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Approach to Equilibrium of Glauber Dynamics in the One Phase Region

II. The General Case

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Abstract: We develop a new method, based on renormalization group ideas (block decimation procedure), to prove, under an assumption of strong mixing in a finite cube Λ_0, a Logarithmic Sobolev Inequality for the Gibbs state of a discrete spin system. As a consequence we derive the hypercontractivity of the Markov semigroup of the associated Glauber dynamics and the exponential convergence to equilibrium in the uniform norm in all volumes Λ "multiples" of the cube Λ_0.

1. Preliminaries, Definitions, and Results

In this paper we analyze the problem of the approach to equilibrium for a general, not necessarily ferromagnetic, Glauber dynamics, i.e. a single spin flip stochastic dynamics reversible with respect to the Gibbs measure of a classical discrete spin system with finite range, translation invariant interaction. We prove that, if the Gibbs measure satisfies a *Strong Mixing Condition* on a large enough finite cube Λ_0, then the Glauber dynamics reaches the equilibrium exponentially fast in time in the *uniform norm,* in any finite or infinite volume Λ, provided that Λ is a "multiple" of the basic cube Λ_0. Such a result has already been proved in our previous papers [MO1, MO2] in the so-called "attractive case" by ad hoc methods. Here we prove the result in greater generality by proving a Logarithmic Sobolev Inequality for the Gibbs measure of the system. We refer to [MO2] for a general introduction to the problem of approach to equilibrium in the one phase region for Glauber dynamics; in particular in [MO2] one finds a critical discussion of the various *finite volume* mixing conditions for the Gibbs state and of the role played by the shape of the volumes involved when getting near to a line of first order phase transition. We also refer the reader to the beautiful series of papers by Zegarlinski [Z1, Z2, Z3] and Zegarlinski and Stroock [SZ1, SZ2, SZ3], where the theory of the Logarithmic Sobolev Inequality for Gibbs states was developed and its role in the proof of fast convergence to equilibrium of general, not

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necessarily attractive, Glauber dynamics was clarified. We suggest in particular the interested reader look at the nice review in [S].

In order to precisely state our results and for the reader's convenience, we recall here the model and the notation of the first paper of this series.

1. The Model. We will consider lattice spin systems with a finite single spin state space S. We take for simplicity $S = \{-1, +1\}$ and we will denote by $\sigma \equiv \sigma_\Lambda$ an element of the configuration space $\Omega_\Lambda = S^\Lambda$ in a subset $\Lambda \subset \mathbf{Z^d}$. The symbol σ_x or $\sigma(x)$ will always denote the value of the spin at the site $x \in \Lambda$ in the configuration σ.

The *energy* associated to a configuration $\sigma \in \Omega_\Lambda$ when the boundary condition outside Λ is $\tau \in \Omega_{\Lambda^c}$ is given by:

$$H_\Lambda^\tau(\sigma) = H_\Lambda(\sigma\,|\tau) = \sum_{X:\, X\cap\Lambda\neq\emptyset} U_X \prod_{x\in X} (\sigma\tau)_x\,, \tag{1.1}$$

where, in general, $\sigma\tau$ denotes the configuration:

$$\begin{aligned}(\sigma\tau)_x &= \sigma_x\,, \qquad x\in\Lambda\,,\\ (\sigma\tau)_x &= \tau_x\,, \qquad x\in\Lambda^c\,,\end{aligned} \tag{1.2}$$

and the potential $U = \{U_X, X \subset\subset \mathbf{Z^d}\}$, where $X \subset\subset \mathbf{Z^d}$ means that X is a finite subset of $\mathbf{Z^d}$, satisfies the following hypotheses:

H1. *Finite range:* $\exists r > 0 : U_X \equiv 0$ *if* diam $X > r$ *(we use Euclidean distance denoted, in the sequel, by* dist*).*

H2. *Translation invariance*

$$\forall X \subset\subset \mathbf{Z^d}\,, \qquad \forall k \in \mathbf{Z^d}\,, \qquad U_{X+k} = U_X\,.$$

Because of the hypothesis H1, $H_\Lambda^\tau(\sigma)$ depends only on τ_x for x in $\partial_r^+\Lambda$:

$$\partial_r^+\Lambda = \{x \notin \Lambda\colon \mathrm{dist}(x,\Lambda) \le r\}\,. \tag{1.3}$$

With the energy function $H_\Lambda^\tau(\sigma)$ we construct the usual *Gibbs measure* in Λ with b.c. $\tau \in \Omega_{\Lambda^c}$ given by:

$$\mu_\Lambda^\tau(\sigma) = \frac{\exp(H_\Lambda^\tau(\sigma))}{Z_\Lambda^\tau}\,, \tag{1.4}$$

where the normalization factor, or *partition function*, is given by

$$Z_\Lambda^\tau = \sum_{\sigma\in\Omega_\Lambda} \exp(H_\Lambda^\tau(\sigma))\,. \tag{1.5}$$

If there exists a unique limiting Gibbs measure for $\Lambda \to \mathbf{Z^d}$, independent on τ, it will be denoted by μ.

Remark. Notice that, for future notation convenience, we have included the usual $-\beta$ factor in the Boltzman weight (1.4) directly in the energy $H_\Lambda^\tau(\sigma)$.

Next we define the stochastic *jump* dynamics, given by a continuous time Markov process on $\Omega = S^{\mathbf{Z^d}}$ that will be studied in the sequel. Discrete time versions can also be considered.

Given $\Lambda \subset\subset \mathbf{Z^d}$ let

$$D(\Lambda) = \{f : \Omega \to R : f(\eta) = f(\sigma) \text{ if } \eta_x = \sigma_x \ \forall x \in \Lambda\}$$

be the set of *cylindrical functions* with support Λ. The set

$$D = \bigcup_{\Lambda} D(\Lambda)$$

is the set of cylindrical functions and by $C(\Omega)$ we denote the set of all continuous functions on $\Omega = \prod_x S_x$, $S_x \equiv S$ with respect to the product topology of discrete topologies on S.

The dynamics is defined by means of its *generator* L which is given, for $f \in D$, by:

$$Lf(\sigma) = \sum_{x,a} c_x(\sigma, a)\,(f(\sigma^{x,a}) - f(\sigma))\,, \tag{1.6}$$

where $\sigma^{x,a}$ is the configuration obtained from σ by setting the spin at x equal to the value a and the non-negative quantities $c_x(\sigma, a)$ are called "jump rates."

We will also consider the Markov process associated to the above described jump rates in a *finite volume* Λ with boundary conditions τ outside Λ. By this we mean the dynamics on Ω_Λ generated by L^τ_Λ defined as before starting from the jump rates

$$c_x^{\tau,\Lambda}(\sigma, a) \equiv c_x(\sigma\tau, a)\,,$$

where, given $\tau \in \Omega_{\Lambda^c}$ and $\sigma \in \Omega_\Lambda$, $\sigma\tau$ has been defined in (1.2).

The general hypotheses on the jump rates, that we shall always assume, are the following ones:

H3. *Finite range r. This means that $\eta(y) = \sigma(y)\ \forall x, y : |y - x| \leq r$ implies $c_x(\sigma, a) = c_x(\eta, a)$.*

H4. *Translation invariance. That is if $\eta(y) = \sigma(y+x)\ \forall y$, then $c_{x+y}(\sigma, a) = c_y(\eta, a)$.*

H5. *Positivity and boundedness. There exist two positive constants k_1, k_2 such that*

$$0 < k_1 \leq \inf_{\sigma,x,a} c_x(\sigma, a) \leq \sup_{\sigma,x,a} c_x(\sigma, a) \leq k_2\,.$$

H6. *Reversibility with respect to the Gibbs measure μ (in finite or infinite volume):*

$$\begin{aligned} &\exp\Big(\sum_{X \ni x} U_X \prod_{y \in X} \sigma_y\Big) c_x(\sigma, a) \\ &= \exp\Big(\sum_{X \ni x} U_X \prod_{y \in X} (\sigma^{x,a})_y\Big) c_x(\sigma^{x,a}, \sigma_x) \quad \forall x \in \mathbf{Z^d}\,. \end{aligned} \tag{1.7}$$

A similar equation holds in finite volume Λ with boundary conditions τ, provided that we replace in (1.7) σ with the configuration $\sigma\tau$.

It is immediate to check that, in finite volume, reversibility implies that the unique invariant measure of the dynamics coincides with the Gibbs measure μ^τ_Λ. This important fact holds also in infinite volume provided that there exists a unique Gibbs measure in the thermodynamic limit and that all the invariant measures of the dynamics are Gibbsian.

It is well known (see [L]) that under the above conditions $L(L^\tau_\Lambda)$ generates a unique positive selfadjoint contraction semigroup on the space $L^2(\Omega, d\mu)\,(L^2(\Omega_\Lambda, d\mu^\tau_\Lambda))$ that will be denoted by T_t or $T_t^{\Lambda,\tau}$.

2. The Logarithmic Sobolev Inequality and Hypercontractivity of T_t. In order to introduce the Logarithmic Sobolev Inequality (LSI) we have to define the differentiation operator on the functions of the spin configurations. We set:

$$\partial_{\sigma_x} f(\sigma) = f(\sigma) - \tfrac{1}{2}\,[f(\sigma^{x,+1}) + f(\sigma^{x,-1})]\,, \tag{1.8}$$

where $\sigma^{x,+1}$, respectively $\sigma^{x,-1}$, is the configuration obtained from σ by setting the spin at x equal to $+1$, respectively -1. Given a subset A of the lattice $\mathbf{Z^d}$ the symbol $(\nabla_A f)^2$ will be a shorthand notation for the expression $\sum_{x\in A} (\partial_{\sigma_x} f(\sigma))^2$.

Finally we define the "standard" Logarithmic Sobolev Constant $c_s(\nu)$ for an arbitrary measure ν on Ω_Λ as the smallest number c such that for any non-negative function $f:\Omega_\Lambda \to \mathbf{R}$ the following inequality holds:

$$\nu(f^2 \log(f)) \leq c\nu((\nabla_\Lambda f)^2) + \nu(f^2)\log((\nu(f^2))^{1/2})\,, \tag{1.9}$$

where $\nu(f)$ denotes the average of the function f with respect to the measure ν.

In the sequel we will refer to (1.9) as the "standard" Logarithmic Sobolev Inequality for ν. It is very important to observe that if we denote by

$$\mathscr{E}^\tau_\Lambda(f,f) = -\mu^\tau_\Lambda(fL^\tau_\Lambda f)$$

the Dirichlet form associated to the generator L^τ_Λ and we take in (1.9) $\nu = \mu^\tau_\Lambda$, then the term:

$$\mu^\tau_\Lambda((\nabla_\Lambda f)^2)$$

satisfies the estimate:

$$\Big(4 \max_{x,a,\sigma} c_x^{\tau,\Lambda}(\sigma,a)\Big)^{-1} \mathscr{E}^\tau_\Lambda(f,f) \leq \mu^\tau_\Lambda((\nabla_\Lambda f)^2) \leq \Big(4 \min_{x,a,\sigma} c_x^{\tau,\Lambda}(\sigma,a)\Big)^{-1} \mathscr{E}^\tau_\Lambda(f,f)\,.$$

Therefore, if μ^τ_Λ satisfies the "standard" Logarithmic Sobolev Inequality (1.9), with standard logarithmic Sobolev constant $c_s(\mu^\tau_\Lambda)$, then it also satisfies the Logarithmic Sobolev Inequality for the semigroup $T_t^{\Lambda,\tau}$ relative to the measure μ^τ_Λ:

$$\mu^\tau_\Lambda(f^2\log(f)) \leq c_{\mathscr{E}}(\mu^\tau_\Lambda)\mathscr{E}^\tau_\Lambda(f,f) + \mu^\tau_\Lambda(f^2)\log((\mu^\tau_\Lambda(f^2))^{1/2}) \tag{1.10}$$

with logarithmic Sobolev constant $(4k_2)^{-1}c_s(\mu^\tau_\Lambda) \leq c_{\mathscr{E}}(\mu^\tau_\Lambda) \leq (4k_1)^{-1}c_s(\mu^\tau_\Lambda)$ because of H5. Thus the "standard" Logarithmic Sobolev Inequality and the Logarithmic Sobolev Inequality for the Dirichlet form $\mathscr{E}$ are equivalent and in the sequel, whenever confusion does not arise, we will call both of them the Logarithmic Sobolev Inequality.

