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ORIGINAL ARTICLE

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Approach à la Piola for the equilibrium problem of bodies with second gradient energies. Part II: Variational derivation of second gradient equations and their transport

Dedicated to the 90th birthday of Professor Giulio Maier.

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Abstract After the wide premise of Part I, where the equations for Cauchy's continuum were retrieved through the energy minimization and some differential geometric perspectives were specified, the present paper as Part II outlines the variational derivation of the equilibrium equations for second gradient materials and their transformation from the Eulerian to the Lagrangian form. Volume, face and edge contributions to the inner virtual work were provided through integration by parts and by repeated applications of the divergence theorem extended to curved surfaces with border. To sustain double forces over the faces and line forces along the edges, the role of the third rank hyperstress tensor was highlighted. Special attention was devoted to the edge work, and to the evaluation of the variables discontinuous across the edge belonging to the contiguous boundary faces. The detailed expression of the contact pressures was provided, including multiple products of normal vector components, their gradient and a combination of them: in particular, the dependence on the local mean curvature was shown. The transport of the governing equations from the Eulerian to the Lagrangian configuration was developed according to two diverse strategies, exploiting novel differential geometric formulae and revealing a coupling of terms transversely to the involved domains.

Keywords Continuum mechanics · Second gradient materials · Differential geometry · Piola transformation · Lagrangian and Eulerian formulation · Double force · Edge force

1 Introduction

Cauchy's first gradient theory represented for more than one century the prevailing paradigm of continuum mechanics, in the sense of Kuhn [1]. Irony of fate, the scientific journal founded by Piola (titled "Opuscoli matematici e fisici di diversi autori"), published in two volumes in Milan during 1832-34, contributed decisively to the diffusion of the most important Cauchy's scientific works translated in Italian. In the academic community, the main assumption of Cauchy's theory, namely the dependence of the traction vector on the point and on the local normal vector (Cauchy's postulate), was presented along several decades as the unique reasonable choice to model every material at each observation scale, hiding its true nature of constitutive hypothesis, see, e.g. [2,3]. However, as discussed in [4–7], in the presence of second gradient variations the internal work cannot be represented through the Cauchy stress tensor alone, unable to sustain line forces along the edges, or double forces over the faces, see, e.g. [8–11]. Moreover, the first gradient theory can describe neither size effects, not including a microstructural length scale, see, e.g. [12], nor the dependence of the contact pressures

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on the curvature of the boundary face [13]. The particular features of second gradient materials require new ingredients, like a third-order hyperstress (or double stress) tensor, and supplementary boundary conditions, which at a first sight may seem less intuitive. Due to this enrichment, second gradient modelling is able to detect boundary layers [14], can describe the mechanical response of bodies with a highly heterogeneous and anisotropic reinforcement [15,16], deals with specific loading conditions in crack and dislocation problems [17,18] where the first gradient theory predicted singularities is suitable to detect dispersion effects in wave propagation problems [19]. For such a class of materials existence and stability results were provided [20,21].

The diffusion of novel, generalized scientific theories, leading to the falsification of the previous paradigm or at least to its downsizing, is usually endowed by bitter and divisive controversies, *querelles des anciens et des modernes* which can be met frequently along the history. Such disputes tend to quench when old, scientific traditions and past contributions, ostracized and forgotten, are rediscovered as vital also within the novel perspectives. In this respect, the principle of virtual work offered an extraordinary framework to generalize Cauchy–Navier models to second and higher gradient continua, see, e.g. [22]. Additional boundary conditions, including unexpected expressions for the contact pressures, are naturally provided by such a variational approach, which seems therefore more powerful of the balance principles in which the class of contact interactions is postulated a priori. Moreover, from the mathematical standpoint the virtual work terms represent linear functionals over the space of functions with compact support (i.e. the virtual displacements), and hence, can be characterized as (volume, surface and line) distributions in the sense of Schwartz, exploiting representation theorems and other meaningful results, see [23,24].

The present paper as Part II was organized as follows. In Sects. 2 and 3, the governing equations were derived via variational calculus from energy densities depending on both the first and the second gradients of the placement map. Section 4 highlighted the dependence of the contact pressures on the product of normal vector components, on their gradient and on a combination of them: in particular, on the mean curvature of the boundary face. In Sect. 5, the selection of work-conjugate variables was discussed, allowing the internal virtual work to be represented by the same mathematical structure in both the Eulerian and the Lagrangian configuration, leading to governing equations formally identical. Section 6 gathered the external work contributions and the Lagrangian equations in strong form. Section 7 was devoted to the transport of the governing equations from the Eulerian to the Lagrangian configuration, making extensive use of the novel transformation formulae for vectors and projectors provided in Part I, and revealing a coupling of terms transversely to the involved domain. Advanced tools resting on differential geometry and tensor algebra, which were discussed in Part I, such as the pullback metric tensor, the covariant derivative and a revisited formulation of the divergence theorem for curved surfaces with border, were utilized for the present transport procedure. Section 8 included some closing remarks, emphasizing future prospects of this research not only theoretical. In this manuscript, the Equations of the companion paper [25] were cited in the form [Part I, Eq. number]. Preliminary results included in this paper were announced in [26]. With respect to [27], herein the focus was on the differential geometric perspectives mentioned above, opening alternative paths for the problem in point.

Notation Symbols and conventions will be consistent with those adopted in Part I. In particular, recourse will be made to index, component-wise notation for the involved equations, although sometimes the relevant matrix or tensorial expressions will be reported, see [28]. Classical syntaxis of tensor algebra will be adopted (see [29,30]), with the Einstein convention on the implicit sum of repeated indices. As far as possible, Lagrangian quantities will be denoted by uppercase symbols, and their Eulerian counterparts by lowercase symbols. In tensor calculus, to distinguish valences acting on Lagrangian vectors from those specifying Eulerian spaces, e.g. as in F^a_A, the former will be indicated by uppercase letters, i.e. $A, B, \cdots$, the latter by lowercase ones, $a, b, \cdots$. The Lagrangian gradient will be denoted by symbols $\nabla \equiv \frac{\partial}{\partial X^A}$, with the obvious extension to k-th order gradients as $\nabla^{(k)} = \nabla\, \nabla^{(k-1)}$.

2 Second gradient energy

In some works of Gabrio Piola, see, e.g. [31–33], the equilibrium problem of a continuous medium was investigated with reference to general expressions of the deformation energy density, depending not only on the local deformation gradient $\mathbf{F} = \partial \boldsymbol{\chi} / \partial \mathbf{X}$ but also on its higher order spatial derivatives $\nabla^{(k)}\mathbf{F}$, with $k \geq 1$. In this paper, we considered energy densities in the form $\mathrm{W}(\mathbf{F}, \nabla\mathbf{F})$. Accordingly, the placement map $\boldsymbol{\chi}(\mathbf{X})$, defined in the Lagrangian or material domain $\Omega_\star \subset \mathcal{R}^3$ [45], must belong at least to the Sobolev space $[\mathcal{H}^2\,(\Omega_\star)]^3$, also to guarantee the trace regularity, see, e.g. [34]. The objectivity of the energy can be ensured by prescribing a dependence on the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and on its gradient $\nabla\mathbf{C}$ (or on $\mathbf{C}^{-1}$ and $\nabla\mathbf{C}^{-1}$), as illustrated in [Part I, Eq. (8)], see also [35]. To attain the equilibrium configuration of a deformable body, the minimum of the following energy functional is sought

$$\hat{\boldsymbol{\chi}} = \arg\min_K \left\{ \mathcal{E}^{\mathrm{TOT}}(\boldsymbol{\chi}) = \int_{\Omega_\star} \mathrm{W}(\mathbf{F},\ \nabla\mathbf{F})\ d\Omega_\star - \mathcal{E}^{\mathrm{EXT}}(\boldsymbol{\chi},\ \nabla\boldsymbol{\chi}) \right\} \tag{1}$$

where symbol $\mathcal{E}^{\mathrm{EXT}}$ denotes the external work, and the feasible functional subset $K \subset [\mathcal{H}^2(\Omega_\star)]^3$ incorporates the essential boundary conditions on the placement map and on its normal derivative. The volume integral of the stored energy density in Eq. (1) will be denoted by symbol $\mathcal{E}^{\mathrm{DEF}}$. When prescribing the stationarity condition, the first variation of the above functional, discussed in [Part I, Section 3], can be evaluated herein with reference to such a more general energy density, as follows ([36])

$$\begin{aligned}\delta \int_{\Omega_\star} \mathrm{W}(\mathbf{F},\ \nabla\mathbf{F})\ d\Omega_\star &= \int_{\Omega_\star} \left(\frac{\partial \mathrm{W}}{\partial \mathbf{F}} : \delta\mathbf{F} + \frac{\partial \mathrm{W}}{\partial \nabla\mathbf{F}} \vdots\, \delta(\nabla\mathbf{F}) \right) d\Omega_\star = \\ &= \underbrace{\int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_A}\, \delta F^i{}_A\, d\Omega_\star}_{=\delta\mathcal{E}_I^{\mathrm{DEF}}} + \underbrace{\int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{A,B}}\, \delta F^i{}_{A\,,B}\, d\Omega_\star}_{=\delta\mathcal{E}_{II}^{\mathrm{DEF}}}\end{aligned} \tag{2}$$

where symbols $:$ and $\vdots$ denote the usual double dot product and the triple contraction, respectively (see, e.g. [28]). The contributions of the external virtual work $\delta\mathcal{E}^{\mathrm{EXT}}$ will be specified later. The first addend $\delta\mathcal{E}_I^{\mathrm{DEF}}$ was integrated in [Part I, Eq. (10)]. Now, let us focus on the term involving the second gradient of the placement map, above specified as $\delta\mathcal{E}_{II}^{\mathrm{DEF}}$. Considering that $\delta F^i{}_{A,B} = \frac{\partial^2 \delta\chi^i}{\partial X^A \partial X^B}$, the above equation can be further reduced. Moreover, due to the Schwarz theorem one has $F^i{}_{A,B} = F^i{}_{B,A}$, and comma can be omitted without ambiguity. By commuting the derivative and the first variation, integrating by parts one has

$$\begin{aligned}\delta\mathcal{E}_{II}^{\mathrm{DEF}} &= \int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\, \delta F^i{}_{AB}\, d\Omega_\star = \int_{\Omega_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \frac{\partial}{\partial X^B} \left(\delta F^i{}_A\right) d\Omega_\star = \\ &= \int_{\Omega_\star} \left[\frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \delta F^i{}_A \right) - \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \right) \delta F^i{}_A \right] d\Omega_\star\end{aligned} \tag{3}$$

Applying the Gauss–Ostrogradsky divergence theorem to the first addend, and setting $\Sigma_\star = \partial\Omega_\star$, one obtains

$$\delta\mathcal{E}_{II}^{\mathrm{DEF}} = \underbrace{\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \delta F^i{}_A\, N_B\, d\Sigma_\star}_{(\blacklozenge)} \underbrace{- \int_{\Omega_\star} \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \right) \delta F^i{}_A\, d\Omega_\star}_{(\ddagger)} \tag{4}$$

whilst in the volume integral the test function can be further reduced as follows

$$\begin{aligned}(\ddagger) &= -\int_{\Omega_\star} \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \right) \frac{\partial}{\partial X^A} \delta\chi^i\, d\Omega_\star = \\ &= -\int_{\Sigma_\star} \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \right) \delta\chi^i\, N_A\, d\Sigma_\star + \\ &\quad + \int_{\Omega_\star} \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^B} \left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \right) \delta\chi^i\, d\Omega_\star\end{aligned} \tag{5}$$

Let us notice that the surface term $(\blacklozenge)$ in Eq. (4) still includes $\delta F^i{}_A$. If we neglect for a whilst such a boundary term and we add the first gradient contributions [Part I, Eq. (10)], we find

$$
\begin{aligned}
&\delta\mathcal{E}_{I}^{\mathrm{DEF}} + \delta\mathcal{E}_{II}^{\mathrm{DEF}} \; - (\blacklozenge) = \\
&= \int_{\Sigma_\star} \left\{ \frac{\partial \mathrm{W}}{\partial F^i_{\ A}} - \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}}\right)\right\} N_A \delta\chi^i \, d\Sigma_\star + \\
&\quad - \int_{\Omega_\star} \frac{\partial}{\partial X^A}\left\{ \left(\frac{\partial \mathrm{W}}{\partial F^i_{\ A}}\right) - \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}}\right)\right\} \delta\chi^i \; d\Omega_\star
\end{aligned} \tag{6}
$$

Surprisingly, the "truncated" second gradient formulation of Eq. (6) (i.e. without the term $\blacklozenge$) exhibits the same form of the first gradient solution, provided that an augmented stress tensor is introduced, by setting

$$
P_i^{\ A} = \frac{\partial \mathrm{W}}{\partial F^i_{\ A}} - \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}}\right); \qquad \mathbf{P} = \frac{\partial \mathrm{W}}{\partial \mathbf{F}} - \mathrm{DIV}\left(\frac{\partial \mathrm{W}}{\partial \nabla \mathbf{F}}\right); \tag{7}
$$

However, the presence of a stress-like flux by itself does not alter Cauchy's formulation of the equilibrium problem. Something similar occurs in buckling problems when the second-order terms are included in the kinematical description, leading to a geometric stiffness competing with the elastic one.

