework describes five levels of development of images of covariation. These images of covariation are described in terms of the mental actions supported by each image.

Level 1. Coordination.
At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).

Level 2. ***Direction.***
At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions MA1 and MA2 are BOTH supported by Level 2 image.

Level 3. Quantitative Coordination.
At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions MA1, MA2 and MA3 are supported by Level 3 image.

Level 4. Average Rate.
At the average rate level, the images of covariation can support the mental actions of coordinating the average rate-of-change of the function with uniform changes in the input variable. The average rate-of-change can be unpacked to coordinate the amount of

change of the output variable with changes in the input variable. The mental actions MA1 through MA4 are supported by Level 4 image.

Level 5. Instantaneous Rate.

At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate-of-change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate-of-change resulted from smaller and smaller refinements of the average rate-of-change. It also includes awareness that the inflection point is where the rate-of-change changes from increasing to decreasing, or decreasing to increasing. The mental actions MA1 through MA5 are supported by Level 5 image.

* levels are local, just MAs

We note that some students have been observed exhibiting behaviors that gave an appearance of engaging in a specific mental action; however, when probed, did not provide evidence of possessing an understanding that supported the behavior. We refer to this as pseudo-analytical behavior (i.e., the underlying understanding for performing the specific behavior is not present (Vinner, 1997)) and describe the mental action that produced the behavior as a pseudo-analytical mental action. (Pseudo-analytic behavior is produced by pseudo-analytical thought processes (Vinner, 1997)). We also re-emphasize that a student is only classified as having a specific covariational reasoning ability level (say, Level 5) if he/she is able to perform the mental action associated with that level (MA5) and all lower numbered mental actions (MA1 through MA4). In other words it is possible for a student to exhibit mental action 5 without applying Level 5 covariational reasoning (see Bottle Problem Interview Excerpts, Student C).

The proposed covariation framework provides an analytical tool with which to analyze covariational thinking to a finer degree than has been done in the past. In addition, it provides a structure and language for classifying covariational thinking in the context of responding to a specific problem, and for describing a student's general covariational reasoning abilities (i.e., developmental level in the framework).

Covariational Reasoning in a Graphical Context

Students' covariational reasoning abilities are important for interpreting and representing graphical function information. Since this is the context in which we initially observed student difficulties, this is where much of our work has focused. A close look at students' covariational reasoning in the context of a graph reveals that students who exhibit behaviors supported by MA1 typically recognize that the value of the *y*-coordinate changes with changes in the value of the *x*-coordinate. (Typically, the *x*-coordinate plays the role of the independent variable, although we have observed students treating the *y*-coordinate as the independent variable). This initial coordination of the variables is commonly revealed by a student labeling the coordinate axes of the graph, followed by utterances that demonstrate recognition that as one variable changes the other variable changes. Attention to the direction of change (in the case of an increasing function) involves the formation of an image of the *y*-values getting higher as the graph moves from left to right (MA2). The common behavior displayed by students at this level has been the construction of a line that rises as one moves to the right on the graph or utterances that suggest an understanding of the direction of change of the output variable while considering increases in the input variable (e.g., as more water is added, the height goes up). MA3 involves the coordination of the relative magnitudes of change in the *x* and *y* variables. In this context, students have been observed partitioning the *x*-axis into intervals of fixed lengths (e.g., x_1, x_2, x_3, x_4), while considering the amount of change in the output for each new interval of the input. This behavior has been commonly followed by the student constructing points on the graph (the points are viewed as representing amounts of change of the output while considering equal amounts of the input) and is followed by the construction of lines to connect these points. Activity at the rate level involves recognition that the amount of change of the output variable with respect to a uniform increment of the input variable expresses the rate of change of the function for an interval of the function's domain. This is typically revealed by the student sketching secant lines on a graph or by carrying out the mental computation or estimation of the slope of a graph over small intervals of the domain (the sketching of these lines would result from the student imagining and adjusting slopes for different intervals of the domain). It is noteworthy that mental actions 3 and 4 may both result in the construction of secant lines; however, the type of reasoning that produces these constructions is different (i.e., MA 3 focuses on the *amount* of change of the output (height) while considering changes in the input; and MA4 focuses on the *rate of change* of the output with respect to the input for uniform increments of the input). Attention to continuously changing *instantaneous rate* (MA5) is revealed by the construction of an accurate curve and includes an understanding of the changing nature of the instantaneous rate of change for the entire domain. It should be noted that a student may perform MA5 without demonstrating an understanding that the instantaneous rate of change resulted from examining smaller and smaller intervals of the domain. However, the developmental nature of the framework indicates that only students who are able to unpack mental action 5 (build from MA1-MA4) would receive a Level 5 covariational reasoning classification. This Level 5 image has been shown to support an understanding of *why* a concave up graph conveys where the "rate of change" is increasing, and *why* the inflection point relates to the point on the graph where the "rate of change" changes from increasing to decreasing, or decreasing to increasing.

Using the Framework

This section provides a dynamic situation (Figure 1) that illustrates common covariational reasoning behaviors that have been expressed by students when responding to a specific task (Carlson, 1998; Carlson & Larsen, 2001). The mental actions supported by each image of covariation are followed by a description of specific behaviors that have been observed in students, and their corresponding classifications using the framework.