Remark. From the above discussion it also immediately follows that, if $c_x^{\tau,\Lambda}(\sigma,a)$ and $\tilde{c}_x^{\tau,\Lambda}(\sigma,a)$ are two different jump rates satisfying H3...H6, and if $\mathscr{E}$ and $\tilde{\mathscr{E}}$ are the corresponding Dirichlet forms, then we have:

$$c_{\mathscr{E}}(\mu^\tau_\Lambda) \leq \max_{x,a,\sigma} \frac{c_x^{\tau,\Lambda}(\sigma,a)}{\tilde{c}_x^{\tau,\Lambda}(\sigma,a)}\, c_{\tilde{\mathscr{E}}}(\mu^\tau_\Lambda) \tag{1.11}$$

and

$$c_{\mathscr{E}}(\mu^\tau_\Lambda) \geq \max_{x,a,\sigma} \frac{c_x^{\tau,\Lambda}(\sigma,a)}{\tilde{c}_x^{\tau,\Lambda}(\sigma,a)}\, c_{\tilde{\mathscr{E}}}(\mu^\tau_\Lambda)\,. \tag{1.12}$$

Remark. In the case when the single spin space consists of N elements with $N > 2$, the definition of the differentiation operator is no longer so clear. One possibility (see [SZ3]) is to set:

$$\partial_{\sigma_x} f(\sigma) = f(\sigma) - \langle f\rangle_x\,, \tag{1.13}$$

where $\langle f\rangle_x$ is the average with respect to the uniform measure on S of the function f, considered as a function of the single spin σ_x. Another possibility is to order the elements of S as $s_1, \ldots, s_N$ and to set:

$$\partial_{\sigma_x} f(\sigma) = \frac{f(s_{i+1}) - f(s_i)}{2} \tag{1.14}$$

if $\sigma_x = s_i$ with $s_{N+1} \equiv s_1$.

Both definitions are reasonable and equivalent in the sense that:

i) $\langle \partial_{\sigma x} f\rangle = 0$,

ii) there exists a finite positive constant k_0 (in general depending on N) such that:

$$(k_0)^{-1}\mathscr{E}_\Lambda^\tau(f,f) \le \mu_\Lambda^\tau((\nabla_\Lambda f)^2) \le k_0 \mathscr{E}_\Lambda^\tau(f,f),$$

where the definition of the generator L and of the Gibbs measure in this more general setting is the obvious one.

Although, in some sense, the choice of the differentiation operator reflects the choice of the dynamics for the single spin at "infinite temperature" (a uniform sampling in the first case or a symmetric random walk in the second case), because of ii) above, one is free to choose whatever definition is more suited to the methods of proof. In particular if, as we do, one wants to treat ∂_{σ_x} as much as possible as a continuous derivative, then the second definition seems more suited.

The above "ambiguity" points out that the logarithmic Sobolev constant is not intrinsically associated to the Gibbs measure but, rather, to the pair (μ, ∇).

As it is well known since the basic work by Gross [G1] (see also [G2]), the Logarithmic Sobolev Inequality for the Gibbs state μ_Λ^τ is strictly connected with the hypercontractivity properties of the Markov semigroup $T_t^{\Lambda,\tau}$, where:

$T_t^{\Lambda,\tau}$ is hypercontractive with respect to μ_Λ^τ if there exists a constant $c(\Lambda,\tau)$ such that

$$|T_t^{\Lambda,\tau}(f)|_{L^q(\mu_\Lambda^\tau)} \le |f|_{L^p(\mu_\Lambda^\tau)} \forall (p,q,t) \quad \textit{with} \quad p \le q \le 1 + (p-1)e^{\frac{2t}{c(\Lambda,\tau)}}. \tag{1.15}$$

More precisely, Gross' Theorem states that the constant $c(\Lambda,\tau)$ in (1.15) can be taken equal to the logarithmic Sobolev constant $c_{\mathscr{E}}(\mu_\Lambda^\tau)$.

Besides its intrinsic interest, hypercontractivity of the Markov semigroup $T_t^{\Lambda,\tau}$ or of T_t is a fundamental tool to transform, in the general case, exponential convergence to equilibrium of the Glauber dynamics in the $L^2(d\mu_\Lambda^\tau)$-sense into exponential convergence to equilibrium in the L^∞-sense. More exactly, for an interaction U_X satisfying the general hypotheses H1, H2, one can easily prove the following theorem (see [SZ2] Lemma 2.9 and Lemma 1.8 there):

Theorem 1.1. *Let Γ be a, finite or infinite, class of subsets of $\mathbf{Z}^\mathrm{d}$ and suppose that there is a constant c_0 such that:*

$$\sup_{\tau,\Lambda\in\Gamma} c_{\mathscr{E}}(\mu_\Lambda^\tau) \le c_0 .$$

Then there exists a positive constant m and for any cylindrical (i.e. depending on finitely many spins) $f : \Omega_{\mathbf{Z}^\mathrm{d}} \to R$ there exists a finite constant C_f such that:

$$\sup_{\tau,\Lambda\in\Gamma} |P_t^{\tau,\Lambda} f - \mu_\Lambda^\tau(f)|_\infty \le C_f \exp(-mt).$$

Thus, in this way, the problem of proving exponentially fast approach to equilibrium in the uniform norm for a Glauber dynamics is reduced to proving a bound on the logarithmic Sobolev constant of the Gibbs state.

The real breakthrough in this direction was made a few years ago by Zegarlinski [Z1, Z2, Z3], who proved, in particular, that, if the Gibbs state satisfies a weak coupling condition similar to the old uniqueness Dobrushin's condition (single site) [D1], then the logarithmic Sobolev constant in a set Λ with boundary conditions τ is finite uniformly in Λ and τ. In [Z2], in order to treat the one dimensional case for all temperatures, the author uses intervals instead of points. This result was then extended and generalized by Stroock and Zegarlinski [SZ1, SZ2, SZ3] to spin systems with single spin space given either by a compact Riemannian manifold or by a finite discrete set. The main result of the above mentioned works is the equivalence of the existence of a finite logarithmic Sobolev constant, independent of the volume and of the external spin configuration, and the Dobrushin-Shlosman complete analyticity [DS1, DS2] of the Gibbs state μ. Although the Stroock-Zegarlinski result yielded very important progress in the general problem of relating the fast convergence to equilibrium of the dynamics to "mixing" properties of the Gibbs measure, it cannot, in general, be used concretely in order to establish results close to a line of a first order phase transition since it requires good mixing properties of the measure μ_Λ^τ in volumes Λ of *arbitrary shape* (see [MO2], Sect. 2, for more details). As we have discussed in detail in the first work of this series [MO2], if one is willing to produce results really close to a line of a first order phase transition, one should consider the Gibbs measure only on "fat" volumes like cubes or parallelipipeds with large enough shortest side. This is exactly the subject of the present work.

3. The Results. In order to present the new results of the present work, we need to recall the *finite volume* mixing condition that already played an important role in the first paper of this series [MO2].

We say that a Gibbs measure μ_Λ on Ω_Λ satisfies a *strong mixing* condition with constants C, γ if for every subset $\Delta \subset \Lambda$:

$$\sup_{\tau,\tau^{(y)}\in\Omega_{\Lambda^c}} \mathrm{Var}(\mu^\tau_{\Lambda,\Delta}, \mu^{\tau^{(y)}}_{\Lambda,\Delta}) \le Ce^{-\gamma\,\mathrm{dist}(\Delta,y)}, \tag{1.16}$$

where $\mu^\tau_{\Lambda,\Delta}$ denotes the *relativization* (or projection) of the measure μ^τ_Λ on Ω_Δ, Var is the variation distance and $\tau^{(y)}_x = \tau_x$ for $x \neq y$.

We denote this condition by $SM(\Lambda, C, \gamma)$.

In [MO2] a lower bound on the gap in the spectrum of the generator L^τ_Λ was derived (see Theorem 1.2 below) under the assumption $SM(\Lambda_0, C, \gamma)$, where Λ_0 is a cube of side L_0, provided that, given C and γ, L_0 is sufficiently large.

In what follows Γ will denote the class of all subsets of $\mathbf{Z^d}$ given by the union of translates of the cube Λ_0 such that their vertices lay on the rescaled lattice $L_0\mathbf{Z^d}$. The constants C and γ appearing in our mixing condition will be fixed once and for all.

Theorem 1.2. *There exists a positive constant $\bar{L}$ depending only on the range of the interaction and on the dimension d such that if $SM(L_0, C, \gamma)$ holds with $L_0 \ge \bar{L}$ then:*
i) *there exists a positive constant m_0 such that for any $\Lambda \in \Gamma$ and for any function f in $L^2(d\mu^\tau_\Lambda)$:*

$$\|T_t^{\Lambda,\tau}(f) - \mu^\tau_\Lambda(f)\|_{L^2(d\mu^\tau_\Lambda)} \le \|f - \mu^\tau_\Lambda(f)\|_{L^2(d\mu^\tau_\Lambda)} \exp(-m_0 t).$$

ii) *There exist constants C' and γ' such that for any $\Lambda \in \Gamma$, $SM(\Lambda, C', \gamma')$ holds.*

Corollary 1.1. *There exists a positive constant $\bar{L}$ depending only on the range of the interaction and on the dimension d such that if $SM(L_0, C, \gamma)$ holds with $L_0 \geq \bar{L}$ then there exists a positive constant m such that for any pair of cylindrical functions f and g with supports S_f and S_g and for any $\Lambda \in \Gamma$, with S_f, $S_g \subset \Lambda$, one has:*

$$\mu_\Lambda^\tau(f,g) \leq |f|_\infty |g|_\infty |S_f|\,|S_g| \exp(-m\,\mathrm{dist}(S_f, s_g)),$$

where $|X|$ denotes the cardinality of the set $X \subset \mathbf{Z^d}$.

Here we considerably strengthen part i) of Theorem 1.2 by proving the following theorem:

Theorem 1.3. *There exists a positive constant $\bar{L}$ depending only on the range of the interaction and on the dimension d, such that if $SM(L_0, C, \gamma)$ holds with $L_0 \geq \bar{L}$ then there exists a positive constant c_0 such that for any $\Lambda \in \Gamma$ and any boundary condition τ the logarithmic Sobolev constant $c_{\mathscr{E}_\Lambda^\tau}$ is bounded by c_0.*

Moreover, there exists a positive constant m such that for any $\Lambda \in \Gamma$, any cylindrical function f there exists a positive constant C_f, depending only on f, such that:

$$|P_t^{\tau,\Lambda} f - \mu_\Lambda^\tau(f)|_\infty \leq C_f \exp(-mt).$$

The second part of the above theorem proves Theorem 4.2 of [MO2] in the non-attractive case.

4. The Strategy of the Proofs. We conclude this introduction by describing the ideas behind the proof of Theorem 1.3 and by comparing them with the Stroock-Zegarlinski's approach.

Our proof is divided into two distinct parts:
1) In this first part (see Sect. 2) we show that any Gibbs measure ν on a set Λ which is the (finite or infinite) union of certain "blocks" $\Lambda_1 \ldots \Lambda_j \ldots$ (e.g. cubes of side l or single sites of the lattice $\mathbf{Z^d}$) has a logarithmic Sobolev constant which is not larger than a suitable constant depending on the maximum size of the blocks provided that the interaction (not necessarily of finite range) *between* the blocks is very weak in a suitable sense. A simple example of such a situation is represented by a Gibbs state at high temperature, but the result is more general since we do not assume that the interaction *inside* each block is weak.