A further remark is needed. The most general quadratic energy density for the second gradient continua can be expressed additively as $\mathrm{W}(\mathbf{F}, \nabla\mathbf{F}) = \mathrm{W}^{I}(\mathbf{F}) + \mathrm{W}^{II}(\mathbf{F}, \nabla\mathbf{F})$, see, e.g. [36]. The contribution indicated by superscript I depends exclusively on the first gradient, whilst the contribution with superscript II depends on both first and second gradients, thus allowing one to describe not only the truly second gradient response but also a possible coupling between them. For the sake of simplicity, the above specification was omitted in what follows, but this choice does not affect the generality of the results. For instance, when considering the above decomposition of the energy, Eq. (6) assumes the form

$$
\begin{aligned}
&\int_{\Sigma_\star} \left\{ \left(\frac{\partial \mathrm{W}^{I}}{\partial F^i_{\ A}} + \frac{\partial \mathrm{W}^{II}}{\partial F^i_{\ A}}\right) - \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}^{II}}{\partial F^i_{\ AB}}\right)\right\} N_A \delta\chi^i \, d\Sigma_\star + \\
&\quad - \int_{\Omega_\star} \frac{\partial}{\partial X^A}\left\{ \left(\frac{\partial \mathrm{W}^{I}}{\partial F^i_{\ A}} + \frac{\partial \mathrm{W}^{II}}{\partial F^i_{\ A}}\right) - \frac{\partial}{\partial X^B}\left(\frac{\partial \mathrm{W}^{II}}{\partial F^i_{\ AB}}\right)\right\} \delta\chi^i \; d\Omega_\star
\end{aligned} \tag{8}
$$

3 Higher order boundary terms

To progress toward a solution for the boundary term ($\blacklozenge$) of Eq. (4), more than one century was needed from the studies of Piola: to our knowledge, a specific strategy was proposed by Paul Germain, see, e.g. [37,38], following a tradition set in the French school by André Lichnerowicz. In fact, by utilizing the relationship $\delta^C_A = [M_\parallel]^C_A + [M_\perp]^C_A$ involving the surface projectors [Part I, Appendix A], the gradient of the virtual placement map in Lagrangian form can be additively decomposed into an orthogonal and a tangential contribution, namely

$$
\begin{aligned}
(\blacklozenge) &= \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, \delta F^i_{\ A} N_B \, d\Sigma_\star = \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \left[\frac{\partial}{\partial X^A}\,\delta\chi^i\right] N_B \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, N_B \, \delta^C_A \left[\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star = \\
&= \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, N_B \left([M_\parallel]^C_A + [M_\perp]^C_A\right)\left[\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star = \\
&= \underbrace{\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, N_B \, [M_\perp]^C_A \left[\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star}_{(\perp)} + \\
&\quad + \underbrace{\int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, N_B \, [M_\parallel]^C_A \left[\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star}_{(\parallel)}
\end{aligned} \tag{9}
$$

The addend ($\perp$) including the orthogonal projector cannot be further reduced: in the second gradient theory, in fact, the virtual work is represented by means of two (independent) test functions, the virtual placement vector itself, and its normal derivative. Hence, recalling that $[\mathbf{M}_\perp]^C_A = N^C N_A$ [Part I, Eq. (82)], from Eq. (9) one can write

$$(\perp) = \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} N_B\, N^C\, N_A \left[\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star = \\ = \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} N_B\, N_A \left[N^C\,\frac{\partial}{\partial X^C}\,\delta\chi^i\right] d\Sigma_\star = \\ = \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} N_B\, N_A \left[\frac{\partial}{\partial N}\,\delta\chi^i\right] d\Sigma_\star \tag{10}$$

This surface integral represents the virtual work between the projection of the virtual placement gradient normally to the face (i.e. the normal derivative of $\delta\chi^i$), and the hyperstress tensor contracted with two components of the normal vector (as for the Lagrangian indices). Such an Eulerian vector (index i being free) has been referred to in the literature as inner double force (in Italian *biforza*), see, e.g. [37,38]. Dimensionally it is an energy per unit surface, equivalently a force per unit length, or also a pressure multiplied by a length (which better suggests the analogy with a dipole). Other authors preferred the term double traction or hypertraction (see, e.g. [10,39]), to trace the mutual relationship between the stress tensor and the traction vector. In principle, the term hyperstress does not seem suitable for a generalization to the higher order gradients when needed, on the contrary of the double stress (e.g. triple stress, etc.). However, in this study the term hyperstress was adopted to indicate the third rank tensor alone, and inner double forces for its doubly contracted contributions over the surface and along the edges. The surface double force can then be expressed as follows:

$$\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\right) N_B\, N_A = \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} [\mathbf{N}\otimes\mathbf{N}]^C_B\, g_{CA} = g_{CA}\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} [M_\perp]^C_B \tag{11}$$

Several ways have been proposed to interpret physically and represent graphically the hyperstress components and forces generated by its contraction, see [19,40,41], usually with reference to the Eulerian configuration. Here, we can try to provide a general rule, although less intuitive since two indices of the second gradient are Lagrangian, and the area element deforms when passing from the Lagrangian to the Eulerian configuration. In practice, we are going to approximate the differential relationship of Eq. (7) or (32) at the macroscale, by means of fluctuations at a lower scale. To this purpose, let us consider an infinitesimal cube of material, with sides parallel to the reference frame, and let us assume that, closely to each face, the boundary layer can be discretized by ordered particles at interatomic distances, mutually exchanging forces (per unit area). Pairs of equal opposite forces are supposed to act on contiguous particles. We can specify such pairs of particles in the Lagrangian configuration, for instance the former located just over the cube face with normal vector $\mathbf{e}_1$, the latter separated from the former by a lever arm along the direction $\mathbf{e}_2$ (and hence located over the same face). Of course, the limit scenario must be imagined, with the traction intensity tending to infinity and the mutual distance along the lever arm tending to zero whilst keeping unchanged the orientation, thus obtaining a finite value from their product. According to this description, the individual entry $\partial \mathrm{W}/\partial F^i_{12}$, dimensionally a pressure multiplied by a length, can be thought at varying i as mechanically equivalent to pairs of equal opposite Eulerian traction vectors (see Sect. 5), applied to the particles which are the Eulerian counterparts (via placement map) of the extremes of the lever arms mentioned above. Such particles are close to a face whose normal vector is the Eulerian counterpart of $\mathbf{e}_1$. Of course, since $\partial \mathrm{W}/\partial F^i_{12} = \partial \mathrm{W}/\partial F^i_{21}$, the role of the two Lagrangian indices can be permuted. For the inner double force in Eq. (11), doubly contracted with the normal vector, the lever arm is orthogonal to the face with normal N_A, and the corresponding pairs of equal opposite traction vectors in the Eulerian configuration act closely to the face with normal vector $n_k = (\mathbf{F}^{-1})^A_k N_A/\|\mathbf{F}^{-T}\mathbf{N}\|$. When passing to the limit as suggested above, of course no couples can be generated. Inner double force generates work only if the gradient of the virtual placement vector $\delta\chi^i$ along the direction normal to the face does not vanish. For this reason, the double force plays a crucial role, for instance, in the debonding process for the joined assemblies, or in fluid mechanics problems including the surface tension. The inner double force acting along the edge will be discussed later.

The surface term (‖) in Eq. (9) including the tangential projector $[\mathbf{M}_\parallel]$ can be further reduced. Exploiting the idempotence of the projector, we can duplicate it and utilize the ambient variables for both the divergence operator and the virtual placement map, namely $[M_\parallel]^C_A = [M_\parallel]^D_A [M_\parallel]^C_D$, thus avoiding the intrinsic representation of the surface. Through integration by parts, one obtains

$$(\parallel) = \int_{\Sigma_\star} \frac{\partial \mathrm{W}}{\partial F^i_{AB}} N_B [M_\parallel]^D_A \left(\frac{\partial}{\partial X^C}\delta\chi^i\right)[M_\parallel]^C_D \, d\Sigma_\star =$$
$$= \int_{\Sigma_\star} \left\{ \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{AB}} N_B [M_\parallel]^D_A \delta\chi^i\right) + -\frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{AB}} N_B [M_\parallel]^D_A\right)\delta\chi^i \right\}[M_\parallel]^C_D \, d\Sigma_\star \tag{12}$$

The first addend, once attained the format suitable for the surface divergence theorem [Part I, Eq. (74)] (see also [42]), can be transported to the border edge $L_\star \equiv \partial\Sigma_\star\ (= \partial\partial\Omega_\star)$, namely

$$= \underbrace{\int_{L_\star} \frac{\partial \mathrm{W}}{\partial F^i_{AB}} N_B B_A \delta\chi^i \, dL_\star}_{(\models)} + \underbrace{-\int_{\Sigma_\star} \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i_{AB}} N_B [M_\parallel]^D_A\right)[M_\parallel]^C_D \delta\chi^i \, d\Sigma_\star}_{(\Box)} \tag{13}$$

whilst the latter will be discussed later. In the first addend (⊨), we have set $[M_\parallel]^C_A B_C = B_A$, being the edge normal vector tangent to the boundary face. Such an addend, represented by a line integral and including the first variation of the placement map, is now complete. It is worth emphasizing that the expression work-conjugate to $\delta\chi^i$ is identical to the inner double force of Eq. (11) except the fact that herein the Lagrangian hyperstress was contracted with the components of two diverse vectors, the face normal N_B and the edge normal B_A, respectively, see [41]: equivalently, with $\mathbf{N}\otimes\mathbf{B}$. We call this term (inner) edge force, to distinguish it from its surface counterpart. According to the mechanical scheme imagined above, the pair of interacting particles is now close to the face with normal $\mathbf{N}$, and the lever arm between them lies along the direction of the edge normal $\mathbf{B}$. Since the Lagrangian hyperstress is symmetric wrt its contravariant indices, in the double contraction only the symmetric part $N_{(B}B_{A)}$ can play a role, see, e.g. [18]. Moreover, along each edge two geometrical contributions must be added, relevant to the contiguous faces, and their superposition turns out again to be symmetric, as already discussed in [Part I, Eqs. (18)–(25)].