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.

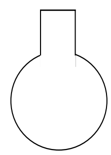

Figure 1. The Bottle Problem.

Th*e Coordination Level (L1)* supports the mental action (MA1) of coordinating the height with changes in the volume. Mental action 1 has been identified by observing students label the axes and by hearing them express an awareness that as one variable changes the other variable changes (e.g., as volume changes, height changes). The student does not necessarily attend to the direction, amount, or rate of change.

The *Direction Level (L2)* supports MA1 and the mental action of coordinating the direction (increasing) of change of the height while considering changes in the volume. Mental action 2 has been identified by observing students construct an increasing straight line, or by verbalizing that as more water is added, the height of the water in the bottle increases.

The *Quantitative Coordination Level (L3)* supports MA1, MA2 and the mental action of coordinating the amount of change of the height with the amount of change of the volume while imagining changes in the volume. Mental action 3 has been identified by observing students place marks on the side of the bottle (with each increment successively smaller until reaching the middle and successively larger from the middle to the neck). MA3 has also been identified by observing students plot points on the graph or by hearing remarks that express an awareness of how the height changes while considering increases in the amount of water.

The *Average Rate Level* (L4) supports MA1, MA2, MA3 and the mental action of coordinating the average rate of change of the height with respect to the volume for equal amounts of the volume. Mental action 4 has been identified in students by observing the construction of contiguous line segments on the graph with the slope of each segment adjusted to reflect the (relative) rate for the specified amount of water; or by hearing remarks that express an awareness of the rate of change of the height with respect to the volume while considering equal amounts of water. (Note that some students have been observed initially constructing line segments that were not contiguous and some students have been observed switching the roles of the independent (volume) and dependent (height) variable several times in the context of discussing the thinking they used to construct the graph for this task.)

Instantaneous Rate Level (L5) supports MA1 through MA4 and the mental action of coordinating the instantaneous rate of change of the height (with the respect to volume) with changes in the volume. Mental action 5 has been identified in students by observing the construction of a smooth curve that is concave down, then concave up, then linear; and by hearing remarks that suggest an understanding that the smooth curve resulted from considering the changing nature of the rate while imagining the water changing continuously. It is noteworthy that a student would only receive an *Instantaneous Rate Level* classification if he/she demonstrated an understanding that the instantaneous rate resulted from considering smaller and smaller amounts of water (built on the reasoning exhibited in MA4). The image that supports Level 5 reasoning would also support behaviors that demonstrate an understanding of *why* an inflection point conveys the exact point where the rate of change of the height (with respect to volume) changed from decreasing to increasing, or increasing to decreasing.

Some students have been observed exhibiting behaviors that gave the appearance of engaging in mental action 5 (e.g., construction of a smooth curve). When asked to provide a rationale for their construction, however, they expressed that they had relied on memorized facts to guide their construction. This behavior was classified as *pseudo-analytical* and the mental action that supported this behavior was classified as *pseudo-analytic* MA5 (Vinner, 1997).

METHODS FOR THE STUDY

Participants

Twenty students who had recently completed second-semester calculus with a course grade of A were asked to respond to five items that involved an analysis of covariant aspects of dynamic events (e.g., water filling a spherical shaped bottle, temperature changing over time, a ladder sliding down a wall). These students represented most of the A students from five different sections, each section with a different teacher. The course materials used in teaching the sections were traditional, with lecture the primary mode of instruction. Calculators were not allowed on either homework or exams. The 20 students received payment for their time in completing the five-item quantitative assessment. Six of these students were subsequently invited to participate in 90-minute clinical interviews in which they were also paid for their time. The selection of the interview subjects was based on assembling a collection of individuals who had provided diverse responses on the written instrument.

Procedures

The five-item instrument was completed by each of the 20 subjects within a week of completing the final exam. It was administered in a monitored setting (with no time restriction) and the students were asked to provide their answers in writing. Written items were then scored using carefully developed and tested rubrics (Carlson, 1998) and the percentage (of the twenty students) who provided each response-type for each item was determined.

The six interviews were conducted within two days of the students' completing the five-item written instrument. Although the interviews were primarily unstructured, with the interviewer spontaneously reacting to the student's description of her/his solution, prepared interview questions imposed some structure. During the interview, the researcher initially read each question aloud and made general reference to the student's response. The student was then given a few minutes to review her/his written response and was subsequently prompted to verbally describe and justify her/his solution. After the student summarized her/his written response, the researcher made general inquiries, such as, "explain" or "clarify", and continued to ask more specific questions until a response was elicited or it appeared that all relevant knowledge had been communicated. This process was repeated for each item.

Analysis of the interview results involved an initial reading of each interview transcript to determine the general nature of the response. This was followed by numerous careful readings to classify the behaviors and responses of each student on each item using the mental actions described in the covariation framework. Subsequent to labeling the mental actions (e.g., MA1, MA2, MA3) associated with the various behaviors exhibited for a single item, the entire response for that item was reviewed to determine the covariational reasoning level (e.g., L3) that supported the identified mental actions that were displayed. Illustrations of select quantitative data and coded interview excerpts are followed by a discussion of the students' responses to three (of the five) covariational reasoning tasks.