The result is a perturbative one since, as it is well known [G1], if there is no interaction between the blocks then the logarithmic Sobolev constant of ν is not larger than

$$\sup_{j,\tau} c_s(\nu_{\Lambda_j}^\tau).$$

2) In the second part (see Sect. 3) of our approach we use renormalization group, in the form known as decimation (i.e. integration over a certain subset of the variables σ_x), to show that, in the assumption of the theorem, the Gibbs state μ_Λ^τ after a finite ($\leq 2^d$) number of decimations becomes a new Gibbs measure exactly of the type discussed in part 1). It is then a relatively easy task to derive the boundedness of the logarithmic Sobolev constant of μ_Λ^τ.

As it is well known since the work of Olivieri [O] and Olivieri and Picco [OP], the mixing condition $SM(\Lambda_0, C, \gamma)$ implies that if the decimation is done over blocks of a sufficiently large size (see for instance [EFS] for pathologies that may occur if the size is not large enough) then it is possible to control, e.g. by a converging cluster expansion, the effective potential of the renormalized measure and to show that it satisfies the weak coupling condition needed in part 1). This is, however, more than what it is actually needed, since the hypotheses of part 1 are fulfilled by the renormalized measure as soon as the truncated two point correlation functions of the *original* Gibbs measure μ^{τ}_{Λ} decay exponentially fast. This is exactly the content of Corollary 1.1 above; therefore the method can avoid the lengthy computations of the cluster expansion.

We want to notice at this point a difference in the role played by the DLR structure of Gibbs measures in our approach and in the one used by Zegarlinski and Stroock-Zegarlinski. For simplicity let us consider the case when the Dobrushin's uniqueness condition holds true. For instance in [Z1] Zegarlinski uses in a crucial way the following property of the Gibbs local specification operation $\mathbf{E}_{\Lambda}$:

$$\lim_{n\to\infty} \mathbf{E}_{i_n}, \ldots, \mathbf{E}_{i_1} f = \mu(f),$$

where $\{i_k \in \mathbf{Z^d}\}_{k\in\mathrm{N}}$ is a suitable sequence going infinitely many times through each site of the lattice and μ is the unique infinite volume Gibbs state. A similar property is used in [SZ3] in the case when the Dobrushin-Shlosman complete analyticity condition holds true.

In the present paper, on the contrary, we use the following simple general property valid for any probability measure ν on a finite space Ω:

$$\nu(A_1 \cap \ldots \cap A_n) = \nu(A_1 \mid A_2 \cap \ldots \cap A_n)\nu(A_2|A_3 \cap \ldots \cap A_n)\ldots \nu(A_{n-1} \mid A_n)\nu(A_n).$$

Clearly, if the measure ν is a Gibbs measure corresponding to a given potential, then the DLR property enters in the explicit computation of the conditional probabilities $\nu(A_i \mid A_{i+1} \cup \ldots \cup A_n)$.

After the completion of this work we learned that also Lu and Yau, in their work on the gap for the Kawasaki dynamics for the Ising model [LY], obtained, by martingale techniques, a uniform bound on the Logarithmic Sobolev Constant of μ^{τ}_{Λ}, where Λ is an arbitrary cube of the lattice, under the assumption that $SM(\Lambda, C, \gamma)$ holds for all finite generalized cubes Λ.

2. Logarithmic Sobolev Inequality for Weakly Coupled Gibbs Measures

In this section we prove two results that will play a crucial role in the derivation of the logarithmic Sobolev inequality (LSI) for Gibbs measures satisfying a finite volume mixing condition. In order to present our results we need to precisely define the setting of the problem and the notation that we will adopt in the sequel. We warn the reader that in this section we *do not assume* the finite range condition (H1); in this situation we will denote the potential by Φ instead of U.

1. The Setting of the Problem. Let Λ be a finite subset of the lattice $\mathbf{Z^d}$ such that $\Lambda = \Lambda_1 \cup \Lambda_2 \cup, \ldots, \cup \Lambda_N$ with $\Lambda_i \cap \Lambda_j = 0$ if $i \neq j$, $\sup_{i,N} |\Lambda_i| < \infty$ and let Ω_Λ be the space of configurations $\Omega_\Lambda = \{-1, +1\}^{\Lambda}$. Let also Φ_X, $X \subset \mathbf{Z^d}$, be a "potential" such that for any "boundary" configuration $\eta \in \Omega_{\mathbf{Z^d}\setminus\Lambda}$ and any configuration $\sigma \in \Omega_\Lambda$, the following Hamiltonian is finite:

$$H^{\eta}(\sigma) = \sum_{X\cap\Lambda\neq 0} \Phi_X \prod_{x\in X} (\sigma\eta)_x. \tag{2.1}$$

If for convenience we denote by $H^{\eta}_{\Lambda_i}(\sigma_{\Lambda_i})$ the sum:

$$H^{\eta}_{\Lambda_i}(\sigma_{\Lambda_i}) = \sum_{\{X\cap\Lambda\neq 0; X\cap\Lambda_i = X\cap\Lambda\}} \Phi_X \prod_{x\in X}(\sigma\eta)_x\,, \tag{2.2}$$

then the total Hamiltonian can be written as:

$$H^{\eta}(\sigma) = \sum_i H^{\eta}_{\Lambda_i}(\sigma_{\Lambda_i}) + W^{\eta}(\sigma)\,, \tag{2.3}$$

where the term $W^{\eta}(\sigma)$ represents now the interaction between the sets Λ_i, $i = 1\dots N$.

Given the Hamiltonian $H^{\eta}(\sigma)$, we will denote by μ^{η} the corresponding Gibbs measure.

In the sequel, together with the measure μ^{η}, we will need also other Gibbs measures that are obtained from μ^{η} by integrating out one by one the variables σ_{Λ_i}, $j = 1, 2\dots$ (decimation procedure). More precisely for any $i = 1,\dots,N$ we define a new Gibbs measure $\mu^{\eta}_{\geq i}$ on the space

$$\Omega^{(\geq i)}_{\Lambda} = \{-1,+1\}^{\Lambda_i\cup\Lambda_{i+1}\cup,\dots,\cup\Lambda_N}$$

as the relativization to $\Omega^{(\geq i)}_{\Lambda}$ of the measure μ^{η}:

$$\mu^{\eta}_{\geq i}(\sigma_{\Lambda_i},\dots,\sigma_{\Lambda_n}) = \sum_{\sigma_{\Lambda_1},\dots,\sigma_{\Lambda_{i-1}}} \mu^{\eta}(\sigma_{\Lambda_1},\dots,\sigma_{\Lambda_N})\,. \tag{2.4}$$

Obviously the measure $\mu^{\eta}_{\geq i}$ is also a Gibbs measure with a new Hamiltonian:

$$\begin{aligned}\bar{H}^{\eta}_{(\geq i)}(\sigma_{\Lambda_i},\dots,\sigma_{\Lambda_N} &= \log(Z^{\eta,\sigma_{\Lambda_i},\dots,\sigma_{\Lambda_N}}_{\Lambda_1,\dots,\Lambda_{i-1}}) \\ &\quad + \sum_{X: X\cap\left(\bigcup_{j=1}^{N}\Lambda_j\right)\neq\emptyset;\, X\cap\left(\bigcup_{j=1}^{i-1}\Lambda_j\right)=\emptyset} \Phi_X \prod_{x\in X}(\sigma\eta)_x\,,\end{aligned} \tag{2.5}$$

where $Z^{\eta,\sigma_{\Lambda_i},\dots,\sigma_{\Lambda_N}}_{\Lambda_1,\dots,\Lambda_{i-1}}$ is the partition function in $\Lambda_1\cup\dots\cup\Lambda_{i-1}$ with boundary conditions $\eta, \sigma_{\Lambda_i},\dots,\sigma_{\Lambda_N}$.

Finally, we will denote by $\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}$ the measure on Ω_{Λ_i} obtained from $\mu^{\eta}_{\geq i}$ by conditioning to the event that the spin configurations $\sigma_{\Lambda_{i+1}},\dots,\sigma_{\Lambda_N}$ are equal to the spin configurations $\tau_{i+1},\dots,\tau_N$. We will write $\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}$ as:

$$\nu_i^{\eta,\tau_{i+1},\dots,\tau_N} = \frac{\exp(W^{\eta}_{(i)}(\sigma_{\Lambda_i}(\sigma_{\Lambda_i};\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N}))}{Z_i}\,, \tag{2.6}$$

where $Z_i \equiv Z_i(\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N},\eta)$ is a normalization factor and the "effective" interaction

$$W^{\eta}_{(i)}(\sigma_{\Lambda_i};\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N}) \equiv W^{\eta}_{(i)}(\sigma)$$

is given by:

$$\begin{aligned}&W^{\eta}_{(i)}(\sigma_{\Lambda_i};\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N}) \\ &= \bar{H}^{\eta}_{(\geq i)}(\sigma_{\Lambda_i};\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N}) - \bar{H}^{\eta}_{(\geq i)}(\tilde{\tau}_{\Lambda_i},\tau_{\Lambda_{i+1}},\dots,\tau_{\Lambda_N})\,,\end{aligned} \tag{2.7}$$

where $\tilde{\tau}_{\Lambda_i}$ is a given reference configuration in Λ_i (e.g. all spins up).

2. Assumptions and Results. We are now in a position to precisely state the hypotheses on the "potential" Φ_X that we need in order to prove the main results of this section.

Assumptions. a) There exists a positive constant ε such that:

$$\sup_{(\eta,i,N)} \sum_{j=1}^{i-1} \sup_{x\in\Lambda_i} |\partial_{\sigma_x} W^{\eta}_{(j)}(\sigma)| \le \varepsilon \tag{2.8}$$

and

$$\sup_{(\eta,k,N)} \sum_{j=k+1}^{N} \sup_{x\in\Lambda_i} |\partial_{\sigma_x} W^{\eta}_{(k)}(\sigma)| \le \varepsilon\,. \tag{2.9}$$

b) There exists a positive constant α_0 such that:

$$\sup_{(\eta,N)} \sum_{j=1}^{N} \sup_{x\in\mathbf{Z^d}\setminus\Lambda} |\partial_{\eta(x)} W^{\eta}_{(j)}(\sigma)|_{\infty} \le \alpha_0 \tag{2.10}$$

and

$$\sup_{(\eta,N)} \sup_{k} \sum_{x\in\mathbf{Z^d}\setminus\Lambda} |\partial_{\eta(x)} W^{\eta}_{(k)}(\sigma)|_{\infty} \le \alpha_0 \sup_{i} |\Lambda_i|\,.$$

c) There exists a positive constant α_1 such that for any N, any η, any $i = 1,\ldots,N$, any $f \in L^2(\Omega_{\Omega_i}, d\nu_i^{\eta,\tau_{i+1},\ldots,\tau_N})$ and any value of the conditioning spins $\tau_{i+1},\ldots,\tau_N$ one has:

$$\sum_{\sigma_{\Lambda_i},\hat\sigma_{\Lambda_i}} \nu_i^{\eta,\tau_{i+1},\ldots,\tau_N}(\sigma_{\Lambda_i})\nu_i^{\eta,\tau_{i+1},\ldots,\tau_N}(\hat\sigma_{\Lambda_i})[f(\sigma_{\Lambda_i}) - f(\hat\sigma_{\Lambda_i})]^2$$
$$\le \alpha_1^2 \nu_i^{\eta,\tau_{i+1},\ldots,\tau_N}(((\nabla_{\Lambda_i} f)^2))\,. \tag{2.11}$$

Remark. Assumptions a) and b) are clearly expressing some decay property of the effective interaction of the Gibbs measures $\mu^{\eta}_{\ge i}$. The reason why in this section we *do not* require finite range of the interaction is that after a few steps of our decimation procedure, even a Gibbs state corresponding to a finite range interaction will be transformed into a new Gibbs measure corresponding to an effective interaction with unbounded range at least in some directions.