Recalling the description of the frame vectors in a neighborhood of a border edge outlined in [Part I, Section 4], the addend (⊨) in Eq. (13) can be evaluated as follows

$$(\models) = \sum_{f=1}^{n_{\text{faces}}} \sum_{e=1}^{n^{(f)}_{\text{edge}}} \left\{ \int_{\overrightarrow{L^{(fe)}_\star}} \frac{\partial \mathrm{W}}{\partial F^i_{AB}} [N^{(f)}]_B [B^{(fe)}]_A \delta\chi^i \, dL_\star \right\} =$$
$$= \sum_{f=1}^{n_{\text{faces}}} \sum_{e=1}^{n^{(f)}_{\text{edge}}} \left\{ \int_0^{\tilde{L}^{(fe)}_\star} \frac{\partial \mathrm{W}}{\partial F^i_{AB}} \delta\chi^i \left([N^{(f)}]_B [B^{(fe)}]_A\right)_{\boldsymbol{\mu}(s)} ds \right\} \tag{14}$$

being $\boldsymbol{\mu}(s)$ the representation of the edge with respect to the curvilinear abscissa, with unit Jacobian and the integration domain equal to the rectified length of the curve, say $\tilde{L}^{(fe)}_\star$. All terms within the integral are intended to be restricted to such a curve: however, the dependence on $\boldsymbol{\mu}(s)$ was indicated explicitly only where it can affect the sign. The double loop in Eq. (14) implies that the integral along one edge appears two times, since the edge support is in common between two faces. When, for instance, two contiguous faces with an edge in common are parallel (with an identical outward normal $\mathbf{N}^{(f)}$), the contribution to the overall sum provided by that edge vanishes, since vectors $\mathbf{B}^{(fe)}$ relevant to the contiguous faces are equal opposite (as well as the edge tangent $\mathbf{T}^{(fe)}$). In fact, that edge actually does not represent a discontinuity locus for the normal vector field, and the two contiguous faces constitute indeed a unique regular region.

To evaluate the edge contribution, the above approach Eq. (14) resting on a double loop (the latter nested in the former) does not seem very straightforward. If instead we orderly gather all the edges, counted once, into a unique vector sized $\mathrm{ntot}_{\mathrm{edge}}$, run by a unique index tote, i.e. according to an edge driven approach, we can easily include in the same integral pairs of quantities discontinuous across that edge and belonging to contiguous faces (i.e. having the support of that edge as common border). By formulae

$$(\models) = \sum_{\mathrm{tote}=1}^{\mathrm{ntot}_{\mathrm{edge}}} \int_0^{\tilde{L}_\star^{\mathrm{tote}}} \frac{\partial \mathrm{W}}{\partial F^i_{\ QP}} \left([N^+]_P \, [B^+]_Q + [N^-]_P \, [B^-]_Q \right) \delta\chi^i \, ds \tag{15}$$

As already discussed in [Part I, Section 4], sign $+$ (or $-$) indicates here the face at the left (resp. at the right) of the edge support crossed in the positive direction $\mathbf{T}^+$, so that $\mathbf{B}^+$, $\mathbf{T}^+$ and $\mathbf{N}^+$ constitute a right handed frame along the edge (counterclockwise). Since the above algebraic sum is commutative and the Jacobian of the line integral is not affected by the curve orientation, the selection of the positive edge tangent $\mathbf{T}^+$ for Eq. (15) is purely conventional and can be left to the user. It is worth noting that the tensorial sum within parenthesis in Eq. (15) is symmetric wrt indices P and Q, see [Part I, Eqs. (18)–(25)], and in any case it is symmetrized by the double dot product with a symmetric tensor.

4 Dependence of contact pressures on local curvature

At this stage, we can analyse the surface contribution in Eq. (13) including the tangential projectors. By the Leibniz product rule, one has

$$\begin{aligned}
(\square) &= -\int_{\Sigma_\star} \frac{\partial}{\partial X^C} \left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, N_B \, [M_\parallel]^D_A \right) \delta\chi^i \, [M_\parallel]^C_D d\Sigma_\star = \\
&= -\int_{\Sigma_\star} \left\{ \frac{\partial}{\partial X^C} \left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \right) N_B \, [M_\parallel]^D_A + \right. \\
&\quad \left. + \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \, \frac{\partial}{\partial X^C} \left(N_B \, [M_\parallel]^D_A \right) \right\} \delta\chi^i \, [M_\parallel]^C_D d\Sigma_\star
\end{aligned} \tag{16}$$

The tangential projectors appearing in the above equation can be expressed as functions of the normal vector components [Part I, Appendix A]. Let us start from the derivative included in the last addend

$$\begin{aligned}
\frac{\partial}{\partial X^C} \left(N_B \, [M_\parallel]^D_A \right) &= \frac{\partial}{\partial X^C} \left(N_B \, [\delta^D_A - N^D N_A] \right) = \\
&= \frac{\partial N_B}{\partial X^C} \, \delta^D_A - \frac{\partial}{\partial X^C} \left(N_B \, N^D \, N_A \right);
\end{aligned} \tag{17}$$

which can be substituted into Eq. (16) obtaining

$$\begin{aligned}
(\square) &= -\int_{\Sigma_\star} \left\{ \frac{\partial}{\partial X^C} \left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \right) N_B \, [M_\parallel]^D_A + \right. \\
&\quad \left. + \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \left(\frac{\partial N_B}{\partial X^C} \, \delta^D_A - \frac{\partial}{\partial X^C} \left(N_B \, N^D \, N_A \right) \right) \right\} \delta\chi^i [M_\parallel]^C_D d\Sigma_\star = \\
&= \int_{\Sigma_\star} -[M_\parallel]^C_A \frac{\partial}{\partial X^C} \left(\frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} \right) N_B - \frac{\partial \mathrm{W}}{\partial F^i_{\ DB}} \, [M_\parallel]^C_D \, \frac{\partial N_B}{\partial X^C} + \\
&\quad + \frac{\partial \mathrm{W}}{\partial F^i_{\ AB}} [M_\parallel]^C_D \frac{\partial}{\partial X^C} \left(N_B \, N^D \, N_A \right) \delta\chi^i \, d\Sigma_\star
\end{aligned} \tag{18}$$

By following the same strategy for the remaining projectors in Eq. (18), being $[M_\parallel]^D_A [M_\parallel]^C_D = [M_\parallel]^C_A$ due to the idempotence, one finds

$$
\begin{aligned}
= \int_{\Sigma_\star} \Bigg\{ & - \frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\right) N_B + \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\right) N_B N^C N_A + \\
& - \frac{\partial \mathrm{W}}{\partial F^i{}_{DB}} \frac{\partial N_B}{\partial X^D} + \underbrace{\frac{\partial \mathrm{W}}{\partial F^i{}_{DB}} N^C N_D \frac{\partial N_B}{\partial X^C}}_{=0} + \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} \underbrace{\frac{\partial}{\partial X^D}\left(N_B N^D N_A\right)}_{=N_B N_A\, \partial N^D/\partial X^D} + \\
& \underbrace{- \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} N^C N_D \frac{\partial}{\partial X^C}\left(N_B N^D N_A\right)}_{=0} \Bigg\} \delta\chi^i \, d\Sigma_\star
\end{aligned} \tag{19}
$$

where recourse was made to the following relationships

$$
\frac{\partial N_D}{\partial X^B} N^D = 0; \qquad \frac{\partial N_D}{\partial X^B} N^B = 0; \tag{20}
$$

Finally, we can provide a detailed expression for the virtual work over the boundary face in Eq. (16), namely

$$
\begin{aligned}
= \int_{\Sigma_\star} \Bigg\{ & -\frac{\partial}{\partial X^A}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\right) N_B - \frac{\partial \mathrm{W}}{\partial F^i{}_{DB}} \frac{\partial N_B}{\partial X^D} + \\
& + \frac{\partial}{\partial X^C}\left(\frac{\partial \mathrm{W}}{\partial F^i{}_{AB}}\right) N_B N^C N_A + \frac{\partial \mathrm{W}}{\partial F^i{}_{AB}} N_B N_A \frac{\partial N^D}{\partial X^D} \Bigg\} \delta\chi^i \, d\Sigma_\star
\end{aligned} \tag{21}
$$

Equation (21) reveals for second gradient materials a complex dependence of the contact pressures (work-conjugate to $\delta\chi^i$) on the shape of the boundary face. In fact, beside an addend linearly dependent on one normal vector component, analogous to that at the basis of Cauchy's first gradient theory, we can notice a differential relationship including the normal vector components (through the Lagrangian gradient), an algebraic polynomial of normal vector components of third degree (through a triple product of normal components), a combination of the above two expressions (through a double product of normal components multiplied by the divergence of the normal vector). As well known, the divergence of the normal vector equals the mean curvature H of the face (up to a factor -2).[1] If a boundary face is flat, only the contributions with an algebraic dependence on the normal vector components are present.

5 Work-conjugate pairs

We derived the inner virtual work through the first variation of the Lagrangian energy functional $\mathcal{E}^{\mathrm{DEF}}$ in the reference configuration $\Omega_\star$: integrating by parts, through the divergence theorem we provided volume, surface and edge contributions. So far we utilized the partial derivatives of the energy density with respect to the first

[1] The mean curvature H of a surface in $\mathcal{R}^3$ is related to the trace of the Weingarten shape operator S^β_α ($\alpha, \beta = 1, 2$, see, e.g. [43,44]) and to the normal vector $\mathbf{N}$ by the following relationships

$$
\frac{\partial N^D}{\partial X^D} = \mathrm{DIV}_X \mathbf{N} = -\mathrm{tr}(S^\beta_\alpha) = -2H = -\frac{2}{R_m};
$$

being R_m the radius of mean curvature. The entries of the self-adjoint, second rank shape tensor S^β_α, expressed in intrinsic coordinates, can be evaluated on the basis of the first and the second fundamental forms of the surface. The eigenvalues of the Weingarten shape operator represent the principal curvatures: their sum equals the trace (i.e. 2 H), whilst their product, i.e. the determinant, provide the Gauss curvature. Surprisingly, only the Gauss curvature is invariant by local isometries: in fact, it can be proven (Theorem Egregium) that the Gaussian curvature depends only on the first form coefficients and on their derivatives, therefore on the local metrics which is preserved by local isometries.

and the second Lagrangian gradient of the placement map, namely F^i_A and F^i_{AB}, without making more explicit their role. At this stage, we can analyse in depth these terms by setting

$$P^A_{1\,i} = \frac{\partial \mathrm{W}}{\partial F^i_A}; \qquad P^{AB}_{2\,i} = \frac{\partial \mathrm{W}}{\partial F^i_{AB}}; \tag{22}$$

Symbol $P^A_{1\,i}(\mathbf{X})$ represents a stress-like tensor, referred to as first Piola–Kirchhoff stress, with a leg in the Eulerian configuration and another one in the Lagrangian configuration. $P^{AB}_{2\,i}(\mathbf{X})$ represents a third rank tensor, two times contravariant (Lagrangian indices) and one time covariant (Eulerian). We will refer to it as Piola hyperstress. The subscripts 1 and 2 remark that the relevant tensors were provided by differentiation wrt the first and the second gradient, respectively. At this point, we can everywhere substitute the symbols $P^A_{1\,i}$ and $P^{AB}_{2\,i}$, without any ambiguity. For instance, the Lagrangian inner virtual work in Eq. (2) can now be expressed as follows

$$\delta\mathcal{E}^{\mathrm{DEF}} = \int_{\Omega_\star} P^A_{1\,i}\,\delta F^i_A\, d\Omega_\star + \int_{\Omega_\star} P^{AB}_{2\,i}\,\delta F^i_{AB}\, d\Omega_\star;$$
$$\delta F^i_A = \frac{\partial}{\partial X^A}\delta\chi^i(\mathbf{X}); \quad \delta F^i_{AB} = \frac{\partial^2}{\partial X^A \partial X^B}\,\delta\chi^i(\mathbf{X}); \tag{23}$$

It must be stressed that the analytical developments through the integration by parts were made possible by the above representation of the internal work, including δF^i_A and δF^i_{AB}. By a trivial change of variables, the same functional Eq. (23) can be referred to the Eulerian configuration Ω: however, whilst the Lagrangian virtual work represents the first variation of an energy functional, the Eulerian form does not, since the integration volume changes along the deformation process.