RESULTS FROM THE STUDY

The bottle problem (see Figure 1) prompted students to construct a graph of a dynamic situation with a continuously changing rate, and with an instance of the "rate" changing from decreasing to increasing (i.e., an inflection point). The temperature problem (see Figure 3) presented the "rate of change" of temperature for an eight-hour period, and prompted students to construct the temperature graph. This problem required students to directly interpret rate information displayed in the form of a graph and to use this information to graph the original function (i.e., the temperature function). The ladder problem (see Figure 5), a modification of a problem reported by Monk (1992), prompted students to select a means of representing a dynamic situation (i.e., a ladder moving down a wall), and to use that representation to describe the behavior of the top of the ladder.

When responding to the written assessment for the bottle problem (Figure 1), only 25% (five) of these high-performing second semester calculus students provided an acceptable solution, while 70% (14 out of 20) constructed an increasing graph that was strictly concave up or concave down (Table 3).

Table 3
Bottle Problem Quantitative Results

Student Responses	Number of students out of 20 providing each response type
Constructed a line segment with positive slope	1
Constructed an increasing concave up graph	11
Constructed an increasing concave down graph	3
Acceptable graph, except for slope of segment	3
All aspects of graph were acceptable	2

When prompted during the follow-up interview to describe the graph's shape, the six interview subjects provided varied responses (see *Interview Excerpts*). Only two of the interview subjects provided a response that suggested an image of continuously changing instantaneous rate (MA5) for this situation. When prompted to explain the rationale for her acceptable graph, Student A initially stated, "if you look at it as putting the same amount of water in each time and look at how much the height would change, the height would be changing more quickly, and in the middle if you add the same amount of water, the height would not change as much as it would at the bottom." (MA3). (See student A's graph in Figure 2). When prompted to explain why she had constructed a smooth curve, student A responded, "I imagined the height changing as the water was pouring in at a steady rate" (MA3). She also characterized the inflection point as the point, "where the rate at which it was filling goes from decreasing to increasing" (suggestive of MA5). On the other hand, Student C, who also constructed an acceptable graph by sketching a smooth curve over her previously constructed contiguous line segments, justified the construction by saying, "I just know that it must be smooth because this is what these graphs always look like, not these connected line segments." Even though her initial construction of a smooth curve was suggestive of MA5 and appeared to depict an image of continuously changing rate, further probing revealed that this student's answer expressed only an opinion of how the graph should look, rather than an emergent representation of how the variables changed. This response was therefore classified as pseudo-analytic MA5.

When analyzing the thinking of the three students who provided either a concave up or concave down construction, it appeared that two of these students (students B and E), at times during the interview, constructed images of the height changing at a varying rate (e.g., "as you go up a little more height increases and the volume increases quite a bit) (MA3). However, inconsistencies in their reasoning appeared to result in their constructing an incorrect graph. In the case of student B, he justified his concave down construction by saying, “every time you have to put more and more volume in to get a greater height towards the middle of the bottle” (MA3). (Notice that this illustrates a situation where the student switched the roles of the independent and dependent variables—i.e., the amount of change of the volume was considered while considering uniform changes in the height). He subsequently failed to continue thinking about the relative changes in the volume and height for the top half of the bottle. Student F justified his concave up construction by saying, “as I add more water, it still gets higher and higher” (suggestive of MA2). Although both of these students appeared to possess initial images of height changing as more water was added, at some point during the interview they appeared to focus on incorrect information or had difficulty representing their correct reasoning patterns using a graph. The remaining interview subject, student D, provided an increasing straight line and stated with confidence, “As the volume comes up, the height would go up at a steady rate…it would be a straight line” (MA2). He appeared to only notice that the height increased while considering increases in the volume (MA2).

Both the quantitative and qualitative data for this item support the finding that very few of these high performing second semester calculus students were able to form accurate images of the continuously changing instantaneous rate (MA5) for this dynamic function event.

The Bottle Problem Interview Excerpts

Student A constructed an acceptable graph (Figure 2). During the course of the interview she initially focussed on the amount of change of the height while considering fixed increases in the volume (MA3). This was followed by discussions about the slope and rate changes for fixed

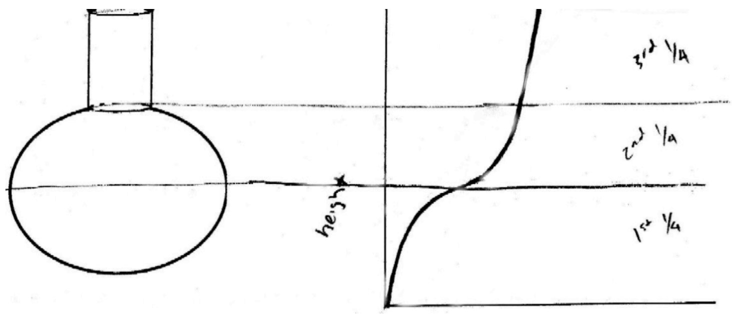

Figure 2. Student A's written response.

amounts of water (MA4). When prompted for more explanation, her responses eventually moved to comments that conveyed she was attending to the continuously changing rate, while imagining the bottle filling with water (MA5). She appeared to understand the information conveyed by the inflection point and appeared to have a mature image of changing rate over the domain of the function. The behaviors exhibited by this student when responding to this task suggested an *Instantaneous Rate* (Level 5) covariational reasoning ability.