Assumption c) looks somewhat more mysterious but nevertheless plays an important role. In some sense c) is a hypothesis of rapid approach to equilibrium for a heat bath or Metropolis dynamics in Λ_i, reversible with respect to the Gibbs measure $\nu_i^{\eta,\tau_{i+1},\ldots,\tau_N}$. Using the arguments of Sect. 1, § 2, the constant α_1^2 becomes in fact proportional to the inverse of the gap of the generator of the dynamics, i.e. α_1^2 is proportional to the relaxation time in the "block" Λ_i. In the perturbation argument given below, the constant ε, which expresses the weak coupling between the blocks $\Lambda_1\ldots\Lambda_N$, will always appear multiplied by the constant α_1^2, and therefore the true "small" parameter of the analysis becomes the "coupling among the blocks $\times$ the relaxation time in a single block."

We will see in the next section that all the above assumptions follow from the finite volume mixing condition $SM(C,\gamma,L)$ defined in Sect. 1 provided that L is large enough and that the set Λ consists of a union of sufficiently "fat" subsets of the lattice $\mathbf{Z^d}$.

Under the above assumptions the following two theorems hold.

Theorem 2.1. *Given the constant α_1, for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\alpha_1, \delta)$ such that if $\varepsilon \sup_i |\Lambda_i| \leq \varepsilon_0$, then:*

$$\sup_\eta c_s(\mu^\eta) \leq (1+\delta) \sup_i \sup_{\eta, \tau_{i+1}, \ldots, \tau_N} c_s(\nu_i^{\eta, \tau_{i+1}, \ldots, \tau_N}) .$$

Remark. If we denote by ϱ the uniform measure on the single spin state space S, it is rather easy to see that

$$c_s(\nu_i^{\eta, \tau_{i+1}, \ldots, \tau_N}) \leq \left| \frac{\exp(W_{(i)}^\eta)}{Z_i} \right|_\infty c_s(\varrho)$$

(see e.g. Lemma 5.1 in [HS]).

In turn, using the very definition of the measure $\nu_i^{\eta, \tau_{i+1}, \ldots, \tau_N}$, it is not difficult to check that

$$\left| \frac{\exp(W_{(i)}^\eta}{Z_i} \right|_\infty \leq \exp\left(2 \sum_{X \cap \Lambda_i \neq \emptyset} |\Phi_X| \right).$$

Thus the logarithmic Sobolev constant $c_s(\nu_i^{\eta, \tau_{i+1}, \ldots, \tau_N})$ is finite uniformly in i and N if one has a good control on the interaction Φ_X since, as it is well known, $c_s(\varrho)$ is finite.

Theorem 2.2. *Given the constants α_0 and α_1 there exists $\varepsilon_0 = \varepsilon_0(\alpha_0, \alpha_1)$ such that if $\varepsilon \sup_i |\Lambda_i| \leq \varepsilon_0$, then there exist two constants α_2, α_3 depending on α_0 and α_1 such that for any function $f : \Omega_{\mathbf{Z^d}} \to R$ the following inequality holds:*

$$(\nabla_{\mathbf{Z^d} \setminus \Lambda} (\mu^\eta(f^2))^{1/2})^2 \leq \alpha_2 \mu^\eta((\nabla_{\mathbf{Z^d} \setminus \Lambda} f)^2) + \alpha_3 \sup_i |\Lambda_i| \mu^\eta((\nabla_\Lambda f)^2) .$$

Theorem 2.2 is a technical result which will be needed in Sect. 3.

Remark. Notice that in Theorem 2.2 the function f is a function from $\Omega_{\mathbf{Z^d}}$ to R and thus it may depend also on the boundary spins η. Therefore the expression $(\mu^\eta(f^2))^{1/2}$ may depend on the spins η which are involved in the derivatives $\nabla_{\mathbf{Z^d} \setminus \Lambda}$ in two ways: through the Gibbs measure μ^η and through the function f.

3. Proof of Theorem 2.1. If there was no interaction among the sets Λ_i then the total Gibbs measure μ^η would have been a product measure and the proof of the theorem would be a very simple exercise. However, our hypotheses say that the mutual interaction among the sets Λ_i is very weak in a suitable sense and it is therefore natural to try to make some perturbation theory around the non-interacting case. Because of the structure of the LSI, we found convenient first of all to write the average $\mu^\eta(f)$ of an arbitrary function f in a form that resembles as much as possible that of the average of f over a product measure. This form is as follows:

$$\mu^\eta(f) = \nu_N^\eta(\nu_{N-1}^{\eta, \tau_N}(\ldots(\nu_1^{\eta, \tau_2, \ldots, \tau_N}(f)\ldots) . \tag{2.12}$$

If we now apply this representation of $\mu^\eta(f)$ to the function $f^2 \log(f)$ we get:

$$\begin{aligned}
\mu^\eta(f^2 \log(f)) &= \nu_N^\eta(\nu_{N-1}^{\eta, \tau_N}(\ldots(\nu_1^{\eta, \tau_2, \ldots, \tau_N}(f^2 \log(f))\ldots) \\
&\leq c_1 \nu_N^\eta(\nu_{N-1}^{\eta, \tau_N}(\ldots(\nu_1^{\eta, \nu_2, \ldots, \tau_N}((\nabla_{\Lambda_1} f)^2)) \\
&\quad \nu_N^\eta(\nu_{N-1}^{\eta, \tau_N}(\ldots(\nu_2^{\eta, \tau_3, \ldots, \tau_N}(\nu_1^{\eta, \tau_2, \ldots, \tau_N}(f^2) \\
&\quad \times \log(\nu_1^{\eta, \tau_2, \ldots, \tau_N}(f^2))^{1/2}))) ,
\end{aligned} \tag{2.13}$$

where

$$c_1 = \sup_{\tau_2,\dots,\tau_N,\eta} c_s(\nu_1^{\eta,\tau_2,\dots,\tau_N}).$$

Next we define the new function

$$g_1 = (\nu_1^{\eta,\tau_2,\dots,\tau_N}(f^2))^{1/2},$$

and, more generally:

$$g_i = (\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(g_{i-1}^2))^{1/2}. \tag{2.14}$$

With these notation if we iterate (2.13) we obtain:

$$\mu^\eta(f^2 \log(f)) \le \sum_i c_i \mu^\eta((\nabla_{\Lambda_i} g_{i-1})^2) + \mu^\eta(f^2)\log((\mu^\eta(f^2))^{1/2}), \tag{2.15}$$

where

$$c_i = \sup_{\tau_{i+1},\dots,\tau_N,\eta} c_s(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}).$$

We are thus left with the estimate of the term:

$$\sum_i c_i \mu^\eta((\nabla_{\Lambda_i} g_{i-1})^2). \tag{2.16}$$

This is done in the next proposition where we show that, by paying a small price if the constant ε is small enough, one can safely replace in (2.16) the functions g_{i-1} with the function f.

Proposition 2.1. *Given the constant α_1, for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\alpha_1, \delta)$ such that if $\varepsilon \sup_i |\Lambda_i| \le \varepsilon_0$ then:*

$$\sum_i \mu^\eta((\nabla_{\Lambda_i} g_{i-1})^2) \le (1+\delta) \sum_i \mu^\eta((\nabla_{\Lambda_i} f)^2).$$

Before giving the proof of the proposition let us finish the proof of the theorem. If we use the result of the proposition we see that for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\alpha_1, \delta)$ such that if $\varepsilon \sup_i |\Lambda_i| \le \varepsilon$, then the r.h.s. of (2.15) can be bounded above by:

$$(1+\delta)\sup_i c_i \sum_i \mu^\eta((\nabla_{\Lambda_i} f)^2) + \mu^\eta(f^2)\log((\mu^\eta(f^2))^{1/2}), \tag{2.17}$$

i.e.

$$\sup_\eta c_s(\mu^\eta) \le (1+\delta) \sup_i \sup_{\tau_{i+1},\dots,\tau_N} c_s(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}),$$

and the theorem follows.

Proof of Proposition 2.1. One technical complication of working with discrete spins and discrete derivatives is that the latter do not enjoy exactly the same properties of the usual continuous derivatives like Leibniz rule and so forth.

Therefore before entering into the details of the proof let us give some elementary results concerning discrete derivatives that will be frequently used later on. Properties a), b), c), d), e) follow only from the definition of ∂_{σ_x} and are true in general whereas property f) is a consequence of assumption c) above on the interaction. A proof can be found in the Appendix.

In what follows $\langle f\rangle_x$ will denote the average of the function $f(\sigma_x)$ with respect to the measure $\frac{1}{2}(\delta_{+1}+\delta_{-1})$ and, for any given $x \in \Lambda_j$ with $j > i$,

$$\lambda_i^x = \left|\frac{d\nu_i^{\eta,\tau_{i+1}^{x,+1},\dots,\tau_N^{x,+1}}}{d\nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}}\right|_\infty \vee \left|\frac{d\nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}}{d\nu_i^{\eta,\tau_{i+1}^{x,+1},\dots,\tau_N^{x,+1}}}\right|_\infty .$$

Then we have:

a) $\partial_{\sigma_x} f^2 = 2\langle f\rangle_x \partial_{\sigma_x} f$.

b) $\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\langle|f|\rangle_x) \le \lambda_i^x \langle \nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(|f|)\rangle_x$.

c) If f is non-negative then $f(\sigma_x) \le 2\langle f\rangle_x$.

d) Given $x \in \Lambda_j$ with $j > i$:

$$\begin{aligned}
&|\partial_{\tau_x}[\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f)\dots)]| \\
&\quad \le |[\partial_{\tau_x}(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)| \\
&\qquad + \prod_{j=i}^{1} \lambda_j^x \nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(|\partial_{\tau_x} f|)\dots),
\end{aligned}$$

where $|[\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)|$ is a convenient way to denote the expression:

$$\frac{|\nu_i^{\eta,\tau_{i+1}^{x,+1},\dots,\tau_N^{x,+1}}(\dots(\nu_1^{\eta,\tau_2^{x,+1},\dots,\tau_N^{x,+1}}(f(\sigma\tau)\dots) - \nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}(\dots(\nu_1^{\eta,\tau_2^{x,-1},\dots,\tau_N^{x,-1}}(f(\sigma\tau)\dots)}{2}$$

e) ("Leibniz rule"). Let f be non-negative. Then, for $i \ge 2$:

$$\begin{aligned}
&|[\partial_{\tau_x}(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N})](f)| \\
&\quad \le |[\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}](\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f)\dots)| \\
&\qquad + \lambda_i^x \nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(|[\partial_{\tau_x}(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N})((f)|
\end{aligned}$$

f)

$$\begin{aligned}
&|[\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}](f^2)| \\
&\quad \le 2\alpha_1\lambda_i^x|\partial_{\tau_x} W_{(i)}^\eta(\sigma\tau)|_\infty [\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(f^2)^{1/2}[\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}((\nabla_{\Lambda_i} f)^2)]^{1/2} .
\end{aligned}$$

We are now ready to start our computations.