The Eulerian counterpart of the virtual placement map can be defined through the composition with the inverse mapping, namely $\delta\boldsymbol{\chi}(\mathbf{x}) = \delta\boldsymbol{\chi} \circ \boldsymbol{\chi}^{-1}(\mathbf{x})$. Accordingly, we can express the first and in turn the second (spatial) gradient of the Eulerian test function through their Lagrangian counterparts, namely

$$\delta D^i_j = \frac{\partial}{\partial x^j}\delta\chi^i(\mathbf{x}) = \frac{\partial \delta\chi^i}{\partial X^A}\frac{\partial X^A}{\partial x^j} = \delta F^i_A\left(\mathbf{F}^{-1}\right)^A_j; \tag{24}$$

and

$$\begin{aligned}\delta D^i_{k\,j} &= \frac{\partial^2}{\partial x^j x^k}\delta\chi^i(\mathbf{x}) = \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^j}\delta\chi^i(\mathbf{x})\right) = \\ &= \frac{\partial}{\partial x^k}\left(\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A\right) = \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \left(\mathbf{F}^{-1}\right)^A_j \frac{\partial}{\partial x^k}\delta F^i_A = \\ &= \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^A_j \delta F^i_A + \left(\mathbf{F}^{-1}\right)^A_j \delta F^i_{AB}\left(\mathbf{F}^{-1}\right)^B_k;\end{aligned} \tag{25}$$

Rearranging Eqs. (24) and (25), firstly one can write

$$\left(\mathbf{F}^{-1}\right)^A_j\left(\mathbf{F}^{-1}\right)^B_k \delta F^i_{AB} = \delta D^i_{k\,j} - \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L; \tag{26}$$

and then, through the relationship (being $F^j_A\left(\mathbf{F}^{-1}\right)^L_j = \delta^L_A$)

$$F^j_A \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j = -\frac{\partial}{\partial x^k}\left(F^j_A\right)\left(\mathbf{F}^{-1}\right)^L_j \qquad (\dagger)$$

one obtains

$$\begin{aligned}\delta F^i_{AB} &= F^j_A F^k_B\,\delta D^i_{k\,j} - F^k_B \underbrace{F^j_A \frac{\partial}{\partial x^k}\left(\mathbf{F}^{-1}\right)^L_j}_{\text{via }(\dagger)} \delta F^i_L = \\ &= F^j_A F^k_B \delta D^i_{k\,j} + F^k_B \frac{\partial}{\partial x^k}\left(F^j_A\right)\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L =\end{aligned}$$

$$= F^j_A F^k_B \,\delta D^i_{k\,j} + F^j_{AC} \underbrace{F^k_B \left(\mathbf{F}^{-1}\right)^C_k}_{=\delta^C_B} \left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L =$$

$$= F^j_A F^k_B \,\delta D^i_{k\,j} + F^j_{AB} \left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L \tag{27}$$

The Lagrangian inner virtual work of Eq. (23), once referred to the spatial configuration, must equal its Eulerian counterpart, expressed through properly selected dual variables. One finds

$$\begin{aligned}
\delta\mathcal{E}^{\mathrm{DEF}} &= \int_\Omega J^{-1} P^A_{1\,i}\, \delta F^i_A \, d\Omega + \int_\Omega J^{-1} P^{AB}_{2\,i}\, \delta F^i_{AB}\, d\Omega = \\
&= \int_\Omega J^{-1} P^A_{1\,i} \underbrace{F^j_A \delta D^i_j}_{\delta F^i_A} d\Omega + \\
&\quad + \int_\Omega J^{-1} P^{AB}_{2\,i} \Big[F^j_A F^k_B \,\delta D^i_{k\,j} + F^j_{AB} \underbrace{\left(\mathbf{F}^{-1}\right)^L_j \delta F^i_L}_{\delta D^i_j} \Big] d\Omega = \\
&= \int_\Omega T^j_{1\,i}\, \delta D^i_j \, d\Omega + \int_\Omega T^{jk}_{2\,i}\, \delta D^i_{jk}\, d\Omega;
\end{aligned} \tag{28}$$

It is worth emphasizing that the Eulerian test functions appearing in the last row, namely δD^i_j and δD^i_{jk}, result from the transformation of the Lagrangian test functions δF^i_A and δF^i_{AB}, respectively, when referred to the current configuration. On the basis of Eq. (28), it is now possible to specify relationships between the Eulerian and the Lagrangian dual quantities, distinguishing first and second gradient contributions, namely

$$\begin{aligned}
T^j_{1\,i} &= J^{-1}\, P^A_{1\,i}\, F^j_A + J^{-1} P^{AB}_{2\,i}\, F^j_{AB}; \\
T^{jk}_{2\,i} &= J^{-1}\, P^{AB}_{2\,i}\, F^j_A F^k_B;
\end{aligned} \tag{29}$$

Once selected the above work-conjugate pairs, the mathematical structure of the governing equations remains the same in the Eulerian and in the Lagrangian configuration, and the integration by parts can be carried out once in an abstract setting, without duplicating the procedure.

6 External work and strong form

At this stage, we can specify the contributions of the external loading to the Lagrangian virtual work for a second gradient material, namely

$$\begin{aligned}
\delta\,\mathcal{E}^{\mathrm{EXT}} &= \int_{\Omega_\star} \mathcal{F}^{\mathrm{ext}}_{\Omega_\star\,i}\, \delta\chi^i \, d\Omega_\star + \int_{\Sigma_\star} \mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\,i}\, \delta\chi^i \, d\Sigma_\star + \\
&\quad + \int_{\Sigma_\star} \mathcal{F}^{\mathrm{ext}}_{\perp\Sigma_\star\,i}\, \frac{\partial\,\delta\chi^i}{\partial N}\, d\Sigma_\star + \int_{L_\star} \mathcal{F}^{\mathrm{ext}}_{L_\star\,i}\, \delta\chi^i \, dL_\star;
\end{aligned} \tag{30}$$

The above symbols possess the following meaning: $\mathcal{F}^{\mathrm{ext}}_{\Omega_\star\,i}(\mathbf{X})$, $\mathcal{F}^{\mathrm{ext}}_{\Sigma_\star\,i}(\mathbf{X})$ and $\mathcal{F}^{\mathrm{ext}}_{L_\star\,i}(\mathbf{X})$ denote Eulerian vectors defined in the Lagrangian domain ($\Omega_\star$), over its boundary face ($\Sigma_\star$) and along its border edges ($L_\star$), dimensionally equal to force densities per unit volume, per unit surface and per unit length, respectively; $\mathcal{F}^{\mathrm{ext}}_{\perp\Sigma_\star\,i}(\mathbf{X})$ indicates an Eulerian vector field defined over the Lagrangian boundary surfaces, referred to as external surface double force, dimensionally equal to a force per unit length. The above contributions to the external virtual work sustainable by a second gradient body emanate directly from the initial assumption on the energy density of the material, Eq. (1), and correspond to the irreducible terms derived so far from the inner virtual work.

According to the fundamental lemma of the calculus of variations, we derived the strong form of the Lagrangian equilibrium equations by selecting the test functions $\delta\boldsymbol{\chi}$ with their compact support localized in turn within the volume interior, over the boundary faces (excluding their border edges) and along the edges (excluding the wedges, if any), whilst for $\partial\delta\boldsymbol{\chi}/\partial N$ it was localized within the boundary faces, excluding their border. For the sake of simplicity, with a slight abuse of notation the difference (in the sense of set theory) between such domains and their differential border was herein denoted by the symbol $\mathring{D}$. Hence, from Eqs. (5), (10), (15), (21) one can write

$$
\begin{aligned}
& -\frac{\partial}{\partial X^A}\left(P_{1\,i}^{A}\right)+\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^A}\left(P_{2\,i}^{AB}\right)-\mathcal{F}^{\text{ext}}_{\Omega_\star\,i}(\mathbf{X})=0 \qquad \mathbf{X}\in\mathring{\Omega}_\star;\\
& P_{1\,i}^{A}N_A-\frac{\partial P_{2\,i}^{AB}}{\partial X^B}N_A-\frac{\partial P_{2\,i}^{AB}}{\partial X^A}N_B-P_{2\,i}^{AB}\frac{\partial N_A}{\partial X^B}+\frac{\partial P_{2\,i}^{AB}}{\partial X^C}N_BN^CN_A+\\
& \qquad +P_{2\,i}^{AB}N_BN_A\frac{\partial N^D}{\partial X^D}-\mathcal{F}^{\text{ext}}_{\Sigma_\star\,i}(\mathbf{X})=0 \qquad \mathbf{X}\in\mathring{\Sigma}_\star;\\
& P_{2\,i}^{AB}\,N_AN_B-\mathcal{F}^{\text{ext}}_{\perp\Sigma_\star\,i}(\mathbf{X})=0 \qquad \mathbf{X}\in\mathring{\Sigma}_\star;\\
& P_{2\,i}^{AB}\,\left([B^+]_A[N^+]_B+[B^-]_A[N^-]_B\right)-\mathcal{F}^{\text{ext}}_{L_\star\,i}(\mathbf{X})=0 \qquad \mathbf{X}\in\mathring{L}_\star;
\end{aligned}
\tag{31}
$$

As already discussed in Eq. (7), the first two equations can assume a slightly shorter form by entering an effective Lagrangian stress thus defined

$$
P_i^{\text{eff}\,A}=P_{1\,i}^{A}-\frac{\partial P_{2\,i}^{AB}}{\partial X^B}
\tag{32}
$$

The equation in the second row is often expressed in a more compact form by means of the tangential projectors and the surface divergence, see Eq. (16), namely

$$
\left(P_{1\,i}^{A}-\frac{\partial P_{2\,i}^{AB}}{\partial X^B}\right)N_A-[M_\parallel]_E^C\frac{\partial}{\partial X^C}\left([M_\parallel]_B^E\,P_{2\,i}^{AB}\,N_A\right)-\mathcal{F}^{\text{ext}}_{\Sigma_\star\,i}(\mathbf{X})=0 \quad \mathbf{X}\in\mathring{\Sigma}_\star;
\tag{33}
$$

Hence, the governing equations in the Eulerian configuration can be derived from Eqs. (31)–(33), by formally substituting the material coordinates of differential and integral operators with the spatial ones ($X^A\to x^r$) and relevant integration domains, the Piola stress and hyperstress tensors with their Cauchy's counterparts, namely $P_{1\,i}^{A}\to T_{1\,i}$ and $P_{2\,i}^{AB}\to T_{2\,i}^{jk}$.

7 Transport of governing equations

After several intermediate results, it is now possible to transform the governing equations for second gradient continua from the Eulerian to the Lagrangian form. It is worth emphasizing that the transport of the governing equations involves the entire set of equations, and therefore, the final balance has to be achieved at a global level, and not necessarily term by term. However, due to practical reasons, the problem was tackled separately for the diverse contributions to the virtual work, grouped according to some (reasonable but questionable) criterion. Two diverse strategies were outlined to transform the surface contribution including the contact pressures.

7.1 Volume terms

We considered firstly the volume terms, extracted from Eq. (5). As well known, Piola's volume formula (see, e.g. [36,45,46]) allows one to transform the Eulerian divergence of a suitable expression involving a vector field, namely $J^{-1}F_B^aV^B$, into the Lagrangian divergence of the same vector field V^B [Part I, Eq. (37)]. It can be formulated also for the inverse transformation. The availability of Piola's bulk formula has made the Eulerian/Lagrangian transport of the volume contributions relatively easy. On the contrary, the transport of the face and edge contributions, to be discussed in the next paragraphs, exhibited huge difficulties which have required novel strategies.