Student A

Int.: Describe how you sketched the graph (graph is acceptable).

A: I knew it changed different for the bottom part because it's circular and the top part has straight walls. If you look at it as putting the same amount of water in each time and look at how much the height would change, that's basically what I was trying to do. So for the first part, the height would be changing more quickly, and in the middle if you add the same amount of water, the height would not change as much as it would at the bottom (MA3). It's symmetric.

Int.: How does that affect the graph?

A: Higher slope in the beginning, then it levels out, then a higher slope again (MA4). Then for the neck part it's basically a straight line because you're filling the same area with each amount (MA3).

Int.: Can you tell what happened at this point (pointing to the inflection point).

A: That is where the point of symmetry is. I guess it would also be where the second derivative is equal to 0, which is where the rate at which it was filling goes from decreasing to increasing (MA5).

Int.: Why did you draw a smooth curve through the lines?

A: Well, I imagined the slope changing as the water was pouring in at a steady rate (MA5).

Int.: Do you have anything else to add? What is the slope of the straight line?

A: About like the curve right here (pointing to the junction of the curve and line).

Student B constructed a concave down graph for the entire domain of the function. During the interview, he initially focussed on the direction of the change of the height, as revealed by his comment, "the more water, the higher the height" (MA2). Further prompting revealed that he was able to coordinate changes in the height with changes in the amount of water (i.e., As you go up, a little more height increases and the volume increases quite a bit) (MA3). The statement, "you have to put more and more volume in to get a greater height toward the middle" is also indicative of MA3. His expressed behaviors were suggestive of a *Quantitative Coordination* (Level 3) covariational reasoning ability for this task. His concave down construction appeared to result from his failure to continue coordinating changes in height with changes in the amount of water.

Student B

Int.: Explain your solution (student has a concave down graph for the entire domain).

B: This is my least favorite problem. I tried to solve for height in terms of volume and it was a mess.

Int.: Can you analyze the situation without explicitly solving for h?

B: OK, the more water, the higher the height would be (MA2). In terms of height of the water, that is what we are talking about. If you are talking about the height left over, that is basically decreasing. Right here the height will be zero and the volume is zero. As you go up, a little more height increases and the volume increases quite a bit (MA3), so the amount by which the height goes up is not as fast (MA3). Once you get there (pointing to halfway up the spherical part of the bottle), the height increases even slower (MA3). I guess from here to there height increases the same as the volume increases, and once you get here it increases slower (MA3). No, I am wrong. So, every time you have to put more and more volume in to get a greater height towards the middle of the bottle and once you get here, it would be linear, probably (pointing to the top of the spherical portion). So, it's always going up (he traces his finger along the concave down graph), then it would be a line.

Int.: So, what does the graph look like?

B: Like this (pointing to the concave down graph he has constructed), but it has a straight line at the end.

Student C produced a graph with only minor errors. Her initial justification that "it is going to be filling rapidly, so you are going to have greater slope" focussed on the relative slope for a section of the graph. This was immediately followed by the justification, "as you increase the volume you are going to get less height" (MA3) and her final justification was a statement of rules learned in calculus. Her responses suggested that although she was able to associate greater slope with filling rapidly and appeared at times to be imagining continuously changing instantaneous rate (MA5), she did not appear to understand how the instantaneous rates were obtained (i.e., was not able to unpack MA5). Even with direct probing, she could not explain what the inflection point conveyed. As a result, she was not classified as having a level 5 covariational reasoning ability. When responding to same this task, student C appeared to predominantly use Level 3 reasoning along with memorized rules learned in calculus. This combination of abilities appeared to be adequate for the construction of an acceptable graph.

Student C

Int.: Explain how you obtained your graph (final graph is acceptable).

C: I knew it was filling at a cubic rate somehow, so it would have something like a cubic equation. When you take the inverse of that equation it whips it like that. But I was also able to see here that when you start out, it's going to be filling rapidly, so you are going to have a greater slope (MA5—the student appears to know that "rapidly" and "more steep" are connected, but does not demonstrate an understanding of how the instantaneous rate was obtained). (Some confusion and continues.) But as you increase the volume, you're going to get less of a height change until you get up to here (MA3). As you get past the halfway point, it's going to go from concave down to concave up and you're going to have an inflection point. For this cylinder part, I know it's going to be linear, since for the cylinder it's related by volume, which equals area times height. And so we have area as a constant. So what we have is a linear equation for height as it's related to volume.

Int.: Can you tell me why you drew the smooth curve through the line segments that you had constructed?

C: Well (a long pause)…..I just know that it must be smooth because this is what these graphs always look like, not these connected line segments (pseudo-analytic MA5).

Int.: Can you tell me why it changed concavity there (pointing to the inflection point)?

C: Because if you take the second derivative of this volume in terms of height, you'll get a 0. On this side you have a negative acceleration. But once you reach the halfway point, then you start becoming a positive second derivative.