Since we have to estimate terms like $(\nabla_{\Lambda_i} g_{i-1})^2$, we start to estimate the following quantity:

$$|\partial_{\tau_x}((\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(g_{i-2}^2))^{1/2})| ,$$

where x is an arbitrary site in the set Λ_i. Following Zegarlinski and using a) above, we observe that it is sufficient to estimate

$$|\partial_{\tau_x}((\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(g_{i-2}^2))|$$

by

$$2\langle(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(g_{i-2}^2))^{1/2}\rangle_x[\dots]$$

to get that

$$|\partial_{\tau_x}(((\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(g_{i-2}^2))^{1/2})| \le [\dots] . \tag{2.18}$$

Using d) and e) above with f replaced by f^2, we get:

$$
\begin{aligned}
&|\partial_{\tau_x}[\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)]| \\
&\le \left(\prod_{j=i-1}^{1}\lambda_j^x\right)\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(|(\partial_{\tau_x}f^2)|)\ldots) \\
&\quad+\sum_{k=1}^{i-1}\left(\prod_{j=i-1}^{k+1}\lambda_j^x\right)\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots\nu_{k+1}^{\eta,\tau_{k+2},\ldots,\tau_N}(|[\partial_{\tau_x}\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}] \\
&\quad\times\nu_{k-1}^{\eta,\tau_2,\ldots,\tau_N}(\ldots\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)|)\,.
\end{aligned}
\tag{2.19}
$$

The first term in the r.h.s. of (2.19), using the Schwartz inequality, and properties a) and b), is bounded from above by:

$$
\begin{aligned}
&2\left(\prod_{j=i-1}^{1}\lambda_j^x\right)^{3/2}\langle(\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)^{1/2}\rangle_x \\
&\quad\times\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}((\partial_{\tau_x}f)^2)\ldots)^{1/2}\,.
\end{aligned}
\tag{2.20}
$$

Using f) above, the second term in the r.h.s. of (2.19) is bounded from above by:

$$
\begin{aligned}
&\sum_{k=1}^{i-1}\left(\sum_{j=i-1}^{k+1}\lambda_j^x\right)2\alpha_1\lambda_k^x|\partial_{\sigma(x)}W^{\eta_{(k)}(\sigma\tau)}|_\infty \\
&\quad\times\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_{k+1}^{\eta,\tau_{k+2},\ldots,\tau_N}([\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}(g_{k-1}^2)]^{1/2} \\
&\quad\times[\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}k(((\nabla_{\Lambda_k}g_{k-1})^2))]^{1/2}\,.
\end{aligned}
\tag{2.21}
$$

Using the definition of the function g_{k-1} we have:

$$
[\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}(g_{k-1}^2)]^{1/2}=[\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)]^{1/2}\,,
$$

and thus, using again a) and the Schwartz inequality, we get that (2.21) is bounded above by:

$$
\begin{aligned}
&2\langle[\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)]^{1/2}\rangle_x\sum_{k=1}^{i-1}\left[\left(\prod_{j=i-1}^{k+1}\lambda_j^x\right)|\partial_{\sigma(x)}W_{(k)}^\eta(\sigma\tau)|_\infty 2\alpha_1\lambda_k^x\right] \\
&\times\{\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(((\nabla_{\Lambda_k}g_{k-1})^2))\}^{1/2}\,.
\end{aligned}
\tag{2.22}
$$

We now define:

$$
V_{i,k}=\sup_{x\in\Lambda_i}\left[\left(\prod_{j=i-1}^{k+1}\lambda_j^x\right)|\partial_{\sigma(x)}W_{(k)}^\eta(\sigma\tau)|_\infty 2\alpha_1\lambda_k^x\right]
$$

and

$$
B_{i-1}=\sup_{x\in\Lambda_i}\left(\prod_{j=1}^{i-1}\lambda_j^x\right)^{3/2}\,.
$$

With this notation, if we put together (2.20) and (2.22), we obtain that:

$$\begin{aligned}
&|\partial_{\tau_x}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f^2)\dots)]| \\
&\quad \le 2\langle(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f^2)\dots)^{1/2})\rangle_x \\
&\quad \times \Bigg\{ B_{i-1}(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\partial_{\tau_x}f)^2\dots)^{1/2} \\
&\quad + \sum_{k=1}^{i-1} V_{i,k}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\nabla_{\Lambda_k}g_{k-1})^2))]^{1/2}\Bigg\}
\end{aligned} \tag{2.23}$$

i.e., using (2.18),

$$\begin{aligned}
&|\partial_{\tau_x}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f^2)\dots)]^{1/2}| \\
&\quad \le B_{i-1}(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\partial_{\tau_x}f)^2)\dots)^{1/2} \\
&\quad + \sum_{k=1}^{i-1} V_{i,k}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\nabla_{\Lambda_k}g_{k-1})^2))]^{1/2}\,.
\end{aligned} \tag{2.24}$$

Thus, using the above bound, we get that

$$\begin{aligned}
&\sum_{x\in\Lambda_i} |\partial_{\tau_x}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f^2)\cdot)]^{1/2}|^2 \\
&\quad \le pB_{i-1}^2(\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\partial_{\Lambda_i}f)^2)\dots) \\
&\quad + |\Lambda_i|q\left(\sum_{k=1}^{i-1}V_{i,k}\right)\sum_{k=1}^{i-1}V_{i,k}[\nu_{i-1}^{\eta,\tau_i,\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}((\nabla_{\Lambda_k}g_{k-1})^2))]
\end{aligned} \tag{2.25}$$

for any $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In conclusion, by summing (2.24) over the index i, we obtain for the initial expression $\sum_i \mu^\eta((\nabla_{\Lambda_i}g_{i-1})^2)$ the bound:

$$\begin{aligned}
\sum_i \mu^\eta((\nabla_{\Lambda_i}g_{i-1})^2) \le \sum_i \Bigg\{ & pB_{i-1}^2\mu^\eta(((\nabla_{\Lambda_i}f)^2)^2) \\
&+ |\Lambda_i|q\left(\sum_{k=1}^{i-1}V_{i,k}\right)\sum_{k=1}^{i-1}V_{i,k}\mu^\eta((\nabla_{\Lambda_k}g_{k-1})^2)\Bigg\}.
\end{aligned} \tag{2.26}$$

At this point we use our decay assumption a) on the interaction in order to estimate the numbers B_i and $V_{i,k}$.

From the definition of λ_j^x one has immediately that for $j \le i$:

$$\sup_{x\in\Lambda_i} \lambda_j^x \le \exp\left(4 \sup_{x\in\Lambda_i} |\partial_{\tau_x} W_{(j)}^\eta|_\infty\right). \tag{2.27}$$

Therefore:

$$B_{i-1} \le \exp\left(8\sum_{j=1}^{i-1} \sup_{x\in\Lambda_i} |\partial_{\tau_x} W_{(j)}^\eta|_\infty\right) \le \exp(8\varepsilon) \le 1 + 10\varepsilon \tag{2.28}$$

if ε is small enough.

Similarly:

$$\sum_{k=1}^{i-1} V_{i,k} \leq C\varepsilon \,,$$

$$\sum_{i=k+1}^{N} V_{i,k} \leq C\varepsilon$$

for a suitable constant C (which depends on α_1 but not on ε) and any sufficiently small ε.

Thus the second term in the r.h.s. of (2.26) is smaller than:

$$\sup_i |\Lambda_i| q(C\varepsilon)^2 \sum_{k=1}^{N} \mu^\eta(((\nabla_{\Lambda_k} g_{k-1})^2)) \,. \tag{2.29}$$

Therefore if $\sup_i |\Lambda_i| q(C\varepsilon)^2 < 1$ we get from (2.26), (2.28) and (2.29) that:

$$\begin{aligned}\sum_i \mu^\eta((\nabla_{\Lambda_i} g_{i-1})^2) &\leq \frac{p(1+10\varepsilon)}{1-\sup_i |\Lambda_i| q(C\varepsilon)^2} \sum_i \mu^\eta((\nabla_{\Lambda_i} f)^2)) \\ &\leq (1+\delta) \sum_i \mu^\eta((\nabla_{\Lambda_i} f)^2))\end{aligned}$$

if we choose for example $q = \frac{1}{\varepsilon}$ and ε sufficiently small. The proposition is proved.

4. Proof of Theorem 2.2. We proceed very similarly to the proof of Proposition 2.1. By doing the same kind of computations as in (2.18), ...,(2.24) we obtain that:

$$\begin{aligned}&\{|\partial_{\eta_x}[\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}(f^2)\ldots)]^{1/2}|\}^2 \\ &\leq \left\{ 2\left(\sup_{x\in \mathbf{Z^d}\setminus\Lambda} \prod_{j=1}^{N} (\lambda_j^x)^{3/2}\right)^2 (\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}((\partial_{\eta_x} f)^2)\ldots) \right. \\ &\quad \left. + 2\left(\sum_{k=1}^{N} \hat{V}_{x,k}[\nu_{i-1}^{\eta}(\ldots(\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}((\nabla_{\Lambda_k} g_{k-1})^2))]^{1/2}\right)^2 \right\},\end{aligned} \tag{2.30}$$

where

$$\hat{V}_{x,k} = \left[\left(\prod_{j=i-1}^{k+1} \lambda_j^x\right) |\partial_{\eta(x)} W_{(k)}^{\eta}(\sigma\tau)|_\infty 2\alpha_1 \lambda_k^x\right].$$

It is important to observe at this point that in general $\hat{V}_{x,k}$ is not small. Assumption a) (see (2.8) or (2.9)) in fact concerns only the interaction between different blocks while in some sense $\hat{V}_{x,k}$ measures the interaction between one block Λ_k and the boundary spin η_x. However, thanks to assumption b) (see (2.10)) and using (2.27), we have that:

$$\begin{aligned}\sup_{\eta,k} \sum_{x\in\mathbf{Z^d}\setminus\Lambda} \hat{V}_{x,k} &\leq 2\alpha_1\alpha_0 \sup_i |\Lambda_i| \exp\left(\sum_{j=1}^{N} 4|\partial_{\eta(x)} W_{(j)}^{\eta}(\sigma\tau)|_\infty\right) \\ &\leq \sup_i |\Lambda_i| 2\alpha_1\alpha_0 \exp(4\alpha_0) \,.\end{aligned} \tag{2.31}$$

Thus if we sum over $x \in \mathbf{Z^d} \backslash \Lambda$ the second term in the r.h.s. of (2.30), we get, after a Schwartz inequality:

$$\sum_{x\in \mathbf{Z^d}\backslash\Lambda} 2\left(\sum_{k=1}^{N} \hat{V}_{x,k}\right)\left[\sum_{k=1}^{N} \hat{V}_{x,k}\nu_N^{\eta}(\ldots(\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}((\nabla_{\Lambda_k} g_{k-1})^2))\right]$$
$$\leq \sup_i |\Lambda_i| 2\alpha_0^2\alpha_1^2 \exp(8\alpha_0)\left(\sum_{k=1}^{N}\nu_N^{\eta}(\ldots(\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}((\nabla_{\nabla_k} g_{k-1})^2)\right). \quad (2.32)$$

Using the identity:

$$\nu_N^{\eta}(\ldots(\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}(((\nabla_{\Lambda_k} g_{k-1})^2)) = \mu^{\eta}((\nabla_{\Lambda_k} g_{k-1})^2)$$

and Proposition 2.1, we see that if ε is small enough there exists a constant α_3 such that:

$$\sum_{x\in \mathbf{Z^d}\backslash\Lambda} 2\left(\sum_{k=1}^{N} \hat{V}_{x,k}\right)\left[\sum_{k=1}^{N} \hat{V}_{x,k}\nu_N^{\eta}(\ldots(\nu_k^{\eta,\tau_{k+1},\ldots,\tau_N}((\nabla_{\Lambda_k} g_{k-1})^2)\right]$$
$$\leq \sup_i |\Lambda_i| \alpha_3 \mu^{\eta}((\nabla_\Lambda f)^2). \quad (2.33)$$

Analogously if we sum over $x \in \mathbf{Z^d}\backslash\Lambda$ the first term in the r.h.s. of (2.30) we get:

$$\sum_{x\in \mathbf{Z^d}\backslash\Lambda} 2\left(\sup_{x\in \mathbf{Z^d}\backslash\Lambda}\prod_{j=1}^{N}\lambda_j^x\right)^2 (\nu_{i-1}^{\eta,\tau_i,\ldots,\tau_N}(\ldots(\nu_1^{\eta,\tau_2,\ldots,\tau_N}((\partial_{\eta_x} f)^2)\ldots)$$
$$\leq 2\exp(8\alpha_0)\mu^{\eta}(\nabla_{\mathbf{Z^d}\backslash\Lambda}(f)). \quad (2.34)$$

If we finally combine (2.34) and (2.33) we obtain the theorem with $\alpha_2 = 2\exp(8\alpha_0)$.

3. Decimation Approach to the Logarithmic Sobolev Constant

1. Proof of Theorem 1.3. In this section we prove our main result namely Theorem 1.3.

The result holds in any dimension but for the sake of simplicity of the exposition we will discuss explicitly only the two dimensional case. As already announced in Sect. 1, our proof is based upon ideas coming from rigorous renormalization group in classical statistical mechanics in the form known as *decimation.* We begin therefore by illustrating our decimation procedure.