Starting from the Eulerian volume terms, by utilizing the conversion formulae Eq. (29) for the stress and the hyperstress tensors one finds

$$\int_{\Omega}\left\{-\frac{\partial}{\partial x^j}\left(T_{1\,i}^{j}\right)+\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\left(T_{2\,i}^{jk}\right)\right\}\delta\chi^i(\mathbf{x})\,d\Omega=$$
$$=\int_{\Omega}\left\{-\frac{\partial}{\partial x^j}\left(J^{-1}\,P_{1\,i}^{A}\,F_{A}^{j}+J^{-1}P_{2\,i}^{AB}\,F_{AB}^{j}\right)+\right.$$
$$\left.\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\left(J^{-1}\,P_{2\,i}^{AB}\,F_{A}^{j}F_{B}^{k}\right)\right\}\delta\chi^i\,d\Omega \tag{34}$$

Through Piola's volume formula, two Eulerian divergence operators in the above equation are easily transformed into their Lagrangian counterparts; differentiating the products, two contributions could be cancelled, and finally by a change of variables one obtains

$$=\int_{\Omega}\left\{-J^{-1}\underbrace{F_{A}^{j}\frac{\partial}{\partial x^j}}_{=\partial/\partial X^A}\left(P_{1\,i}^{A}\right)-\frac{\partial}{\partial x^j}\left(J^{-1}P_{2\,i}^{AB}\,F_{AB}^{j}\right)+\right.$$
$$\left.+\frac{\partial}{\partial x^j}\left(J^{-1}\underbrace{F_{B}^{k}\frac{\partial}{\partial x^k}}_{=\partial/\partial X^B}\left(P_{2\,i}^{AB}\,F_{A}^{j}\right)\right)\right\}\delta\chi^i\,d\Omega=$$
$$=\int_{\Omega}\left\{-J^{-1}\frac{\partial}{\partial X^A}\left(P_{1\,i}^{A}\right)-\frac{\partial}{\partial x^j}\left(J^{-1}P_{2\,i}^{AB}\,F_{AB}^{j}\right)+\right.$$
$$\left.+\frac{\partial}{\partial x^j}\left(J^{-1}F_{A}^{j}\frac{\partial}{\partial X^B}\left(P_{2\,i}^{AB}\right)+J^{-1}\,P_{2\,i}^{AB}\frac{\partial}{\partial X^B}\left(F_{A}^{j}\right)\right)\right\}\delta\chi^i\,d\Omega=$$
$$=\int_{\Omega_\star}J\left\{-J^{-1}\frac{\partial}{\partial X^A}\left(P_{1\,i}^{A}\right)+J^{-1}\frac{\partial}{\partial X^A}\left(\frac{\partial P_{2\,i}^{AB}}{\partial X^B}\right)+\right.$$
$$\left.\underbrace{-\frac{\partial}{\partial x^j}\left(J^{-1}P_{2\,i}^{AB}\,F_{AB}^{j}\right)+\frac{\partial}{\partial x^j}\left(J^{-1}\,P_{2\,i}^{AB}\,F_{AB}^{j}\right)}_{=0}\right\}\delta\chi^i(\mathbf{X})\,d\Omega_\star=$$
$$=\int_{\Omega_\star}\left\{-\frac{\partial}{\partial X^A}\left(P_{1\,i}^{A}\right)+\frac{\partial}{\partial X^B}\frac{\partial}{\partial X^A}\left(P_{2\,i}^{AB}\right)\right\}\delta\chi^i\,d\Omega_\star \tag{35}$$

The transport of the volume contributions turned out to be self-consistent and did not give rise to residual terms. This circumstance implies also that the virtual work of the external loading per unit volume possibly prescribed in the Eulerian configuration, once carried out the change of variables, must balance exclusively the corresponding volume inner work in the Lagrangian configuration.

7.2 Double force term

The virtual work contribution relevant to the boundary face Σ and including the normal derivative of the test function $\partial\,\delta\chi^i/\partial N$ does not appear in Cauchy's theory, and it is peculiar of the higher order gradient continua. Now, let us consider the transformation of such an Eulerian contribution to the Lagrangian form. It is worth recalling that the change of variables implies for the surface element the transformation $d\Sigma=\|J\,\mathbf{F}^{-T}\mathbf{N}\|\,d\Sigma_\star$, see, e.g. [36]. By converting the hyperstresses through Eq. (29) and utilizing the transformation formula for the covariant components of the normal vector [Part I, Eq. (47)], one finds

$$\int_{\Sigma}T_{2\,i}^{jk}\,\frac{\partial\delta\chi^i}{\partial x^r}n^r\,n_k\,n_j\,d\Sigma=\int_{\Sigma}\left(J^{-1}\,P_{2\,i}^{AB}\,F_{A}^{j}F_{B}^{k}\right)\frac{\partial\delta\chi^i}{\partial x^r}n^r\,n_k\,n_j\,d\Sigma=$$
$$=\int_{\Sigma_\star}J^{-1}\,P_{2\,i}^{AB}\,\frac{N_A}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\frac{N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|}\,\frac{\partial\delta\chi^i}{\partial x^r}g^{rs}\,\frac{\left(\mathbf{F}^{-1}\right)_{s}^{S}\,N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|}\|J\,\mathbf{F}^{-T}\mathbf{N}\|\,d\Sigma_\star=$$

$$
\begin{aligned}
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \underbrace{F_R^r \frac{\partial \delta\chi^i}{\partial x^r}}_{=\partial\cdot/\partial X^R} \frac{g^{rs}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left(\mathbf{F}^{-1}\right)_r^R \left(\mathbf{F}^{-1}\right)_s^S N_S \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^R} \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \underbrace{g^{rs} \left(\mathbf{F}^{-1}\right)_r^R \left(\mathbf{F}^{-1}\right)_s^S}_{=g^{\star RS}} N_S \, d\Sigma_\star =
\end{aligned} \tag{36}
$$

where the Eulerian metric tensor g^{rs} was utilized for raising the index of the normal. In the above equation, we can recognize the contravariant pullback metric tensor $g^{\star RS}$ (see [Part I, Section 5]), divided by the surface element norm squared. As expected, the Eulerian normal derivative of the virtual placement map in Eq. (36) corresponds in the Lagrangian configuration to the directional derivative along the oblique direction $g^{\star RS} N_S$, with non-vanishing projections in both the tangent and the normal space with respect to the (material) boundary face. Therefore, according to a strategy already utilized in this study, one can multiply such a gradient by the unit tensor, which equals the sum of the tangential and the orthogonal projectors to the boundary face [Part I, Appendix A], and then manage separately the two addends. Hence, one can write

$$
\begin{aligned}
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} \delta_R^E \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} \left\{ [M_\perp]_R^E + [M_\parallel]_R^E \right) \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} \left\{ N^E N_R + [M_\parallel]_R^E \right\} \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} N^E \underbrace{\frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_R N_S}_{=1} \, d\Sigma_\star + \\
&\quad + \underbrace{\int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} [M_\parallel]_R^E \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, d\Sigma_\star}_{=(*)}
\end{aligned} \tag{37}
$$

where recourse was made to the relationships $[M_\perp]_R^E = N^E N_R$ and $g^{\star RS} N_R N_S = \|\mathbf{F}^{-T}\mathbf{N}\|^2$. By Eq. (37), we have successfully retrieved as first addend the term of Lagrangian inner virtual work including the double force, whilst the latter contribution, indicated by $(*)$ and including the tangential projector $[M_\parallel]_R^E$, can be further reduced. Exploiting the idempotence of the projector and integrating by parts, one finds

$$
\begin{aligned}
(*) &= + \int_{\Sigma_\star} P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} \left([M_\parallel]_L^E [M_\parallel]_R^L \right) \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, d\Sigma_\star = \\
&= + \int_{\Sigma_\star} \left\{ P_{2i}^{AB} \, N_A N_B \, \frac{\partial \delta\chi^i}{\partial X^E} [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right\} [M_\parallel]_L^E \, d\Sigma_\star = \\
&= + \int_{\Sigma_\star} \left\{ \frac{\partial}{\partial X^E} \left(P_{2i}^{AB} \, N_A N_B [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, \delta\chi^i \right) + \right. \\
&\quad \left. - \delta\chi^i \frac{\partial}{\partial X^E} \left(P_{2i}^{AB} \, N_A N_B \, [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) \right\} [M_\parallel]_L^E \, d\Sigma_\star = \\
&= + \int_{\Sigma_\star} [M_\parallel]_L^E \frac{\partial}{\partial X^E} \left(P_{2i}^{AB} \, N_A N_B [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \, \delta\chi^i \right) d\Sigma_\star + \\
&\quad - \int_{\Sigma_\star} \delta\chi^i \, [M_\parallel]_L^E \frac{\partial}{\partial X^E} \left(P_{2i}^{AB} \, N_A N_B \, [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) d\Sigma_\star =
\end{aligned} \tag{38}
$$

Of the two contributions in Eq. (38), the former gives rise through the surface divergence theorem to a line integral along the border edges (see, e.g. [43]), whilst the latter must be further simplified. Indicating by B_R the normal vector to the border edge which is also tangent to the face (see [Part I, Eq. (74) and Sect. 4]), one has

$$\begin{aligned}
&= + \int_{L_\star} P_{2i}^{AB}\, N_A N_B\, \overbrace{[M_\parallel]_R^L [M_\parallel]_L^E B_E}^{=B_R}\, \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S\, \delta\chi^i\, dL_\star + \\
&\quad - \int_{\Sigma_\star} \delta\chi^i \frac{\partial}{\partial X^E} \left(P_{2i}^{AB}\, N_A N_B\, [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) [M_\parallel]_L^E\, d\Sigma_\star = \\
&= + \int_{L_\star} P_{2i}^{AB}\, N_A N_B \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S B_R\, \delta\chi^i\, dL_\star + \\
&\quad \underbrace{- \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left(P_{2i}^{AB}\, N_A N_B\, [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) [M_\parallel]_L^E \delta\chi^i\, d\Sigma_\star}_{=(\lhd)}
\end{aligned} \tag{39}$$

Recalling that $[M_\parallel]_R^E = \delta_R^E - N^E N_R$, one can write the second addend as follows

$$\begin{aligned}
(\lhd) &= - \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left(P_{2i}^{AB}\, N_A N_B\, [M_\parallel]_R^L \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) [M_\parallel]_L^E \delta\chi^i\, d\Sigma_\star = \\
&= - \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left(P_{2i}^{AB}\, N_A N_B \right) \left\{ \left(\delta_R^E - N^E N_R \right) \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right\} \delta\chi^i d\Sigma_\star + \\
&\quad - \int_{\Sigma_\star} \left(P_{2i}^{AB}\, N_A N_B \right) \frac{\partial}{\partial X^E} \left(\left[\delta_R^L - N^L N_R \right] \frac{g^{\star RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S \right) \times \\
&\quad \times \left[\delta_L^E - N^E N_L \right] \delta\chi^i\, d\Sigma_\star =
\end{aligned} \tag{40}$$

up to attain a very suggestive expression

$$\begin{aligned}
&= - \int_{\Sigma_\star} \frac{\partial}{\partial X^E} \left(P_{2i}^{AB}\, N_A N_B \right) \left\{ \frac{g^{\star ES}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S - N^E \right\} \delta\chi^i\, d\Sigma_\star + \\
&\quad - \int_{\Sigma_\star} \left(P_{2i}^{AB}\, N_A N_B \right) \left\{ \frac{\partial}{\partial X^L} \left(\frac{g^{\star LS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S - N^L \right) + \right. \\
&\quad \left. - N_L N^E \frac{\partial}{\partial X^E} \left(\frac{g^{\star LS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S - N^L \right) \right\} \delta\chi^i\, d\Sigma_\star
\end{aligned} \tag{41}$$

The above surface integral puts in comparison the action of the contravariant pullback metric tensor $g^{\star RS}$ on the components of the normal vector (divided by the squared norm), with the raising of indices for the same components generated by the conventional Lagrangian metrics g^{RS}, see [Part I, Section 5]. Such a difference in Eq. (41) is expressed at order zero (i.e. algebraically), under divergence operator and under doubly contracted sum of derivatives. This expression will be utilized for the transformation of the remaining terms.

It is worth emphasizing that the oblique transformation of the Eulerian normal derivative of the placement map in the double force term Eq. (36) gave rise, through the tangent projection and the integration by parts, to contributions in the Lagrangian virtual work over the boundary face $\Sigma_\star$ and along its border edge $L_\star$, which generate work versus the virtual placement map $\delta\chi^i$.