Student D constructed an increasing straight line for his solution. During the interview, he appeared to only coordinate the direction of the change in the height, while considering changes in the volume (MA2). The behaviors exhibited by this student when responding to this task were suggestive of a *Direction* (Level 2) covariational reasoning ability.

Student D

Int.: Can you explain your solution? (Solution is an increasing straight line.)
D: I tried to solve for h. But I think I need to define it as a piecewise defined function. Maybe then I can figure it out.
Int.: Did you try to get an idea of the general shape of the graph by imagining the bottle filling with water?
D: As the volume comes up, the height would go up at a steady rate (MA1, MA2).
Int.: How would you represent this graphically?
D: It would be a straight line (student passes his hand over the increasing straight line).
Int.: So, the entire graph is a straight line.
D: Yes.

Student E provided a concave up graph for his written solution. When probed to explain his answer, he replied that "as you added more water, the height was going up" (MA2). He then proceeded to explain his concave up graph with the statement, "the amount by which the height goes up is increasing" (MA3); however, his factual information was flawed (the amount of the height change was decreasing). Because he did not consistently exhibit behaviors supported my mental action 3, he was classified as having a *Direction* (Level 2) covariational reasoning ability.

Student E

Int.: Can you describe how you determined your graph? (Student provides a concave up graph.)
E: Well, I knew that as you added more water the height was going to go up (MA2)….um…Then I knew that it would curve up because your graph is getting higher all the time since the height is always increasing (MA3). So it is concave up (points to the concave up graph).
Int.: What is increasing?

E: The amount by which the height goes up is increasing (MA3). This means that it will curve up like this.
Int.: How do you explain what the shape looks like here (pointing to the middle of the bottle)?
E: It is still true here that as you add more water, it will increase in height, so it curves up here too (MA2).

Student F also constructed a concave up graph and appeared to consistently focus on the amount of change of the height while considering changes in the volume (MA3), as revealed by the justification, “as I add more water it still gets higher and higher.” At one point during the interview he indicated that, “height is going up more and more” (MA3). However, he did not persist in resolving the incorrectness of this statement (when imaging water being added to the lower half of the bottle); nor did he follow through in resolving the inconsistency that he noticed later in the interview (see Student F excerpt). He did not show a consistent pattern of behaviors supported by mental action 3. Consequently, the behaviors exhibited by this student when responding to this task were suggestive of *Direction* (Level 2) covariational reasoning ability.

Student F

Int.: Can you explain how you determined your graph (student provides a concave up graph).
F: When you're given a flask like this, the way I thought of it was, you have to start the coordinates at (0,0) with volume equal to 0, and the height equal to 0. When you start filling something that has such a wide base like this, the height is going to increase as fast as the volume (MA1, MA2). Then as more water is added it gets higher and higher, so the graph goes up more and more (pointing to the concave up graph) (MA3).
Int.: What happens at the middle of the spherical portion?
F: Now, I am confused. Will it continue to go up higher and higher? (pause). Well yes, as I look at the part above the middle, as I add more water it still gets higher and higher so yes, it curves up like this (MA3) (again pointing to the concave up graph).

The Temperature Problem Results

The following task (Figure 3) presented a rate of change graph, followed by a prompt to construct the corresponding temperature graph.

The Temperature Problem. Given the graph of the rate of change of the temperature over an eight-hour time period, construct a rough sketch of the graph of the temperature over the eight-hour time period. Assume the temperature at time, $t = 0$, is 0 degrees Celsius.

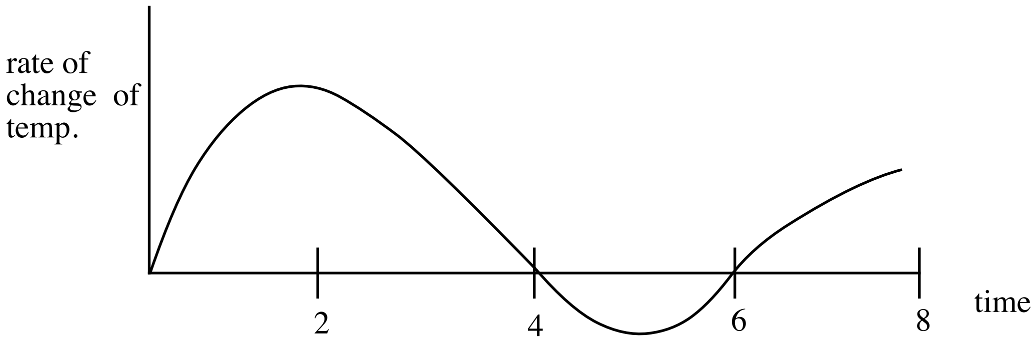

Figure 3. The Temperature Problem.

20% of these high-performing second-semester calculus students constructed an acceptable temperature graph (See student C's response in Figure 4), given the rate of change of temperature for an eight-hour time-period (Figure 3), while 25% of these students produced the same graph for the temperature graph as the one given for the rate of change graph. It was also observed that 30% failed to note the concavity changes when constructing their graphs (Table 4). The 6 students who omitted the concavity changes provided a concave down graph from t = 0 to t = 6, with a maximum value at t = 2.