For any odd integer L_0 let us consider the renormalized lattice $\mathbf{Z^d}(L_0) = L_0\mathbf{Z^d} \subset \mathbf{Z^d}$ and let us define for any $x \in \mathbf{Z^d}(L_0)$ the block $Q_{L_0}(x)$ as the square in the original lattice, centered at x and of side L_0. We will collect the blocks $Q_{L_0}(x)$ into four different families, denoted in the sequel by the letters A, B, C, D, according to whether the coordinates of their centers x are (even, odd), (even, even), (odd, even), or (odd, odd). We will also order in some way the blocks belonging to the same family so that they will be denoted by $A_1, A_2, \ldots, A_n, \ldots$, etc.

Let now $\Lambda(L_0)$ be a finite subset of $\mathbf{Z}^{\mathbf{d}}(L_0)$, let $\Lambda = \bigcup_{x\in\Lambda(L_0)} Q_{L_0}(x)$ and let μ^η be the Gibbs state in Λ corresponding to the Hamiltonian (1.1) with some fixed boundary condition η outisde Λ. Given μ^η we will consider new measures, denoted by:

$$\nu_A, \nu_B^{\tau_A}, \nu_C^{\tau_A \tau_B}, \nu^{\tau_A,\tau_B,\tau_C} \tag{3.1}$$

on the finite sets $\Omega_{A\cap\Lambda}$, $\Omega_{B\cap\Lambda}$, $\Omega_{C\cap\Lambda}$, $\Omega_{D\cap\Lambda}$. Such measures are defined in analogy with the measures $\nu_i^{\tau_{i+1}\dots\tau_N}$ of Sect. 2 as follows:

$\nu_D^{\tau_A,\tau_B,\tau_C}$ is simply obtained from the Gibbs measure μ^η by conditioning the spins in $A\cap\Lambda$, $B\cap\Lambda$, $C\cap\Lambda$ to have the prescribed values τ_A, τ_B, τ_C. To construct $\nu_C^{\tau_A,\tau_B}$ we first integrate out the spins σ_D in μ^η and then we condition to the spins in $A\cap\Lambda$, $B\cap\Lambda$ to have the prescribed values τ_A, τ_B. Similarly to construct $\nu_B^{\tau_A}$ we first integrate out the spins σ_D and σ_C in μ^η and then we condition to the spins in $A\cap\Lambda$ to have the values τ_A. ν_A is simply the relativization of μ^η to $A\cap\Lambda$.

Remark. We observe that by construction the intersection between the family of blocks of type A with the set Λ consists of a finite collection of blocks say $A_1, A_2, \dots, A_{N_A}$ and the same for the other families.

In the notation for the measures $\nu_D^{\tau_A,\tau_B,\tau_C}$, etc. obtained after the decimation, we often omit, for convenience, the superscript η.

We are now in a position to start our calculations. Given an arbitrary function $f:\Omega_\Lambda \to R$ we write, following [OP]:

$$\mu^\eta(f^2\log(f)) = \nu_A(\nu_B^{\tau_A}(\nu_C^{\tau_A,\tau_B}(\nu_D^{\tau_A,\tau_B,\tau_C}(f^2\log(f))))). \tag{3.2}$$

Let us now define $c(L_0)$ to be the largest among the logarithmic Sobolev constant (LSC) of the measures

$$\nu_A, \nu_B^{\tau_A}, \nu_C^{\tau_A,\tau_B}, \nu_D^{\tau_A,\tau_B,\tau_C},$$

more precisely:

$$c(L_0) = \sup_{\tau_A,\tau_B,\tau_C} \max\{c_s(\nu_A), c_s(\nu_B^{\tau_A}), c_s(\nu_C^{\tau_A,\tau_B}), c_s(\nu_D^{\tau_A,\tau_B,\tau_C})\}.$$

With this notation and if we apply the logarithmic Sobolev inequality to $\nu_D^{\tau_A,\tau_B,\tau_C}$, we obtain that the r.h.s. of (3.2) is bounded above by:

$$\begin{aligned} &c(L_0)\nu_A(\nu_B^{\tau_A}(\nu_C^{\tau_A,\tau_B}(\nu_D^{\tau_A,\tau_B,\tau_C}((\nabla_{A_D} f)^2)))) \\ &+ \nu_A(\nu_B^{\tau_A}(\nu_C^{\tau_A,\tau_B}(\nu_D^{\tau_A,\tau_B,\tau_C}(f^2)\log((\nu_D^{\tau_A,\tau_B,\tau_C}(f^2))^{1/2})))). \end{aligned} \tag{3.3}$$

Next we define the new functions:

$$\begin{aligned} g_D &= (\nu_D^{\tau_A,\tau_B,\tau_C}(f^2))^{1/2}, \\ g_C &= (\nu_C^{\tau_A,\tau_B}(g_D^2))^{1/2}, \\ g_B &= (\nu_B^{\tau_A}(g_C^2))^{1/2}. \end{aligned}$$

With these notation, if we iterate (3.3), we obtain:

$$\begin{aligned} \mu^\eta(f^2\log(f)) \le c(L_0)\,&[\mu^\eta((\nabla_{A_A} g_B)^2) + \mu^\eta((\nabla_{A_B} g_C)^2) \\ &+ \mu^\eta((\nabla_{A_C} g_D)^2) + \mu^\eta((\nabla_{A_D} f)^2)] \\ &+ \mu^\eta(f^2)\log((\mu^\eta(f^2))^{1/2}). \end{aligned} \tag{3.4}$$

The proof of Theorem 1.3 will immediately follow from the next result:

Theorem 3.1. *Let us use* $\{*, \tau^*\}$ *to denote a generic pair among:*

$$\{A,\eta\}\ \{B,\tau_A\eta\}\ \{C,\tau_A\tau_B\eta\}\ \{D,\tau_A\tau_B\tau_C\eta\}$$

and let $\nu_*^{\tau*}$ *be the corresponding Gibbs measure. There exists a constant* $\bar{L}$ *such that if* $SM(L,C,\gamma)$ *holds for some* $L \geq \bar{L}$*, then there exists* $\bar{L}_0 > \bar{L}$ *such that if* $L_0 \geq \bar{L}_0$ *then there exist two constants* a_0 *and* a_1 *such that:*

i)

$$c(L_0) < \infty\,,$$

ii)

$$(\nabla_{\Lambda_{\tau(x)}}(\nu_*^{\tau*}(f^2))^{1/2})^2 \leq a_0\nu_*^{\tau*}((\nabla_{\Lambda_{\tau(x)}} f)^2) + a_1\nu_*^{\tau*}((\nabla_{\Lambda*} f)^2)$$

for any $x \notin *$.

Before giving the proof of the above crucial result, let us first complete the proof of the main theorem.

If we apply ii) of Theorem 3.1 to $\mu^\eta((\nabla_{\Lambda_A} g_B)^2)$ we get that:

$$\mu^\eta((\nabla_{\Lambda_A} g_B)^2) \leq a_0\mu^\eta((\nabla_{\Lambda_A} g_C)^2) + a_1\mu^\eta((\nabla_{\Lambda_B} g_C)^2)\,.$$

We have thus succeeded in moving the gradient from the function g_B to the function g_C. If we continue this procedure two more times we end up with all the gradients acting on the original function f. More explicitly, after three repeated applications of ii) of Theorem 3.1, we have that:

$$\mu^\eta((\nabla_{\Lambda_A} g_B)^2) \leq a_2\mu^\eta((\nabla_\Lambda f)^2) \tag{3.5}$$

for a suitable constant a_2. The same estimate of course applies also to the terms:

$$\mu^\eta((\nabla_{\Lambda_B} g_C)^2) \quad \text{and} \quad \mu^\eta((\nabla_{\Lambda_C} g_D)^2)\,.$$

In conclusion we have shown that:

$$\begin{aligned} &c(L_0)\,[\mu^\eta((\nabla_{\Lambda_A} g_B)^2) + \mu^\eta((\nabla_{\Lambda_B} g_C)^2) + \mu^\eta((\nabla_{\Lambda_C} g_D)^2) + \mu^\eta((\nabla_{\Lambda_D} f)^2)] \\ &\quad \leq c'(L_0)\mu^\eta((\nabla_\Lambda f)^2)\,, \end{aligned} \tag{3.6}$$

provided that $SM(L,C,\gamma)$ holds for some L large enough and the size of the blocks of the decimation was also sufficiently large.

Theorem (1.3) is proved.

2. Proof of Theorem 3.1. The main idea of the proof is to show that, if we denote by $\Lambda_1 \ldots \Lambda_{N*}$ the blocks in the family $*$, then, in the hypotheses of the theorem and provided that the parameter L_0 is large enough, the measure $\nu_*^{\tau^*}$ satisfies the assumptions a), b) and c) of Theorems 2.1, 2.2 with constants α_0, α_1 uniformly bounded in the side L_0 of the blocks of the decimation and in the boundary conditions, with the constant ε going exponentially fast to zero as $L_0 \to \infty$.

In order to verify a) and b) we first need to write in a convenient way the derivative with respect to a conditioning spin of the effective potential appearing in any of the measures (3.1). One possibility is to use a cluster expansion to write down the effective potential; there is, however, another way in which the derivative with respect to a conditioning spin of the effective potential becomes essentially a truncated correlation function of a suitable pair of local observables computed with respect to the *original* Gibbs measure μ^η. That is of course very convenient since (see Theorem 1.2) it has been proved that in the hypotheses of the theorem the truncated correlation functions of the measure μ^η decay exponentially fast.

So let us discuss the second way in a rather general setting.

Suppose that we are given a subset $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ of the lattice $\mathbf{Z}^\mathbf{d}$ and a Gibbs measure μ on $\{-1,+1\}^\Lambda$ with Hamiltonian $H(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}, \sigma_{\Lambda_3})$ corresponding to a interaction Φ with finite norm

$$\|\Phi\| = \sum_{O \in X} |\Phi(X)| .$$

Let

$$\hat{H}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) = \log(Z_{\Lambda_3}^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}) \tag{3.7}$$

be the effective Hamiltonian after the integration of the σ_3 variables in Λ_3 and let

$$\hat{W}_1^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1}) \equiv \hat{H}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) - \hat{H}(\tau_1, \sigma_{\Lambda_2}) \tag{3.8}$$

be the effective interaction entering in the conditional Gibbs measure of the spins σ_{Λ_1} given the spins σ_{Λ_2} after the decimation of the spins σ_{Λ_3} [see (2.6)]. In (3.8) τ_1 is an arbitrary reference configuration of the spins σ_{Λ_1}, e.g. all the spins up. Then we have:

Lemma 3.1. *For each $x \in \Lambda_2$ and $y \in \Lambda_1$ there exist two functions $f_x^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3})$ and $g_y^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3})$ with the following properties:*

i) *f and g, as functions of the spins σ_{Λ_3}, have support in a ball centered at x and y respectively with radius equal to the range of the interaction Φ.*

ii)

$$\sup_{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}, \sigma_{\Lambda_3}} |f_x^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3})| \le \exp(2\|\Phi\|) ,$$
$$\sup_{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}, \sigma_{\Lambda_3}} |g_y^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3})| \le \exp(2\|\Phi\|) .$$

iii)

$$\sup_{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}} |\partial_{\sigma_{\Lambda_2}(x)} \hat{W}_1^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1})| \le \sum_{y \in \Lambda_1} \exp(4\|\Phi\|) \sup_{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}} |\langle f_x^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}; g_y^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}} \rangle_{\Lambda_3}^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}|$$
$$+ \sum_{y \in \Lambda_1} \sup_{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}} \left| \log \left(\frac{\langle g_y^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}} \rangle_{\Lambda_3}^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}}{\langle g_y^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}^x} \rangle_{\Lambda_3}^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}} \right) \right| ,$$

where $\langle f \rangle_{\Lambda_3}^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}$ is the conditional average of the observable f with respect to the original Gibbs state given that the spins in Λ_1 and Λ_2 are equal to σ_{Λ_1} and σ_{Λ_2} and $\langle f; g \rangle$ denotes the usual truncated expectation.