7.3 Edge integral

Let us consider the virtual work contribution along the border edges in the Eulerian configuration. By converting the hyperstress tensor through Eq. (29) and using the novel transformation formula for the edge normal in covariant form b_j [Part I, Eq. (64)], one has

$$\int_L T_{2i}^{jk} n_k b_j \, \delta\chi^i \, dL = \int_L \left(J^{-1} \, P_{2i}^{AB} \, F_A^j F_B^k \right) \delta\chi^i n_k b_j \, dL =$$
$$= \int_{L_\star} J^{-1} \, P_{2i}^{AB} \, F_B^k \, n_k \, F_A^j b_j \, \delta\chi^i \, \|\mathbf{F}\mathbf{T}\| \, dL_\star =$$
$$= \int_{L_\star} J^{-1} \, P_{2i}^{AB} \, \frac{N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|} \Bigg[\underbrace{F_A^j \left(\mathbf{F}^{-1}\right)_j^R}_{=\delta_A^R} B_R +$$
$$- \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle} \underbrace{F_A^j \left(\mathbf{F}^{-1}\right)_j^R}_{=\delta_A^R} N_R \Bigg] \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} \, \delta\chi^i \, \|\mathbf{F}\mathbf{T}\| \, dL_\star =$$
$$= \int_{L_\star} P_{2i}^{AB} \, N_B \left[B_A - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} N_A \right] \delta\chi^i \, dL_\star \tag{42}$$

At this point, we recall that a (Lagrangian) line integral was made available by the transport of the double force (see first addend in Eq. 39) and still remains unbalanced. If we add it to the last equation, the correct Lagrangian counterpart for the edge virtual work is retrieved, namely

$$\int_{L_\star} P_{2i}^{AB} \, N_B \left(B_A - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle} N_A \right) \delta\chi^i \, dL_\star +$$
$$+ \int_{L_\star} P_{2i}^{AB} \, N_A N_B \frac{g^{rs}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left(\mathbf{F}^{-1}\right)_r^R \left(\mathbf{F}^{-1}\right)_s^S N_S B_R \, \delta\chi^i \, dL_\star$$
$$= \int_{L_\star} P_{2i}^{AB} \, N_B B_A \, \delta\chi^i \, dL_\star \tag{43}$$

It is worth noting that $\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g = g^{rs} \left(\mathbf{F}^{-1}\right)_r^R \left(\mathbf{F}^{-1}\right)_s^S N_S B_R$ (see [Part I, Eq. (26)]), and therefore, one has

$$\frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle} = \frac{g^{\star\,RS} B_R N_S}{g^{\star\,PQ} N_P N_Q};$$

7.4 Face terms dual of the virtual placement

So far we succeeded in transporting from the Eulerian to the Lagrangian configuration the volume terms, the contribution over the face Σ including the double force and also the line integral along the border edge L. Their Lagrangian counterparts were exactly retrieved although cross-coupling of terms occurred, transversely to the involved domains. In this section, we orderly gathered the remaining contributions to the Eulerian inner virtual work, all generating work versus the variation of the virtual placement map $\delta\chi^i$ over the boundary surface Σ, to prove feasibility of their transformation to the Lagrangian form. We emphasize that, from the previous manipulations, a Lagrangian contribution relevant to $\Sigma_\star$ remained still not balanced, marked by the symbol ($\lhd$) in Eqs. (39)–(41). In this subsection, we have left the projectors within the face contributions to allow for the application of the revisited surface divergence theorem [Part I, Eqs. (80)–(81)], whilst in the next subsection we will express them as functions of the normal vector components, discussing in detail the transformation of the individual terms. After these remarks, we can outline the Eulerian contributions in point, which must equal their Lagrangian counterparts, namely

$$\int_\Sigma T_{1i}^{j} n_j \, \delta\chi^i \, d\Sigma - \int_\Sigma \frac{\partial}{\partial x^k} \left(T_{2i}^{jk} \right) n_j \, \delta\chi^i \, d\Sigma +$$
$$- \int_\Sigma [m_\|]_c^r \frac{\partial}{\partial x^r} \left(T_{2i}^{jk} \, n_k \, [m_\|]_j^c \right) \delta\chi^i \, d\Sigma =$$

$$= + \int_{\Sigma_\star} P_{1\,i}^{A}\, N_A\, \delta\chi^i\, d\Sigma_\star - \int_{\Sigma_\star} \frac{\partial}{\partial X^A} \left(P_{2\,i}^{AB} \right) N_B\, \delta\chi^i\, d\Sigma_\star +$$
$$- \int_{\Sigma_\star} [M_\|]_E^C \frac{\partial}{\partial X^C} \left(P_{2\,i}^{AB}\, N_A [M_\|]_B^E \right) \delta\chi^i\, d\Sigma_\star \quad + (\lhd) \tag{44}$$

For the first two addends at the lhs, linear in the normal vector, the Lagrangian form is trivially retrieved. In fact, by utilizing Eq. (29) and the transformation rule for the covariant normal vector, one finds

$$\int_\Sigma T_{1\,i}^{j} n_j\, \delta\chi^i\, d\Sigma - \int_\Sigma \frac{\partial}{\partial x^k} \left(T_{2\,i}^{jk} \right) n_j \delta\chi^i d\Sigma =$$
$$= \int_{\Sigma_\star} J^{-1} \left(P_{1\,i}^{A}\, F_A^j + P_{2\,i}^{AB}\, F_{AB}^j \right) \delta\chi^i \frac{(\mathbf{F}^{-1})_j^L N_L}{\|\mathbf{F}^{-T}\mathbf{N}\|} \|J\mathbf{F}^{-T}\mathbf{N}\|\, d\Sigma_\star +$$
$$- \int_{\Sigma_\star} \frac{\partial}{\partial x^k} \left(J^{-1}\, P_{2\,i}^{AB}\, F_A^j F_B^k \right) \delta\chi^i \frac{(\mathbf{F}^{-1})_j^L N_L}{\|\mathbf{F}^{-T}\mathbf{N}\|} \|J\mathbf{F}^{-T}\mathbf{N}\|\, d\Sigma_\star =$$
$$= \int_{\Sigma_\star} \Big(P_{1\,i}^{A} \underbrace{F_A^j (\mathbf{F}^{-1})_j^L}_{=\delta_A^L} N_L + P_{2\,i}^{AB}\, F_{AB}^j\, (\mathbf{F}^{-1})_j^L N_L \Big) \delta\chi^i\, d\Sigma_\star +$$
$$- \int_{\Sigma_\star} J \frac{\partial}{\partial x^k} \left(J^{-1}\, P_{2\,i}^{AB}\, F_A^j F_B^k \right) \delta\chi^i\, (\mathbf{F}^{-1})_j^L N_L d\Sigma_\star \tag{45}$$

By applying Piola's transformation within the last addend (with repeated index k), the functional group $J^{-1} F_B^k$ can be shifted out of the derivative and one finds

$$= \int_{\Sigma_\star} \left(P_{1\,i}^{A}\, N_A + P_{2\,i}^{AB}\, F_{AB}^j\, (\mathbf{F}^{-1})_j^L N_L \right) \delta\chi^i\, d\Sigma_\star +$$
$$- \int_{\Sigma_\star} \underbrace{J\, J^{-1}}_{=1} \underbrace{F_B^k \frac{\partial}{\partial x^k}}_{=\partial\cdot/\partial X^B} \left(P_{2\,i}^{AB}\, F_A^j \right) \delta\chi^i\, (\mathbf{F}^{-1})_j^L N_L d\Sigma_\star =$$
$$= \int_{\Sigma_\star} \left(P_{1\,i}^{A}\, N_A + P_{2\,i}^{AB}\, F_{AB}^j\, (\mathbf{F}^{-1})_j^L N_L \right) \delta\chi^i\, d\Sigma_\star +$$
$$- \int_{\Sigma_\star} \Big(\frac{\partial P_{2\,i}^{AB}}{\partial X^B} \underbrace{F_A^j (\mathbf{F}^{-1})_j^L}_{=\delta_A^L} + P_{2\,i}^{AB}\, F_{AB}^j (\mathbf{F}^{-1})_j^L \Big) \delta\chi^i\, N_L d\Sigma_\star =$$
$$= \int_{\Sigma_\star} P_{1\,i}^{A}\, N_A\, \delta\chi^i\, d\Sigma_\star - \int_{\Sigma_\star} \frac{\partial}{\partial X^A} \left(P_{2\,i}^{AB} \right) N_B \delta\chi^i\, d\Sigma_\star \tag{46}$$

With reference to the third Eulerian addend at lhs of Eq. (44) including the tangential projectors, immediately after the change of variables the revisited surface divergence theorem can be applied [Part I, Eq. (81)], by setting $w^d = T_{2\,i}^{dk}\, n_k$. Hence, one finds

$$- \int_\Sigma \frac{\partial}{\partial x^r} \left(T_{2\,i}^{jk}\, n_k\, [m_\|]_j^c \right) \delta\chi^i [m_\|]_c^r\, d\Sigma =$$
$$= - \int_{\Sigma_\star} \|J\mathbf{F}^{-T}\mathbf{N}\| \frac{\partial}{\partial x^r} \left([m_\|]_j^c \left(T_{2\,i}^{jk} n_k \right) \right) [m_\|]_c^r\, \delta\chi^i d\Sigma_\star =$$
$$= - \int_{\Sigma_\star} [M_\|]_S^A \frac{\partial}{\partial X^A} \left(\|J\mathbf{F}^{-T}\mathbf{N}\| [M_\|]_R^S \left(\mathbf{F}^{-1}\right)_d^R\, T_{2i}^{dk} n_k \right) \delta\chi^i\, d\Sigma_\star +$$
$$+ \int_{\Sigma_\star} [M_\|]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| \left[M_\|\right]_R^S \frac{g_V^{\star\,R} N^V}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\, N_W \left(\mathbf{F}^{-1}\right)_d^W\, T_{2i}^{dk} n_k \right\} \delta\chi^i\, d\Sigma_\star = \tag{47}$$

By expressing the Eulerian hyperstress T_{2i}^{dk} as a function of its Lagrangian counterpart P_{2i}^{AB} through Eq. (29), from Eq. (47) one obtains

$$
\begin{aligned}
&= -\int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left(\|J\mathbf{F}^{-T}\mathbf{N}\|[M_\parallel]_R^S \left(\mathbf{F}^{-1}\right)_d^R \left(J^{-1} P_{2i}^{QB} F_Q^d F_B^k n_k\right)\right) \delta\chi^i \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left\{\|J\mathbf{F}^{-T}\mathbf{N}\| \left[M_\parallel\right]_R^S \frac{g_V^{\star\,R} N^V}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \times \right.\\
&\qquad \left. \times N_W \left(\mathbf{F}^{-1}\right)_d^W J^{-1} P_{2i}^{QB} F_Q^d F_B^k n_k\right\} \delta\chi^i \, d\Sigma_\star =
\end{aligned}
\tag{48}
$$

and, through the transport formula for the (covariant) normal vector, one finds

$$
\begin{aligned}
&= -\int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\Big([M_\parallel]_R^S \underbrace{\left(\mathbf{F}^{-1}\right)_d^R F_Q^d}_{=\delta_Q^R} P_{2i}^{QB} N_B\Big) \delta\chi^i \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left\{\left[M_\parallel\right]_R^S \frac{g_V^{\star\,R} N^V}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_W \underbrace{\left(\mathbf{F}^{-1}\right)_d^W F_Q^d}_{=\delta_Q^W} P_{2i}^{QB} N_B\right\} \delta\chi^i \, d\Sigma_\star = \\
&= -\int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left([M_\parallel]_Q^S P_{2i}^{QB} N_B\right) \delta\chi^i \, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left\{\left[M_\parallel\right]_R^S \frac{g_V^{\star\,R} N^V}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_Q P_{2i}^{QB} N_B\right\} \delta\chi^i \, d\Sigma_\star = \\
&= -\int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left([M_\parallel]_Q^S P_{2i}^{QB} N_B\right) \delta\chi^i \, d\Sigma_\star + \\
&\underbrace{+ \int_{\Sigma_\star} [M_\parallel]_S^A \frac{\partial}{\partial X^A}\left\{\left[M_\parallel\right]_R^S \frac{g_V^{\star\,R} N^V}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} P_{2i}^{QB} N_Q N_B\right\} \delta\chi^i \, d\Sigma_\star}_{=-(\lhd)}
\end{aligned}
\tag{49}
$$

We observe that, in the last equality, the first addend coincides with the Lagrangian counterpart Eq. (44), whilst the second addend turns out to be equal opposite to the residual term Eq. (39) reported herein for the reader's convenience:

$$
(\lhd) = -\int_{\Sigma_\star} \frac{\partial}{\partial X^E}\left\{P_{2i}^{AB} N_A N_B \, [M_\parallel]_R^L \frac{g^{\star\,RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S\right\} [M_\parallel]_L^E \delta\chi^i \, d\Sigma_\star;
$$

which therefore does not affect the transformation of face equations expressed through the surface divergence operator.