Table 4

Temperature Problem Quantitative Results

Student Responses	Number of students out of 20 providing each response type
Constructed a strictly concave up graph for the entire domain	1
Constructed the same graph as the temperature graph	5
Omitted the concavity changes at t = 2 and t = 5	6
Reversed the concavity	4
All aspects of graph were acceptable	4

The follow-up interviews revealed that, of those four students who provided an acceptable response (see Figure 4 for student C's response), there was little evidence that they were interpreting the rate information conveyed by the graph. When student C was prompted to justify her acceptable response, she replied, "positive first derivative implies function increasing, negative first derivative implies function decreasing" and "second derivative equal to zero occurs at inflection points." When asked to explain the reasoning that led to this response, she indicated that this is how she had learned it in class and she didn't know how to think about it any other way (pseudo-analytic MA5). It is interesting to note that even when directly probed, she also appeared unable to construct an image of the temperature changing while imagining changes in time (MA3). Her response suggested that a memorized set of rules guided her construction. (This is not surprising if one considers the nature of a traditional calculus course.)

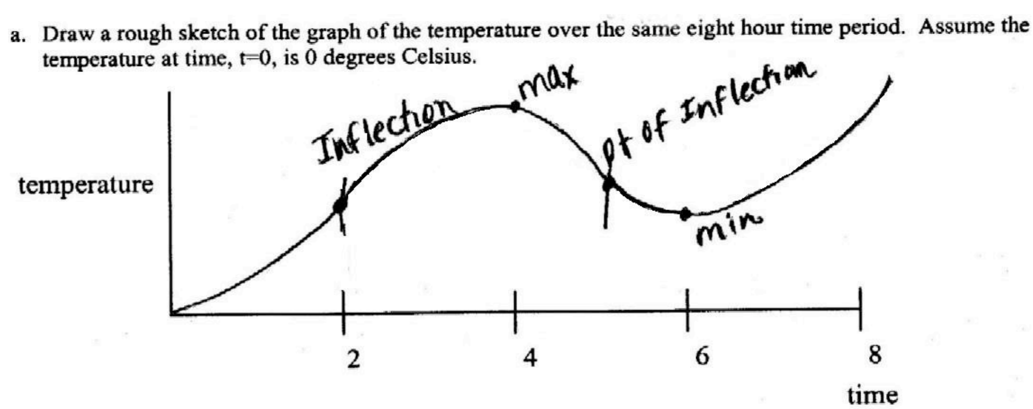

Figure 4. Student C's response to the Temperature Problem.

Moreover, the collection of follow-up interviews revealed that most of these students did not construct an accurate image of the "rate changing" (MA4) as they considered increases in the domain. The two students who constructed an increasing temperature graph (from t=0 to t=4) did not appear to understand what was being conveyed when the rate began to decrease at t=2. When probed to explain, both students indicated that because the rate graph was positive from t=0 to t=4, this conveyed that the temperate graph must be increasing. When specifically asked to explain the behavior of the temperature graph at t = 2, one of these two students explained, "yes at 2 it is also positive, so it will keep curving up until it is 4". Even though these students appeared to have an initial image of the temperature function increasing at an increasing rate (MA5), as suggested by their concave up construction and remarks, their inability to note and represent the rate changing from increasing to decreasing (i.e., the inflection point) suggested weaknesses in their understandings.

Another student who had constructed the same graph for the temperature graph as the given rate of change graph, said "this is hard to think about….it is hard for me not to just draw the shape that I see…it really throws me off." This student appeared to make no attempt to interpret the rate of change information displayed by the graph. Rather, he appeared to want to re-construct the same graph as the one he was observing.

The Ladder Problem Results

When prompted to describe the speed of the top of a ladder as the bottom of the ladder is pulled away from a wall (Figure 5), 40% of these second-semester calculus students provided an accurate justification for the claim that the top of the ladder would speed up as the bottom of the ladder is pulled away from the wall (Table 5). Five additional students (25%) also indicated that the top of the ladder would speed up, but provided no justification to support the claim. In addition, 5 students conveyed that the speed of the top of the ladder would be constant and 2 students (10%) indicated it would slow down.

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The Ladder Problem. From a vertical position against a wall, a ladder is pulled away at bottom, at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.

Figure 5. The Ladder Problem.

The written justifications provided on the written instrument revealed that the eight students who had provided a correct response with a valid justification had imagined a physical enactment of the ladder falling down the wall. This was determined by observing a succession of pictures of the ladder in different positions and/or a written explanation. The follow-up interviews with two of these students supported this observation.

Table 5

Ladder Problem Quantitative Results

Student Responses	Number of students out of 20 providing each response type
Speeds up—valid written justification	8
Speeds up—no justification	5
Stays the same	5
Slows down	2

When one of these students (student B) was prompted to explain his correct response, he performed a physical enactment of the situation using a pencil and book on a table. As he successively pulled the bottom of the pencil away from the book by uniform amounts, he explained, "as I pull the bottom out, the amount by which the top drops gets bigger as it gets closer to the table." (MA3) His comments suggested that he was observing the varying amounts by which the top of the pencil dropped toward the table, as the bottom was pulled out by uniform amounts. His explanation appeared to involve the coordination of an image of the magnitude of the change in the dependent variable with uniform changes in the independent variable (MA3). Student A provided a similar response, except her enactment involved her using her hand and a book to model the situation. She began by pressing her flat hand against the book and successively moved the bottom of her hand away, while watching the amount by which the top of her hand dropped down. In both cases these students appeared to assume that the greater fall implied speeding up, however specific prompts were not offered to verify this assumption; nor were specific prompts made to determine the reasoning behind this deduction.