Proof. Let for $x \in \Lambda_2$ and $y \in \Lambda_1$,

$$\begin{aligned} f_x^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3}) &= \exp(H^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}^x}(\sigma_{\Lambda_3}) - H^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3})) , \\ g_x^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}}(\sigma_{\Lambda_3}) &= \exp(H^{\sigma_{\Lambda_1}^y, \sigma_{\Lambda_2}^x}(\sigma_{\Lambda_3}) - H^{\sigma_{\Lambda_1}, \sigma_{\Lambda_2}^x}(\sigma_{\Lambda_3}^x)) . \end{aligned} \tag{3.9}$$

If we use (3.7), (3.8) and the definition of $\partial_{\sigma(x)}$ we obtain:

$$\begin{aligned}
&\sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} |\partial_{\sigma_{\Lambda_2}(x)} \hat{W}_1^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1})| \\
&\leq 2\sum_{y\in\Lambda_1} \sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} |\partial_{\sigma_{\Lambda_2}(x)}(\hat{H}(\sigma_{\Lambda_1},\sigma_{\Lambda_2}) - \hat{H}(\sigma^y_{\Lambda_1},\sigma_{\Lambda_2}))| \\
&= \sum_{y\in\Lambda_1} \sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} \Bigg| \log(\langle f^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma^x_{\Lambda_2}}) \\
&\quad - \log\left(\frac{\langle g_y^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} f_x^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}}{\langle g_y^{\sigma_{\Lambda_1},\sigma^x_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}}\right)\Bigg| ,
\end{aligned} \tag{3.10}$$

Using $\log(1+x) \leq x$ if $x > 0$ we get immediately that the r.h.s. of (3.10) is bounded by:

$$\begin{aligned}
&\sum_{y\in\Lambda_1} \exp(4\|\Phi\|) \sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} |\langle f_x^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}; g_y^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}| \\
&+ \sum_{y\in\Lambda_1} \sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} \left| \log\left(\frac{\langle g_y^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}}{\langle g_y^{\sigma_{\Lambda_1},\sigma^x_{\Lambda_2}}\rangle_{\Lambda_3}^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}}\right)\right| .
\end{aligned} \tag{3.11}$$

The lemma is proved.

One can now apply the above lemma to verify assumptions a) and b) of Sect. 2, § 2 for each one of the measures (3.1), by conveniently choosing the sets $\Lambda_1, \Lambda_2, \Lambda_3$. Since the discussion is the same for anyone of the measures (3.1), let us treat in detail only one of them, say $\nu_B^{\tau A}$.

Thus let us suppose that we have fixed a block of type B, say B_j, and let us consider the effective interaction $W_{(j)}^{\tau A}(\sigma_{B_j}; \sigma_{B_{j+1}} \ldots)$ obtained from the potential corresponding to the Gibbs measure $\nu_B^{\tau A}$ after the integration of the variables $\sigma_{B_1} \ldots \sigma_{B_{j-1}}$ while keeping the variables $\sigma_{B_{j+1}} \ldots$ fixed. If we set

$$\Lambda_1 = B_j, \qquad \Lambda_3 = C \cup D \cup \left(\bigcup_{k<j} B_k\right) \tag{3.12}$$

and

$$\Lambda_2 = \mathbf{Z^d} \backslash (\Lambda_1 \cup \Lambda_2) = (\mathbf{Z^d} \backslash \Lambda) \cup \left(A \cup \bigcup_{i>j} B_i\right), \tag{3.13}$$

we can write:

$$W_{(j)}^{\tau A}(\sigma_{B_j}; \sigma_{B_{j+1}} \ldots) = W_{\Lambda_1}^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1}),$$

where

$$W_{\Lambda_1}^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1}) = \hat{W}_{\Lambda_1}^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1}) + \tilde{W}_{\Lambda_1}^{\sigma_{\Lambda_2}}(\sigma_{\Lambda_1})$$

with

$$\hat{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma_{\Lambda_1}) = \log(Z^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}_{\Lambda_3}) - \log(Z^{\tau_{\Lambda_1},\sigma_{\Lambda_2}}_{\Lambda_3}),$$

$$\tilde{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma_{\Lambda_1}) = \sum_{X\cap\Lambda_1\neq\emptyset;X\cap\Lambda_3=\emptyset} U_X \prod_{x\in X}\sigma_x - \sum_{X\cap\Lambda_1\neq\emptyset;X\cap\Lambda_3=\emptyset} U_X \prod_{x\in X\cap\Lambda_1}\tau_x \prod_{x\in X\setminus\Lambda_1}\sigma_x,$$

τ_{Λ_1} being a reference configuration, is the effective interaction obtained from the *original* potential $\{U_X\}$ after integration over the variables σ_{Λ_3} while keeping fixed the variables σ_{Λ_2}.

By construction:

$$|\partial_{\sigma_x}\tilde{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma_{\Lambda_1}| \le \|U\| \quad \text{if} \quad \text{dist}(x,\Lambda_2)\le r,$$

$$|\partial_{\sigma_x}\tilde{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma_{\Lambda_1})| = 0 \quad \text{otherwise},$$

where r is the range of the interaction U.

Therefore, since L_0 is larger than r, in order to verify assumption a) (or b)), it is sufficient to estimate

$$|\partial_{\sigma_x}\hat{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma_{\Lambda_1})|.$$

This will be done by means of Lemma 3.1.

Let x be a site in B_i, with $i > j$; in order to apply Lemma 3.1, we observe that since L_0 is greater than the range of the interaction, then the above defined function $g_y^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}$ does not depend on $\sigma(x)$ if $y \in B_j$ and $x \in B_i$. Therefore in this case the second term in the r.h.s. of iii) of the lemma is zero and we get:

$$\sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} |\partial_{\sigma(x)}\hat{W}^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma)| \le \sum_{y\in B_j} \exp(4\|\Phi\|) \sup_{\sigma_{\Lambda_1},\sigma_{\Lambda_2}} |\langle f_x^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}; g_y^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}\rangle^{\sigma_{\Lambda_1},\sigma_{\Lambda_2}}_{\Lambda_3}|. \tag{3.14}$$

Using now Theorem 2.1 we see that there exist constants C and m, depending only on the norm of the original interaction and on its range r, such that for any sufficiently large L_0 the r.h.s. of (3.14) is smaller than:

$$C\sum_{y\in B_j}\exp(-m\,|x-y) \tag{3.15}$$

which implies that:

$$\sup_{(\eta,\sigma_A,j,N_B)} \sum_{i=1}^{j-1} \sup_{x\in B_i} |\partial_{\sigma(x)} W^{\sigma_{\Lambda_2}}_{\Lambda_1}(\sigma)|_\infty \le \varepsilon \tag{3.16}$$

with e.g. $\varepsilon = \exp\left(-\frac{m}{2}L_0\right)$ for any sufficiently large L_0.

In a very similar way one checks the bounds (2.9) and (2.10).

Remark. The conclusion of the above discussion is that the effective potential between any two sites x and y, defined for example as $\partial_{\sigma(x)}\partial_{\sigma(y)}\hat{H}$ with $\hat{H}$ the effective hamiltonian of any of the measures (3.1), decays exponentially fast in the distance between x and y. This implies in particular that in the second inequality in b) of Sect. 2 we can replace $\sup_i |B_i| = L_0^d$ with L_0^{d-1}.

In order to complete the proof of Theorem 3.1 we are left with the problem of verifying assumption c) of Sect. 2. The idea at this point is to show that assumption c) is equivalent to a *lower* bound on the gap of the generator of the Glauber dynamics defined in Sect. 1, reversible with respect to $\nu_*^{\tau*}$. In turn, such a lower bound will follow from Theorem 1.2, i). As before we do the computations only for $\nu_B^{\tau A}$ the other cases being analogous.

We keep the notation (3.12), (3.13) and we denote by ν_1 the conditional distribution on Ω_{Λ_1} of the relativization to $\Lambda_1 \cup \Lambda_2$ of the original Gibbs measure μ^η. For simplicity, in ν_1, we have omitted to specify the boundary conditions τ_1 since all our estimates will hold uniformly in τ_1. First of all we observe that, from assumption H5 on the jump rates and Theorem 1.2 i), it follows, for any $f \in L^2(\Lambda_{\Lambda_1}, d\nu_1)$ with $\nu_1(f) = \mu^\eta(f) = 0$, that:

$$\nu_1(f^2) = \mu^\eta(f^2) \le \frac{\alpha_2}{m_0}\,\mu^\eta((\nabla_{\Lambda_1} f)^2) = \frac{\alpha_2}{m_0}\,\nu_1((\nabla_{\Lambda_1} f)^2)\,. \tag{3.17}$$

Therefore we get immediately that:

$$\sum_{\sigma_{\Lambda_1},\hat\sigma_{\Lambda_1}} \nu_1(\sigma_{\Lambda_1})\nu_1(\hat\sigma_{\Lambda_1})\,[f(\sigma_{\Lambda_1}) - f(\hat\sigma_{\Lambda_1})]^2 \le \frac{2\alpha_2}{m_0}\,\nu_1((\nabla_{\Lambda_1}, f)^2)\,, \tag{3.18}$$

i.e. assumption c) holds true with the constant $\alpha_1 = \left(\dfrac{2\alpha_2}{m_0}\right)^{1/2}$.

Part ii) of the theorem now follows immediately from Theorem 2.2.

To get part i), namely a uniform upper bound on $c(L_0)$, we use Theorem 2.1 to write:

$$c(L_0) \le \sup_{\{*,\tau^*\}} \sup_i c_s(\nu_i^{\tau^{i+1\dots}})\,,$$

where $\nu_i^{\tau^{i+1\dots}}$ is the measure on the i^{th}-block (ordered in some way, e.g. lexicographically) of the family $*$ with boundary conditions τ^*, obtained by integrating out the variables in the first $i-1$ blocks while keeping fixed, equal to $\tau_{i+1}\dots$, the variables in the $i+1\dots$ blocks.

Using the remark after Theorem 2.1 it is sufficient to show that the sup-norm of the Radon-Nikodym derivative of $\nu_i^{\tau_{i+1}\dots}$ with respect to the uniform measure, $\dfrac{\exp(W_{(i)}^{\tau^*,\tau_{i+1}\dots})}{Z_i}$, is bounded uniformly in the boundary spins and in the size of the original volume Λ.

One possibility would be to prove a bound on the interaction of the Gibbs measure $\nu_*^{\tau^*}$. However, by recalling that $\nu_*^{\tau^*}$ itself comes from the decimation of the *original* Gibbs measure μ^η, we easily get that:

$$\left|\frac{\exp(W_{(i)}^{\tau^*,\tau_{i+1}\dots})}{Z_i}\right|_\infty \le \exp\Big(2\sum_{X\cap\Lambda_i\neq\emptyset} |U_X|\Big),$$

where Λ_i is the i^{th}-block of the family $*$ and $\{U_X\}$ is the *original* , finite range potential.

The theorem is proved.

4. An Application to a Non-Ferromagnetic Model

We conclude this paper by considering a non-ferromagnetic model in two dimensions, obtained by adding a small antiferromagnetic next nearest-neighbor coupling to the standard ferromagnetic Ising model with small positive external field.

We will show that if the antiferromagnetic n.n.n. coupling is small with respect to the n.n. ferromagnetic one, then it is possible to find constants C and γ such that for all large enough L and all low enough temperatures, depending on L_0, C and γ, the system satisfies our mixing condition $SM(\Lambda, C, \gamma)$. It is very likely (see e.g. the examples in Sect. 2 of [MO2]) that the system, in the same range of the parameters, does not satisfy the hypothesis of the main theorem in [SZ3].