7.5 Detailed transport of curvature and higher order terms

If inside the surface term of Eq. (47), one makes explicit the dependence of the projectors on the normal vector components, the transport of the higher order contributions can be investigated without the recourse to the extended divergence theorem. By comparing the detailed expressions of the Eulerian and the Lagrangian terms in Eq. (19), one has

$$
\begin{aligned}
&\int_\Sigma \left\{-\frac{\partial}{\partial x^c}\left(T_{2i}^{ck}\right) n_k - T_{2i}^{ck} \frac{\partial n_k}{\partial x^c} + \right.\\
&\quad \left. + \frac{\partial}{\partial x^r}\left(T_{2i}^{jk}\right) n_k n^r n_j + T_{2i}^{jk} n_k n_j \frac{\partial n^r}{\partial x^r}\right\} \delta\chi^i d\Sigma =
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Sigma_\star} \left\{ -\frac{\partial P_{2i}^{AB}}{\partial X^B} N_A - P_{2i}^{AB} \frac{\partial N_A}{\partial X^B} + \right. \\
&\left. + \frac{\partial P_{2i}^{AB}}{\partial X^C} N_B N^C N_A + P_{2i}^{AB} N_B N_A \frac{\partial N^D}{\partial X^D} \right\} \delta\chi^i \, d\Sigma_\star + (\lhd)
\end{aligned} \tag{50}
$$

where symbol $(\lhd)$ indicates the residual term previously discussed, in the form provided by Eq. (41). Now, for the sake of simplicity, let us group the four addends at lhs of Eq. (50) into two derivatives, and let us convert the Eulerian hyperstresses in their arguments through the formulae Eq. (29), as follows

$$
\begin{aligned}
&\int_{\Sigma} \left\{ -\frac{\partial}{\partial x^c} \left(T_{2i}^{ck} n_k \right) + \frac{\partial}{\partial x^r} \left(T_{2i}^{jk} \, n_k n^r n_j \right) \right\} \delta\chi^i d\Sigma = \\
= \; &\int_{\Sigma} \Big\{ \underbrace{-\frac{\partial}{\partial x^c} \left(J^{-1} \, P_{2i}^{AB} \, F_A^c F_B^k n_k \right)}_{=(\bowtie)} + \underbrace{\frac{\partial}{\partial x^r} \left(J^{-1} \, P_{2i}^{AB} \, F_A^j F_B^k n_k n^r n_j \right)}_{=(\Upsilon)} \Big\} \delta\chi^i \, d\Sigma
\end{aligned} \tag{51}
$$

After the change of variables, by applying Piola's transformation and utilizing the transport formula for the covariant normal vector components, one finds for the former addend

$$
\begin{aligned}
(\bowtie) &= \int_{\Sigma_\star} -J^{-1} \underbrace{F_A^c \frac{\partial}{\partial x^c}}_{=\partial(\cdot)/\partial X^A} \left(P_{2i}^{AB} \frac{N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \delta\chi^i \, \|J\mathbf{F}^{-T}\mathbf{N}\| \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i \, \|\mathbf{F}^{-T}\mathbf{N}\| \left\{ -\frac{\partial P_{2i}^{AB}}{\partial X^A} \frac{N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|} - P_{2i}^{AB} \frac{\partial N_B}{\partial X^A} \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} + \right. \\
&\quad \left. - P_{2i}^{AB} N_B \frac{\partial}{\partial X^A} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \right\} d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i \left\{ -\frac{\partial P_{2i}^{AB}}{\partial X^A} N_B - P_{2i}^{AB} \frac{\partial N_B}{\partial X^A} \right\} d\Sigma_\star + \\
&\quad \underbrace{- \int_{\Sigma_\star} \delta\chi^i \, P_{2i}^{AB} N_B \|\mathbf{F}^{-T}\mathbf{N}\| \frac{\partial}{\partial X^A} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) d\Sigma_\star}_{=\mathrm{res}_1}
\end{aligned} \tag{52}
$$

In the above equation, we have successfully retrieved the first two Lagrangian addends at rhs of Eq. (50), plus a residual addend res_1 which will be considered later. As for the second addend of Eq. (51), utilizing the novel transport formula for the contravariant normal vector n^r proposed in [Part I, Eq. (66)], one finds

$$
\begin{aligned}
(\Upsilon) &= \int_{\Sigma_\star} \delta\chi^i \, \|J\mathbf{F}^{-T}\mathbf{N}\| \left\{ \frac{\partial}{\partial x^r} \left(J^{-1} \, P_{2i}^{AB} \frac{N_A N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \frac{g^{\star}{}_S^E F_E^r N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \right\} d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i \, \|J\mathbf{F}^{-T}\mathbf{N}\| \Big\{ J^{-1} \underbrace{F_E^r \frac{\partial}{\partial x^r}}_{=\partial(\cdot)/\partial X^E} \left(P_{2i}^{AB} \frac{N_A N_B}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \frac{g^{\star}{}_S^E N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \Big\} d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i \, \|\mathbf{F}^{-T}\mathbf{N}\| \left\{ \frac{\partial}{\partial X^E} \left(P_{2i}^{AB} \, N_A N_B \frac{g^{\star}{}_S^E N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^3} \right) \right\} d\Sigma_\star =
\end{aligned} \tag{53}
$$

Differentiating the product within parentheses by the Leibniz rule and simplifying whenever possible the surface element norm, one finds

$$
\begin{aligned}
&= \int_{\Sigma_\star} \delta\chi^i \left\{ \frac{\partial\, P_{2\,i}^{AB}}{\partial X^E}\; N_A N_B\, \frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right\}\, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \delta\chi^i \left\{ P_{2\,i}^{AB}\, \frac{\partial}{\partial X^E}\, (N_A N_B)\, \frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \right\}\, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \delta\chi^i\, \|\mathbf{F}^{-T}\mathbf{N}\| \left\{ P_{2\,i}^{AB}\, N_A N_B\, \frac{\partial}{\partial X^E} \left(\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^3} \right) \right\}\, d\Sigma_\star
\end{aligned}
\tag{54}
$$

As already discussed in [Part I, Eqs. (67)–(68)],the Lagrangian vector $g^{\star}{}_{S}^{E} N^S$ possesses non-vanishing components in both the tangent and the normal space, namely

$$
\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} = N^E + \left(\frac{n^t\, \left(\mathbf{F}^{-1}\right)_t^E}{\|\mathbf{F}^{-T}\mathbf{N}\|} - N^E \right);
$$

Hence, since the above pullback metric tensor appears once in each addend, also Equation (54) can be decomposed in the form $(\Upsilon) = (\Upsilon_\perp) + \left(\Upsilon_\|\right)$. Firstly, let us develop the orthogonal contribution

$$
\begin{aligned}
(\Upsilon_\perp) &= \int_{\Sigma_\star} \delta\chi^i \left\{ \frac{\partial\, P_{2\,i}^{AB}}{\partial X^E}\, N_A N_B\, N^E \right\}\, d\Sigma_\star + \\
&+ \underbrace{\int_{\Sigma_\star} \delta\chi^i \left\{ P_{2\,i}^{AB}\, \frac{\partial}{\partial X^E}\, (N_A N_B)\; N^E \right\}\, d\Sigma_\star}_{=0} + \\
&+ \int_{\Sigma_\star} \delta\chi^i\; P_{2\,i}^{AB}\, N_A N_B\, \frac{\partial\, N^E}{\partial X^E}\, d\Sigma_\star + \\
&+ \underbrace{\int_{\Sigma_\star} \delta\chi^i\; P_{2\,i}^{AB}\, N_A N_B N^E\, \frac{\partial}{\partial X^E} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right)\, \|\mathbf{F}^{-T}\mathbf{N}\|\, d\Sigma_\star}_{\mathrm{res}_2};
\end{aligned}
\tag{55}
$$

thus retrieving two Lagrangian addends, plus one additional term res_2 to be discussed later. For the parallel contribution $\left(\Upsilon_\|\right)$ instead, one has:

$$
\begin{aligned}
\left(\Upsilon_\|\right) &= \int_{\Sigma_\star} \delta\chi^i \left\{ \frac{\partial\, P_{2\,i}^{AB}}{\partial X^E}\, N_A N_B \left(\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} - N^E \right) \right\}\, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \delta\chi^i \left\{ P_{2\,i}^{AB}\, \frac{\partial}{\partial X^E}\, (N_A N_B) \left(\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} - N^E \right) \right\}\, d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \delta\chi^i\; P_{2\,i}^{AB}\, N_A N_B\, \frac{\partial}{\partial X^E} \left(\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} - N^E \right)\, d\Sigma_\star + \\
&+ \underbrace{\int_{\Sigma_\star} \delta\chi^i\; P_{2\,i}^{AB}\, N_A N_B \left(\frac{g^{\star}{}_{S}^{E}\, N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} - N^E \right) \frac{\partial}{\partial X^E} \left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right)\, \|\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star}_{=0}
\end{aligned}
\tag{56}
$$

The Lagrangian surface term still unbalanced, which was indicated at rhs of Eq. (50), can be now recalled in the suggestive form of Eq. (41), finding that

$$(\lhd) = -\int_{\Sigma_\star} \frac{\partial}{\partial X^E}\left(P_{2\,i}^{AB}\,N_A N_B\right)\left\{\frac{g^{\star\,ES}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}N_S - N^E\right\}\delta\chi^i\,d\Sigma_\star + \\ -\int_{\Sigma_\star} P_{2\,i}^{AB}\,N_A N_B \frac{\partial}{\partial X^L}\left(\frac{g^{\star\,LS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}N_S - N^L\right)\delta\chi^i\,d\Sigma_\star + \\ +\underbrace{\int_{\Sigma_\star} P_{2\,i}^{AB}\,N_A N_B N_L N^E \frac{\partial}{\partial X^E}\left(\frac{g^{\star\,LS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}N_S - N^L\right)\delta\chi^i\,d\Sigma_\star}_{=0} \tag{57}$$

One can observe that the first two addends of $(\lhd)$ in Eq. (57), after differentiation of the vector $P_{2\,i}^{AB}\,N_A N_B$ by the product rule, cancel out with the first three terms in $\left(\curlyvee_{\|}\right)$ Eq. (56). To proceed further, we are going to prove that the last addend in $\left(\curlyvee_{\|}\right)$ and the third addend in $(\lhd)$ vanish, as annotated $(= 0)$ below the relevant expressions (next points i-ii), and then we will discuss the two residual terms res_1 and res_2 (at point iii).
(i) About the last addend in Eq. (56), recalling formulae for the derivative of the surface element norm [Part I, Eqs. (32)–(35)], we notice that

$$-N^E \frac{\partial}{\partial X^E}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right) + \frac{g^{\star\,E}_{\;\;S} N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\frac{\partial}{\partial X^E}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right) = \\ = -N^E\left(-\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3}\,g^{\star\,RM} N_R\,N_{M|E}\right) + \\ + \frac{g^{\star\,E}_{\;\;S} N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\left(-\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3}\,g^{\star\,RM} N_R\,N_{M|E}\right) = \\ = +\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3}\,g^{\star\,RM} N_R\,N_{M|E} N^E - \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^5}\,g^{\star\,ES} N_S\,g^{\star\,RM} N_R\,N_{M|E} = \\ = -\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3}\,g^{\star\,RM} N_R\,\Gamma^S_{ME} N_S N^E + \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3}\,g^{\star\,RM} N_R\,\Gamma^S_{ME} N_S N^E = 0; \tag{58}$$