Two of the interview subjects provided no justification on the written instrument to explain their correct responses. However, when prompted during the interview to explain their reasoning, student E did provide a valid justification, also using a self-constructed enactment of the situation. Student D indicated that he had "just guessed." It is not clear whether he constructed an image of the situation as a basis for his guess.

The remaining two interview subjects indicated that the speed of top of the ladder would remain the same as the bottom was pulled away from the wall. These students both drew pictures of the ladder in different positions, but modified the length of the ladder so that the amounts of the drop remained the same for each new position of the ladder. When asked to explain their reasoning both students provided responses that indicated they had attempted to enact the situation, but their model was flawed. Student C drew a picture of the successive positions of the ladder as the bottom was pulled out by equal amounts. The drawing also illustrated equal drops of the top of the ladder. She achieved this by adjusting the length of the ladder (violating a condition of the problem). Even though her answer was incorrect, she appeared to engage in a behavior that suggested she was attempting to coordinate the amount of change in the dependent variable with the change in the independent variable (MA3).

The use of physical enactment appeared to provide a powerful representational tool for assisting these students in reasoning about the change in one variable while concurrently attending to the change in the other variable. Further exploration of this observation is needed.

CONCLUSIONS

The students in this study varied in their ability to apply covariational reasoning when analyzing dynamic events. Observed trends suggest that this collection of calculus students had difficulty constructing images of continuously changing rate, with particular difficulties representing and interpreting images of "increasing rate" and "decreasing rate" for a physical situation (MA5). Despite these difficulties, most of the students were able to determine the general direction of the change in the dependent variable with respect to the independent variable (Level 2 reasoning) and were frequently able to coordinate images of the amount of change of the output variable while considering changes in the input variable (MA3). However, we observed weaknesses in their ability to interpret and represent rate of change information (MA4) (Table 3, Table 4). Aided by the use of kinesthetic enactment, however, these students were more often able to observe patterns in the changing magnitude of the output variable (MA3), as well as patterns in the changing nature of the instantaneous rate (MA5). Nonetheless, their difficulty in viewing an instantaneous rate as the result of imagining smaller and smaller refinements of the average rate of change appeared to persist. More importantly, this limitation (an inability to unpack MA5) appeared to create difficulties for them in accurately interpreting and understanding the meaning of an inflection point and in explaining why a curve was smooth. Even direct probing of the few students that were able to engage in mental action 5 revealed that they were not able to explain how the instantaneous rate was obtained. This weakness appeared to result in difficulties for them in bringing meaning to their constructions and graphical interpretations.

Despite the fact that the subjects of our study were high-performing second-semester calculus students who had successfully completed a course emphasizing "rate" and "changing rate", the majority did not exhibit behaviors suggestive of mental action 5 (MA5) while analyzing and representing dynamic function events. They appeared to have difficulty characterizing the nature of change while imagining the independent variable changing continuously. In summary, the majority of these calculus students:

- *were able* to consistently apply Level 3 reasoning. They exhibited behaviors that suggested they were able to coordinate changes in the direction and amount of change of the dependent variable in tandem with an imagined change of the independent variable (MA1, MA2 and MA3);
- *were unable to consistently apply* Level 4 reasoning. They exhibited behaviors that suggested they were unable to consistently coordinate changes in the average rate of change with fixed changes in the independent variable for a function’s domain (MA1-MA4);
- *had difficulty* applying Level 5 reasoning. There were not able to consistently exhibit behaviors that suggested they were able to coordinate the instantaneous rate of change with continuous changes in the independent variable (MA5);
- *had difficulties* explaining why a curve is smooth and what is conveyed by an inflection point on a graph (i.e., Applying *Level 5* covariational reasoning).

Our results support the work of Confrey and Smith (1995) and Thompson (1994a) who revealed similar findings regarding the complexity of reasoning about covarying relationships; however, our study extends what has previously been reported by identifying specific aspects of covariational reasoning that appear to be problematic for college level students. It is also our hope that the study's results and the covariation framework will serve to explicate the cognitive actions involved in reasoning when interpreting and representing dynamic function events.

DISCUSSION

Research has revealed that the basic idea of covariation is accessible to elementary and middle school children (Confrey & Smith, 1994; Thompson, 1994c). It seems reasonable, then, that this same ability would also be accessible to high-performing second semester calculus students. Therefore, the results of this study raise concerns, especially when considering that the selected tasks could be completed successfully by students with no calculus knowledge, but with a strong covariational reasoning ability (Carlson & Larsen; 2001). Since the information assessed in this study was intended to be foundational for building and connecting the major ideas of calculus, we believe these findings suggest a need to monitor the development of students' function understandings and covariational reasoning abilities prior to and during their study of calculus. As this study and others have revealed, even high performing students can emerge from second semester calculus with superficial understandings of ideas that are foundational for future study of mathematics and science. Our failure to monitor these understandings and reasoning abilities portends to have negative consequences for students.