If Λ denotes the square of side L (L odd) centered at the origin of $\mathbf{Z}^2$, then our Hamiltonian reads as follows:

$$\left(\frac{-1}{\beta}\right) H(\sigma) = -\frac{J}{2} \sum_{\langle x,y\rangle \subset \Lambda} \sigma(x)\sigma(y) + \frac{K}{2} \sum_{\langle\langle x,y\rangle\rangle \subset \Lambda} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x) + \text{b.c.}\,, \tag{4.1}$$

where $\sum_{\langle x,y\rangle \subset \Lambda}$ runs over the nearest neighbors pairs in Λ, $\sum_{\langle\langle x,y\rangle\rangle \subset \Lambda}$ runs over the next nearest neighbors pairs in Λ and b.c. contains the interaction with the boundary configuration τ. Notice that, in order to follow our convention (see Sect. 1.4)), we have inserted the factor $-\beta$ directly into the Hamiltonian.

Theorem 4.1. *Let $h > 0$. There exist $C, \gamma, \bar{L}$ such that for any $0 \le K < \frac{J}{4}$ and for any $L \ge \bar{L}, L$ odd, there exists β_0 such that for any $\beta \ge \beta_0$ the mixing condition $SM(\Lambda, C, \gamma)$ holds.*

Proof. Let us denote by dist′ the following distance on $\mathbf{Z}^2$:

$$\text{dist}'(x, y) = \max_i |x_i - y_i|\,, \qquad x, y \in \mathbf{Z}^\mathbf{d}\,.$$

Let

$$\bar{\mathscr{C}} = \{x \in \Lambda;\ \text{dist}'(x,\ \text{corners of } \Lambda) \ge l_0\}$$

with

$$l_0 = [2(J - 2K) + 8K]/h\,.$$

Thus $\bar{\mathscr{C}}$ looks like a cross.

The theorem will immediately follow (see also Sect. 5 of [MO2]) if we can show that for any L large enough, any boundary configuration τ and any $0 \le K < \frac{J}{4}$ the ground state of $H^\tau_\Lambda(\sigma)$ is equal to $+1$ at all the sites x in $\bar{\mathscr{C}}$. Let in fact C and γ be such that:

$$C \exp(-\gamma\sqrt{2}l_0) \ge 1\,. \tag{4.2}$$

If, for any boundary condition τ, the ground state has the structure described above, then, because of the screening effect of the plus spins in $\bar{\mathscr{C}}$ the ground states in each connected component (square) Q_i, $i = 1, \ldots, 4$ of $\Lambda \backslash \bar{\mathscr{C}}$ is only affected by a change of a boundary spin τ_y, $y \in \partial^+_{\sqrt{2}} Q_i \cap \partial^+_{\sqrt{2}} \Lambda$. Therefore we can always estimate the quantity

$$\sup_{\tau,\tau^{(y)} \in \Omega_{\Lambda_0^c}} \operatorname{Var}(\mu^{\tau}_{\Lambda,\Delta} \mu^{\tau^{(y)}}_{\Lambda,\Delta}), \qquad \Delta \subset \Lambda \tag{4.3}$$

appearing in $SM(\Lambda, C, \gamma)$ by 1 when for some $i = 1, \dots, 4$,

$$y \in \partial^+_{\sqrt{2}} Q_i \cap \partial^+_{\sqrt{2}} \Lambda, \qquad \Delta \cap Q_i \neq \emptyset$$

or by $2\mu^{\tau}_{\Lambda}(\sigma_x \neq +1$ for some $x \in \bar{\mathscr{C}})$ otherwise. In both cases we get an estimate smaller than:

$$C \exp(-\gamma \operatorname{dist}(\Delta, y))$$

for large enough β because of our choice of C and γ and the fact that

$$\lim_{\beta \to \infty} \mu^{\tau}_{\Lambda}(\sigma \neq \text{ground state}) = 0 . \tag{4.4}$$

In order to prove the above structure of the set of all the ground states we will use the following "rules":

i) Let N^- be the number of minus spins in a ground state configuration. Then:

$$N^- \leq \frac{4(J + 2K)L + 16K}{h} .$$

ii) In any ground state configuration there exists no horizontal or vertical segment of minus spins (thought of as a thin rectangle) surrounded on two adjacent sides (one of which is a "long" one) by plus spins and with at least a plus spin along a third side.
iii) In any ground state configuration there exists no Peierl's contour with a horizontal or vertical segment of length $l \geq l_0$.
iv) In any ground state configuration if there exists a Peierl's contour γ with a right angle at the side $\left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}\right)$ of the dual lattice, $x \equiv (x_1, x_2) \in \Lambda$, such that the plus spins lay along the exterior of the angle then, starting from $\left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2}\right)$, the contour γ has to reach the vertical and horizontal boundary of Λ without bending.

Rules i), ii), iii) are easily verified by simple energy arguments if $4K < J$. Rule iv) is slightly more complicated. The proof goes as follows.

Without loss of generality let us suppose that the angle has the plus spins at its right and bottom and let us suppose that the contour γ has another right angle at the site $\left(x_1 - n + \frac{1}{2}, x_2 + \frac{1}{2}\right)$ with $x_1 - n \geq -\frac{(L-1)}{2}$. Using ii) the contour γ at the new angle can only bend down; moreover by ii), the minus spins above γ at the sites $(x_1 - j, x_2 + 1)$, $j = 1 \dots n$ have to be surrounded from the left and from above by minus spins. It is easy to see that if the spin at the side $(x_1 + 1, x_2 + 2)$ is minus then it is energetically convenient to flip to plus one all the minus spins at the sites $(x_1 - j, x_2 + 1)$, $j = 1 \dots n$ irrespectively of the value of the spin at the site $(x_1 - n, x_2 + 2)$, and the same if the spin at $(x_1 - n, x_2 + 2)$ is minus. If the spins at $(x_1 - n, x_2 + 2)$ and at $(x_1 + 1, x_2 + 2)$ are both plus then it is energetically convenient to flip to plus all the minus spins at the sites $(x_1 - j, x_2 + 1)$, $j = 1 \dots n$, $(x_1 - j, x_2 + 2)$, $j = 1 \dots n$. In any case the original configuration was certainly not a ground state. Similar arguments cover all the other situations.

It is easy to show, at this point, that, for every τ, the structure of the ground state is the one depicted above.

If L is taken large enough,

$$L^2 > 4(3l_0)^2[(4J+2K)L+16K]/h\,,$$

then, using i), for any ground state it is possible to find a square Q^* of side $l = 3l_0$ completely filled up with pluses and strictly contained in $\bar{\mathscr{C}}$.

Using ii), iii), we know that on top of each face of Q^* there exists a segment of length l_1 larger than l_0 of plus spins. If $l_1 < l$ then there is at least one right angle with exterior + spins at one end of the concerned segment. If not, all the spins adjacent from the exterior to that face of Q^* are plus and we can repeat the argument. Then, continuing in this way, starting from any face of Q^*, either we get to $\partial\Lambda$ on a parallel segment of equal length l or, at a given step, we find a right angle inside Λ. Using iv), by further decreasing the energy, we obtain a configuration containing a cross $\mathscr{C}$ of plus spins centered inside Q^* with width at least l_0. The complement of $\mathscr{C}$ in Λ splits into four disjoint rectangles R_1, R_2, R_3, R_4. Each R_i, by construction, contains a square Q_i of side l_0 having a vertex coinciding with one of the four vertices of Λ. By applying to the faces of R_i, internal to Λ, a construction similar to the one leading to $\mathscr{C}$ it is easy to show that it is energetically convenient to fill by pluses the sets $R_i \backslash Q_i$ so that we end up with a configuration where $\bar{\mathscr{C}}$ is full of pluses.

Appendix

We prove formulae a)...f) given at the beginning of the proof of Proposition 1.2.

The first three ones, a), b), c), are trivially verified. In order to derive d) we assume, without loss of generality, that $\tau_x = +1$ and we let $g_i^x = \dfrac{d\nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}}{d\nu_i^{\eta,\tau_{i+1}^{x,+1},\dots,\tau_N^{x,+1}}}$. Then we write:

$$\begin{aligned}
&|\partial_{\tau_x}[\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(f)\dots)]|\\
&= |\nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}(\dots(\nu_1^{\eta,\tau_2^{x,-1},\dots,\tau_N^{x,-1}}(\partial_{\tau_x}f)\dots)\\
&\quad -[\partial_{\tau_x}(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)|\\
&= \left|\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}\left(\dots\left(\nu_1^{\eta,\tau_2,\dots,\tau_N}\left(\prod_{j=1}^{1} g_j^x\partial_{\tau_x f}\right)\dots\right)\right.\\
&\quad \left. -[\partial_{\tau_x}(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)\right|\\
&\le |[\partial_{\tau_x}(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)|\\
&\quad +\prod_{j=i}^{1}\lambda_j^x\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}(|\partial_{\tau_x}f|)\dots)\,,
\end{aligned}$$

where $|[\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(\dots(\nu_1^{\eta,\tau_2,\dots,\tau_N}](f)|$ denotes the expression:

$$\frac{|\nu_i^{\eta,\tau_{i+1}^{x,+1},\dots,\tau_N^{x,+1}}(\dots(\nu_1^{\eta,\tau_2^{x,+1},\dots,\tau_N^{x,+1}}(f(\sigma\tau)\dots)-\nu_i^{\eta,\tau_{i+1}^{x,-1},\dots,\tau_N^{x,-1}}(\dots(\nu_1^{\eta,\tau_2^{i,-1},\dots,\tau_N^{x,-1}}(f(\sigma\tau)\dots)|}{2}\,. \tag{A1.1}$$

Clearly (A1.1) proves d).

The "Leibniz rule" e) follows by essentially the same argument.

In order to prove f) we follow Zegarlinski [Z1] and we write:

$$4|[\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}](f^2)|$$

as:

$$\begin{aligned}
2|\{\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(f^2) - \nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(g_i^x f^2)\}| \\
= |\nu_i^{\eta,\tau_{i+1},\dots,\tau_N} \times \nu_i^{\eta,\tau_{i+1},\dots,\tau_N} \\
\times ([f^2(\sigma_{\Lambda_i}) - f^2(\tilde{\sigma}_{\Lambda_i})]\,[g_i^x(\sigma_{\Lambda_i}) - g_i^x(\tilde{\sigma}_{\Lambda_i})])|\,,
\end{aligned} \tag{A1.2}$$

where σ_{Λ_i} and $\tilde{\sigma}_{\Lambda_i}$ are two independent replicas in Ω_{Λ_i}.

If we assume, without loss of generality, that $g_i^x(\sigma_{\Lambda_i}) \geq g_i^x(\tilde{\sigma}_{\Lambda_i})$ then, from the definition of g_i^x, we get:

$$g_i^x(\sigma_{\Lambda_i}) - g_i^x(\tilde{\sigma}_{\Lambda_i}) \leq g_i^x(\sigma_{\Lambda_i})4|\partial_{\tau_x} W_{(i)}^\eta(\sigma\tau)|_\infty\,. \tag{A1.3}$$

The Schwartz inequality gives that:

$$\begin{aligned}
|\nu_i^{\eta,\tau_{i+1},\dots,\tau_N} \times \nu_i^{\eta,\tau_{i+1},\dots,\tau_N}([f^2(\sigma_{\Lambda_i}) - f^2(\tilde{\sigma}_{\Lambda_i})])| \\
\leq (\nu_i^{\eta,\tau_{i+1},\dots,\tau_N} \times \nu_i^{\eta,\tau_{i+1},\dots,\tau_N} \\
\times ([f(\sigma_{\Lambda_i}) - f(\tilde{\sigma}_{\Lambda_i})]^2))^{1/2} 2(\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(f^2))^{1/2}
\end{aligned} \tag{A1.4}$$

which, in turn, implies, using (2.11) and (A1.3) above, that

$$\begin{aligned}
4|\partial_{\tau_x}\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}](f^2)| \leq 8\alpha_1\lambda_i^x|\partial_{\tau(x)}W_{(i)}^\eta(\sigma\tau)|_\infty[\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}(f^2)]^{1/2} \\
\times [\nu_i^{\eta,\tau_{i+1},\dots,\tau_N}((\nabla_{\Lambda_i} f)^2)]^{1/2}\,,
\end{aligned}$$

i.e. the inequality f).

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