In the last row of Eq. (58), to simplify and scale the second addend, including the covariant derivative $N_{M|E}$ contracted with two pullback metric tensors, we utilized the remarkable relationship [Part I, Eq. (43)]. Such relationship among Lagrangian variables was proven starting from the condition of vanishing normal derivative for the normal vector in the Eulerian configuration, see [Part I, Eqs. (41)–(44)]. Moreover, in the last passage we have indicated explicitly the Christoffel symbols Γ^S_{ME} (see [Part I, Eqs. (30)–(31)] to emphasize the fact that, after multiplying by N^E the covariant derivative $N_{M|E}$, the addend including the (ordinary) derivative of the normal along the normal direction vanishes, see also Eq. (20).
(ii) As for the vanishing addend of Eq. (57), a few preliminary steps are herein outlined for the reader's convenience. Firstly, let us consider the derivative of $g^{\star\,LS} N_S$ alone. Since the metrics g^{tj} in the Eulerian configuration equals herein the unit tensor, one finds

$$\frac{\partial}{\partial X^E}\left(g^{\star\,LS} N_S\right) = \frac{\partial}{\partial X^E}\left(g^{tj}\left(\mathbf{F}^{-1}\right)^L_t \left(\mathbf{F}^{-1}\right)^S_j N_S\right) = \\ = g^{tj}\frac{\partial\left(\mathbf{F}^{-1}\right)^L_t}{\partial X^E}\left(\mathbf{F}^{-1}\right)^S_j N_S + g^{tj}\left(\mathbf{F}^{-1}\right)^L_t \frac{\partial\left(\mathbf{F}^{-1}\right)^S_j}{\partial X^E} N_S + \\ + g^{tj}\left(\mathbf{F}^{-1}\right)^L_t \left(\mathbf{F}^{-1}\right)^S_j \frac{\partial N_S}{\partial X^E} = \tag{59}$$

and, taking into account the formula (see, e.g. [36])

$$\frac{\partial}{\partial X^E}\left(\mathbf{F}^{-1}\right)^A_a = \frac{\partial}{\partial F^i_M}\left(\mathbf{F}^{-1}\right)^A_a \frac{\partial F^i_M}{\partial X^E} = -\left(\mathbf{F}^{-1}\right)^A_i \left(\mathbf{F}^{-1}\right)^M_a F^i_{ME};$$

utilizing the definition of covariant derivative with the Christoffel symbols [Part I, Eqs. (30)–(31)], from Eq. (59) one has

$$
\begin{aligned}
&= -g^{tj}\left(\mathbf{F}^{-1}\right)_i^L \left(\mathbf{F}^{-1}\right)_t^M F^i_{ME} \left(\mathbf{F}^{-1}\right)_j^S N_S + \\
&\quad - g^{tj}\left(\mathbf{F}^{-1}\right)_t^L \left(\mathbf{F}^{-1}\right)_i^S \left(\mathbf{F}^{-1}\right)_j^M F^i_{ME} N_S + g^{tj}\left(\mathbf{F}^{-1}\right)_t^L \left(\mathbf{F}^{-1}\right)_j^S \frac{\partial N_S}{\partial X^E} = \\
&= -g^{\star MS}\Gamma^L_{ME} N_S - g^{\star LM}\Gamma^S_{ME} N_S + g^{\star LS}\frac{\partial N_S}{\partial X^E} = \\
&= -g^{\star MS}\Gamma^L_{ME} N_S + g^{\star LM} N_{M|E};
\end{aligned} \tag{60}
$$

By adding the surface element norm squared to the denominator in Eq. (59), the above derivative becomes

$$
\begin{aligned}
\frac{\partial}{\partial X^E}\left(\frac{g^{\star LS} N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right) &= \left(g^{\star LM} N_{M|E} - g^{\star MS}\Gamma^L_{ME} N_S\right)\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} + \\
&\quad + g^{\star LS} N_S\left(-\frac{2}{\|\mathbf{F}^{-T}\mathbf{N}\|^4} g^{\star RM} N_R N_{M|E}\right);
\end{aligned} \tag{61}
$$

where the norm derivative was computed by the formula [Part I, Eq. (34)]. Moreover, multiplying Eq. (61) by $N_L N^E = [M_\perp]^E_L$ (i.e. by the orthogonal projector), one finds

$$
\begin{aligned}
& N_L N^E \frac{\partial}{\partial X^E}\left(\frac{g^{\star LS} N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2}\right) = \\
&= N_L N^E \left(g^{\star LM} N_{M|E} - g^{\star MS}\Gamma^L_{ME} N_S\right)\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} + \\
&\quad + \underbrace{\left(N_L g^{\star LS} N_S\right)}_{=\|\mathbf{F}^{-T}\mathbf{N}\|^2} N^E \left(-\frac{2}{\|\mathbf{F}^{-T}\mathbf{N}\|^4} g^{\star RM} N_R N_{M|E}\right) = \\
&= \Big(g^{\star LM} N_L N_{M|E} N^E - g^{\star MS} N_S \Gamma^L_{ME} N_L N^E + \\
&\quad -2 g^{\star RM} N_R N_{M|E} N^E\Big)\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} =
\end{aligned} \tag{62}
$$

As already noticed, the product between N^E and the above covariant derivative $N_{M|E}$ cancels in the latter the addend with the ordinary directional derivative of the normal, see Eq. (20). Therefore, one can indicate explicitly the Christoffel symbols, finding that:

$$
\begin{aligned}
&= \Big(-g^{\star LM} N_L \Gamma^S_{ME} N_S N^E - g^{\star MS} N_S \Gamma^L_{ME} N_L N^E + \\
&\quad +2 g^{\star RM} N_R \Gamma^S_{ME} N_S N^E\Big)\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} = 0;
\end{aligned} \tag{63}
$$

In the same equation (57), the derivative of N^L does not affect the final result, being $N_L N^E \frac{\partial N^L}{\partial X^E} = 0$.
(iii) To complete the transport of the boundary face equations from the Eulerian to the Lagrangian configuration, two terms remain still unbalanced (from Eqs. 56 and 57), and their sum must vanish. By simple manipulations, recalling the covariant derivative of the norm [Part I, Eq. (28)], one has

$$
\begin{aligned}
\mathrm{res}_1 + \mathrm{res}_2 &= -\int_{\Sigma_\star} \delta\chi^i\, P^{AB}_{2i} N_B \|\mathbf{F}^{-T}\mathbf{N}\| \frac{\partial}{\partial X^A}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right) d\Sigma_\star + \\
&\quad + \int_{\Sigma_\star} \delta\chi^i\, P^{AB}_{2i} N_A N_B N^E \frac{\partial}{\partial X^E}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right)\|\mathbf{F}^{-T}\mathbf{N}\|\, d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i\, \|\mathbf{F}^{-T}\mathbf{N}\|\, P^{AB}_{2i} N_B \left\{-\frac{\partial}{\partial X^A}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right) + \right. \\
&\quad \left. + N_A N^E \frac{\partial}{\partial X^E}\left(\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}\right)\right\} d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i\, \|\mathbf{F}^{-T}\mathbf{N}\|\, P^{AB}_{2i} N_B \left\{+\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3} g^{\star RM} N_R N_{M|A} + \right.
\end{aligned}
$$

$$
\begin{aligned}
&+N_A \left(-\frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^3} g^{\star RM} N_R\, N_{M|E} N^E \right) \Bigg\} \, d\Sigma_\star = \\
&= \int_{\Sigma_\star} \delta\chi^i \, \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \, P_{2i}^{AB} N_B \left\{ + g^{\star RM} N_R\, N_{M|A} + \right. \\
&\left. +N_A \left(g^{\star RM} N_R\, \Gamma^S_{ME} N_S N^E \right) \right\} \, d\Sigma_\star
\end{aligned}
\tag{64}
$$

Condition for the above surface integral to vanish ($\forall\, \delta\chi^i,\, F^i_A,\, P^{AB}_{2i}$) is that the two addends in curly brackets satisfy the following equality

$$
+\, g^{\star RM} N_R\, N_{M|A} = -N_A \left(g^{\star RM} N_R\, \Gamma^S_{ME} N_S N^E \right); \tag{65}
$$

By multiplying both sides of the above equation by the contravariant component N^A, one finds

$$
\begin{aligned}
&N^A\, g^{\star RM} N_R\, N_{M|A} = -g^{\star RM} N_R\, \Gamma^S_{MA} N_S N^A = \\
&= -\underbrace{N^A N_A}_{=1} \left(g^{\star RM} N_R\, \Gamma^S_{ME} N_S N^E \right);
\end{aligned}
\tag{66}
$$

The change of sign at lhs was due to the fact that, for the covariant derivative of the covariant vectors, Christoffel symbols are subtracted. Hence, the residual addends in Eq. (64) cancel out. The transformation from the Eulerian to the Lagrangian form of all the contributions to the inner virtual work was finally completed. The transformation of the external work terms will be investigated elsewhere.

8 Closing remarks and future prospects

In this paper, as Part II of a wide research, the virtual work equations for second gradient continua were derived via variational calculus starting from a Lagrangian energy functional. Through the integration by parts and exploiting the divergence theorem for curved surfaces with border, detailed expressions were provided for volume, face and edge contributions. The peculiarities of second gradient formulation with respect to Cauchy's conventional approach were emphasized. Firstly, a third rank hyperstress tensor enabled the material to sustain distributions of the double forces over the boundary faces and of the line forces along the border edges. The double forces over the outer faces turned out to be work-conjugate to the normal derivative of the virtual placement map, whilst the line forces along the border edges generate work versus the virtual placement. Then, the expression of the contact pressures included, besides the term linearly dependent on the normal vector shared with Cauchy's theory, the multiple product of normal vector components, their gradient, and a combination of them. In particular, the dependence on the divergence of the normal vector was highlighted, equaling the local mean curvature of the face. The edge term was expressed as a function of vectors which are discontinuous across the edge and belong to the contiguous faces. By utilizing novel and remarkable formulae for the edge vectors and the surface projectors provided in Part I, the transport of the governing equations from the Eulerian to the Lagrangian configuration was developed according to two different strategies. In the first approach, the contact pressures were represented by means of the tangential projectors, and recourse was made to a novel relationship between Eulerian and Lagrangian expressions derived from the surface divergence theorem. According to the second strategy, the contact pressures were expressed through the normal vector components and their gradients: to prove the transport, recourse was made to the pullback metrics and to the covariant derivatives, outlined in Part I.

The present research was intended to build a sound basis, rooted in the differential geometry, for the study of the higher order gradient continua and for the subsequent development of advanced numerical tools, see, e.g. [47,48]. Most of the existing commercial platforms for the computational mechanics, in fact, based on a finite element discretization of the displacement field, were conceived exclusively for Cauchy's first gradient formulations, and cannot be easily adapted to simulate the response of second gradient continua. Hence, a huge comprehension of the governing equations and of the relevant nonstandard boundary conditions is a prerequisite for the implementation of generalized models, formulated by a variational approach. We expect that also advanced fracture theories, like those regularized by the phase field approach, will soon include second gradient modelling [49]. This opinion is motivated by the need to properly accommodate edge and point contact forces [50] in the presence of geometric nonlinearity.

But that's not all. The present theoretical study was intended to favour also the development of novel mixed experimental-numerical methodologies, see, e.g. [51]. During the last decade, 3D printing and additive manufacturing techniques have made easier, faster and cheaper the realization of new prototypes directly from the digital CAD model: their microstructure can be tailored for the most diverse applications, thus reducing costs and the time to market of novel products, see, e.g. [52]. Such prototypes can be subjected to non-conventional experiments, with the aim to recreate in the laboratory, as far as possible, service and ultimate conditions. To monitor the sample response, a crucial ingredient is represented by kinematic full field measurements, which may concern the outer surface or even the inner bulk of the sample. Such high quality data can be provided, e.g. by Digital Image Correlation procedures, resting on pictures acquired through an optical camera or also on X-ray computed tomographies [53,54]. Novel approaches will be pursued, capable of distinguishing macro- and micro-displacement fields, as in [55,56], or to provide regularized measurements exploiting information provided by the mechanical model, in view of parameter identification [57–59] and model validation procedures.

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