The thinking revealed in this study should prove to be useful for informing the design and development of curricular materials aimed at promoting students' covariational reasoning abilities. The results also underscore the need for students to have opportunities to experience the covariational nature of functions through real-life dynamic events. We recommend that students be given probing lines of inquiry that compel reflections of their own understandings of patterns of change (involving changing rates-of-change). Accordingly, we believe that high school and university level curricula should take into consideration the complexity of acquiring *Level 5* (Instantaneous Rate) reasoning and should provide curricular experiences that sustain and promote this reasoning ability, especially when one considers its importance for understanding major concepts of calculus (e.g., limit, derivative, accumulation) and for representing and understanding models of dynamic changing events.

The theoretical model and results from this study should also be useful for classroom teachers in both identifying and promoting the development of their students' covariational reasoning abilities. In support of the covariational approach to instruction, curricular activities are under development by these authors and have been administered to both preservice secondary teachers in a preservice methods course and first semester calculus students at a large public university in the Southwest United States. The development of these curricular activities were guided by the covariation framework and the insights gained from this study. Preliminary observations of students when working with this curriculum have revealed positive shifts in their covariational reasoning abilities. Although multiple refinements of the curriculum will be needed as students' thinking continues to spawn new ideas for making it better, these observations are encouraging.

The new century offers educators a plethora of technologies (e.g., graphing calculators, geometry software, computer algebra systems, electronic laboratory probes, specialty software such as MathCars (Kaput, 1994)) and specially designed physical devices (e.g., Monk & Nemirovsky, 1994) for studying real-time dynamic events. Rich pedagogical opportunities abound for building on students' intuition about and experience with dynamically changing quantities. Grounded properly and incorporated with sufficient teacher training, these technologies offer valuable tools for students in learning to apply covariational reasoning to analyze and interpret dynamic function situations.

FUTURE RESEARCH

The work from this investigation has resulted in our reflecting on the nature of the reasoning patterns involved in applying covariational reasoning. We claim that the "actions of coordinating the change in one variable with changes in the other variable, while attending to how they change in relation to each other" involves a mental enactment of the operation of coordinating on two objects (these objects are different dependent upon the mental action in the framework). This observation led us to hypothesize that the mental actions involved in applying covariational reasoning are characteristic of *transformational reasoning* as described by Simon (1996). According to Simon (1996),

> *"Transformational reasoning is the mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated"*(p. 201).

We view the mental actions that we have described in *The Covariation Framework* as examples of transformation reproductive images (i.e., the problem solver is able to visualize the transformation resulting from an operator). In our case, the student visualizes the transformation of a dynamic situation as resulting from the operation of coordinating. When engaging in mental action 3 (MA3) the student is able to visualize the transformation of a dynamic situation (e.g., a ladder falling down the wall, a bottle filling with water) by performing a mental enactment of coordinating two objects (the amount of change in one variable with an amount of change in another variable); while mental action 5 involves a mental enactment of coordinating the instantaneous rate of change in one variable with changes in the other variable. In both cases the mental enactment on the objects results in a transformation of the system (e.g., the ladder is envisioned as being in a different position, the bottle is envisioned as containing more water).

Although we claim that we have observed instances of transformational reasoning, we offer no information about the process of coming to generate a particular transformational approach. We concur with Simon (1996) in calling for explorations of this question, as our results also support the notion that appropriate application of transformational reasoning may prove to be extremely powerful for understanding and validating a mathematical system.

Our investigation also calls for an extension of the *Covariation Framework* to include a greater level of epistemological refinement for understanding covarying quantities. Such a framework may include aspects of concept development as it relates to covariational reasoning abilities. It may also include a more finely grained analysis of *Level 5* (Instantaneous Rate) reasoning. Additionally, it could be extended to more clearly articulate the nature of covariational reasoning in the context of working with formulas or the algebraic form of a function.

There are some specific questions for future study on the centrality of continuity and the implicit time variable in covariational reasoning. Our discussion and instruments deal with physical relationships that are inherently continuous. It is unclear to what extent the framework applies to students' study of discontinuous dynamic function events. Furthermore, our experience suggests there is a powerful tendency for students to use time as a variable, even to the point of introducing it in situations like the Bottle Problem, where it is not strictly necessary (or requested). Future research may clarify the role of the implicit time variable in the development of student's covariational reasoning.

Another promising area of research includes investigations of the effectiveness of various curricular interventions in developing students' ability to apply covariational reasoning when solving problems that involve real-world dynamic situations. Such studies may also provide information about the effect of taking a covariational approach to learning functions on students' development of their understanding of the function concept in general.

We have provided examples of students who appeared to be able to apply covariational reasoning for the bottle problem (Table 1) and the ladder problem (Table 5) in a kinesthetic context, but were unable to use the same reasoning patterns when attempting to construct a graph (i.e., reason in the graphing representational context) for these situations. This is important because it suggests that the covariation framework may be used to infer information not just about the developmental level of student images of covariation, but also about the internal structure of these images. If we imagine that a student's overall image of covariation of a dynamic situation contains specific images related to each relevant representation system, we may be able to use the framework to analyze the way these images are connected and coordinated.

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